Kybernetika

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Kybernetika, Vol. 13 (1977), No. 1, (41)--56

Persistent URL: http://dml.cz/dmlcz/125112

KYBERNETIKA — VOLUME 13 (1977), NUMBER 1

On Fuzzy-Quantities with Real and Integer Values

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The presented paper is a free continuation of the author's work [2]. It deals with the application of fuzzy-sets theory, created by Zadeh in [1], to the mathematical models of non-exactly known quantities. The general properties of such quantities, called here fuzzy-quantities, especially the description of their values by means of fuzzy-sets, were investigated in [2]. Results of their addition or of their multiplication by natural numbers are also non-exactly known quantities and their values may be described by fuzzy-sets as shown in [2]. In case the values of fuzzy-quantities are real or integer numbers, some special properties of the model begin to be interesting, and some additional problems appear. These properties and problems are discussed and solved in this paper.

1. INVESTIGATED MODEL

In the whole paper we denote by N, I and R the sets of all natural, integer and real numbers, respectively. By $\mathscr{B}$ we denote the σ-algebra of all Borel sets in R and by $\mathscr{I}$ the σ-algebra of all subsets of I. Finally, by λ we denote the Lebesgue measure on $(R, \mathscr{B})$ and by κ we denote the counting measure on $(I, \mathscr{I})$

$$\kappa(J) = \text{number of elements in } J\,, \quad \text{if} \quad J \subset I, \quad J \text{ is finite}\,,$$

$$\kappa(J) = +\infty\,, \quad \text{if} \quad J \subset I\,, \quad J \text{ is infinite.}$$

It is obvious that both measures λ and κ are σ-finite and that they are Haar measures according to operation of addition

According to Zadeh's work [1] we shall call a *fuzzy-subset* of R or of I any real-valued function f on R or I, respectively, with values $f(x)$ in the closed interval $\langle 0, 1\rangle$ $x \in R$, or $x \in I$. These functions represent a generalization of the classical set characteristic functions with values equal to 0 or 1 only, and they are interpreted so that the value $f(x)$ is the closer to 1 the greater is our expectation that x belongs to the fuzzy-set represented by f. It means that fuzzy-sets may be used for mathematical modelling of quantities the values of which are known only approximately. If $\boldsymbol{a}$ is a quantity the value of which is a non-exactly known real or integer number then it may be described by a fuzzy-subset $f_{\boldsymbol{a}}$ of R or of I, respectively.

Such non-exactly known quantities with values represented by fuzzy-sets are in this paper called *fuzzy-quantities.* By $f_R \equiv 1$ and $f_I \equiv 1$ we denote here the maximal fuzzy-subsets of R and I, respectively, which represent the sets R and I themselves.

If f and g are two $\mathscr{B}$-measurable functions on R then their *convolution* $h = f * g$ is a function

$$h(z) = \int f(z - y)\, g(y)\, dy = \int f(x)\, g(z - x)\, dx \tag{1.1}$$

defined for all $z \in R$ for which this integral exists.

Analogously, if f and g are $\mathscr{I}$-measurable functions on I then their *convolution* $h = f * g$ is a function

$$h(i) = \sum_{j \in I} f(i - j)\, g(j) = \sum_{j \in I} f(j)\, g(i - j) \tag{1.2}$$

defined also for all $i \in I$ for which the sum converges.

A *support-set* of a function f on R or I is the set

$$S_f = \{x \in X : f(x) \neq 0\} \quad \text{for} \quad X = R \quad \text{or} \quad X = I\,, \quad \text{respectively}\,. \tag{1.3}$$

In the whole paper we suppose that the values of all fuzzy-quantities considered here are represented by measurable and integrable functions on $(R, \mathscr{B})$ or $(I, \mathscr{I})$ (namely by fuzzy-subsets of R or I), and that their support sets are bounded subsets of R or I, respectively. Then Statements 2.1 ansd 2.2 from [2] imply that convolutions of fuzzy-subsets of R and I exist and that they are measurable, integrable and bounded functions on R and I, respectively.

2. ADDITION OF FUZZY-QUANTITIES

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are two fuzzy-quantities with values represented by fuzzy-sets $f_{\boldsymbol{a}}$ and $f_{\boldsymbol{b}}$ then their sum $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$ is also a fuzzy-quantity with values represented by fuzzy-set

$$f_{\boldsymbol{c}} = f_{\boldsymbol{a}+\boldsymbol{b}} = \min\{1, f_{\boldsymbol{a}} * f_{\boldsymbol{b}}\}\,. \tag{2.1}$$

Analogously, if $a_1, a_2, \ldots, a_n$ are fuzzy-quantities with values represented by fuzzy-sets $f_{a_1}, f_{a_2}, \ldots, f_{a_n}$ then $c = a_1 + a_2 + \ldots + a_n$ is also a fuzzy-quantity and

$$f_c(x) = f_{a_1+a_2+\ldots+a_n}(x) = \min\{1, [f_{a_1} * f_{a_2} * \ldots * f_{a_n}](x)\} \tag{2.2}$$

is the fuzzy set representing the values of c.

Let $n \in N$, and let a and c be fuzzy-quantities with real or integer values represented by fuzzy-sets f_a and f_c. Let

$$c = \underbrace{a + a + \ldots + a}_{n\text{-}times} = na\,. \tag{2.3}$$

Then (2.2) implies that

$$f_c = f_{na} = \min\{1, [f_a * f_a * \ldots * f_a](x)\}\,. \tag{2.4}$$

Let us suppose, finally, that the natural number n in relation (2.3) is not exactly known and that its possible values are represented by a fuzzy-subset g_n of N. Then we define $f_c = f_{na}$ by the formula

$$\begin{aligned} f_c(x) = f_{na}(x) &= \sup_{m \in N}\{\min\{1, \tilde{f}_{ma}(x)\, g_n(m)\}\} = \\ &= \min\{1, \sup_{m \in N}\{\tilde{f}_{ma}(x)\, g_n(m)\}\}\,, \end{aligned} \tag{2.5}$$

where

$$\tilde{f}_{ma} = \underbrace{f_a * f_a * \ldots * f_a}_{m\text{-}times}\,. \tag{2.6}$$

All these results were obtained and discussed in [2] for fuzzy-quantities with values from general algebraic group. Here we shall more deeply investigate some their properties and further problems which become to be remarkable in the specifical but important cases of fuzzy-quantities with real and integer values.

3. COMBINED "CONVOLUTION" – AN OPEN PROBLEM.

Formulas (1.1) and (1.2) define the convolutions of fuzzy-subsets of R and I, respectively. Let us suppose, now, that a is a fuzzy-quantity with values represented by a fuzzy-subset f_a of R and that b is a fuzzy-quantity with values represented by a fuzzy-subset g_b of I. Then it is possible to consider a fuzzy-quantity $c = a + b$, even if the practical examples of such quantities will be probably very rare. The problem is, how to define a fuzzy-set f_c representing the values of fuzzy-quantity *c*. Namely – on which of the sets R and I it is defined, and which are its values? It seems to be practical to define f_c as a fuzzy-subset of R by the formula

$$(3.1)\qquad f_c(x) = \sum_{i \in I} f_a(x - i)\, g_b(i)\,, \quad x \in R\,,$$

as the contrary procedures

$$f_c(i) = \int f_a(x - i)\, g_b(i)\, \mathrm{d}x$$

or

$$f_c(i) = \int f_a(x)\, g_b(x - i)\, \mathrm{d}x\,, \quad g_b(x - i) = 0 \quad \text{if} \quad x - i \notin I\,,$$

fail to $f_c(i) \equiv 0$. Formula (3.1) is formally similar to the convolution of f_a and g_b, and at least some of its properties are analogous to the general properties of convolutions. It is necessary to check, step by step, which properties of convolutions are preserved even in case of (3.1).

For distinguishing the relation defined by (3.1) from the regular convolutions, we shall call it *combined "convolution"* and denote it by f_a "$*$" g_b.

It is obvious that (3.1) exists if at least one of the support sets of functions f_a and g_b is bounded. But, it is not difficult to show that generally

$$\sum_{i \in I} f_a(x - i)\, g_b(i) \neq \sum_{i \in I} f_a(i)\, g_b(x - i)$$

and, of course, also

$$\sum_{i \in I} f_a(x - i)\, g_b(i) \neq \sum_{i \in I} f_a(x)\, g_b(x - i)\,,$$

as follows from the next example.

Example 3.1. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be fuzzy-quantities with values represented by fuzzy-subsets f_a and g_b of R and I, respectively. Let

$$\begin{aligned} f_a(x) &= \tfrac{1}{2} \quad \text{for} \quad 1 \leqq x < 3\,, \\ &= 0 \quad \text{for} \quad x < 1 \quad \text{or} \quad x \geqq 3\,, \\ g_b(i) &= \tfrac{1}{3} \quad \text{for} \quad i = 0, 1, 2 \\ &= 0 \quad \text{for} \quad i \in I - \{0, 1, 2\}\,. \end{aligned}$$

Then

$$\begin{aligned} \sum_{i \in I} f_a(x - i)\, g_b(i) &= \tfrac{1}{3}(f_a(x) + f_a(x - 1) + f_a(x - 2)) = \\ &= 0 \quad \text{for} \quad x < 1 \quad \text{or} \quad x \geqq 5\,, \\ &= \tfrac{1}{6} \quad \text{for} \quad 1 \leqq x < 2 \quad \text{or} \quad 4 \leqq x < 5\,, \\ &= \tfrac{1}{3} \quad \text{for} \quad 2 \leqq x < 4\,, \end{aligned}$$

$$\sum_{i\in I} f_a(i)\, g_b(x - i) = \tfrac{1}{2}(g_b(x - 1) + g_b(x - 2)) =$$
$$= 0 \quad \text{for} \quad x \neq 1, 2, 3, 4\,,$$
$$= \tfrac{1}{6} \quad \text{for} \quad x = 1, 4,$$
$$= \tfrac{1}{3} \quad \text{for} \quad x = 2, 3,$$
$$\sum_{i\in I} f_a(x)\, g_b(x - i) = 0 \quad \text{for} \quad x \neq 1, 2,$$
$$= \tfrac{1}{2} \quad \text{for} \quad x = 1, 2\,.$$

It means that a very important property of convolutions does not hold. If we define, for example, analogously to the formulas for usual convolutions

$$[f_a \,''{*}''\, g_b]\,(x) = \sum_{i\in I} f_a(x - i)\, g_b(i)\,,$$
$$[g_b \,''{*}''\, f_a]\,(x) = \sum_{i\in I} f_a(i)\, g_b(x - i)\,, \quad g_b(x - i) = 0 \quad \text{if} \quad x - i \notin I$$

then $f_a \,''{*}''\, g_b \neq g_b \,''{*}''\, f_a$. The commutativity of such combined "convolutions" may be preserved only if we include it directly into the definition, and define

$$(3.2) \qquad [f_a \,''{*}''\, g_b]\,(x) = [g_b \,''{*}''\, f_a]\,(x) = \sum_{i\in I} f_a(x - i)\, g_b(i)\,.$$

Under this assumption the associativity of combined "convolution" holds, as follows from the next statement.

Statement 3.1. If ***a***, ***b***, ***c***, ***d*** are fuzzy-quantities with values represented by fuzzy-subsets f_a, f_b of R and g_c, g_d of I, then

$$(f_a \,''{*}''\, g_c) \,''{*}''\, g_d = f_a \,''{*}''\, (g_c * g_d)\,,$$
$$(f_a * f_b) \,''{*}''\, g_c = f_a * (f_b \,''{*}''\, g_c)\,,$$

where $*$ denotes the convolutions defined by (1.1) or (1.2) and $''{*}''$ denotes the combined "convolution" defined by (3.1).

Proof.

$$[(f_a \,''{*}''\, g_c) \,''{*}''\, g_d]\,(x) = \sum_{i\in I} [f_a \,''{*}''\, g_c]\,(x - i)\, g_d(i) =$$
$$= \sum_{i\in I} \sum_{j\in I} f_a(x - i - j)\, g_c(j)\, g_d(i) = \sum_{i\in I} \sum_{k\in I} f_a(x - k)\, g_c(k - i)\, g_d(i) =$$
$$= \sum_{k\in I} f_a(x - k)\, [g_c * g_d]\,(k) = [f_a \,''{*}''\, (g_c * g_d)]\,(x)\,;$$
$$[(f_a * f_b) \,''{*}''\, g_c]\,(x) = \sum_{i\in I} [f_a * f_b]\,(x - i)\, g_c(i) =$$

$$= \sum_{i \in I} \int f_a(x - i - y) f_b(y) \, \mathrm{d}y \, g_c(i) =$$

$$= \sum_{i \in I} \int f_a(x - z) f_b(z - i) \, g_c(i) \, dz = \int f_a(x - z) \left[f_b \,''{*}''\, g_c \right](z) \, \mathrm{d}z =$$

$$= \left[f_a * (f_b \,''{*}''\, g_c) \right](x) .$$

In both parts of the proof we use the assumptions that functions f_a, f_b, g_c and g_d are non-negative and bounded and that their support sets are bounded, which assumptions follow from the assumption about fuzzy-sets formulated at the end of Section 1. □

It follows immediately from the previous statement and from the commutativity of the regular convolutions $*$ that

$$f_a * (f_b \,''{*}''\, g_c) = f_b * (f_a \,''{*}''\, g_c) ,$$

$$(f_a \,''{*}''\, g_c) \,''{*}''\, g_d = (f_a \,''{*}''\, g_d) \,''{*}''\, g_c$$

for f_a, f_b, g_c, g_d fulfilling the assumptions of Statement 3.1.

The brief survey of a few simple properties of the combined "convolutions" presented here shows that its properties really are often analogous to the properties of regular convolutions. But, as their formal structure differs from the structure of convolutions, their detailed investigation is still an open problem. If they are considered useful for calculation with fuzzy-sets and fuzzy-quantities, that investigation will be necessary.

4. MULTIPLICATIVE CONVOLUTIONS

The set R of all real numbers is a commutative group with respect to the addition operation $+$ but $R - \{0\}$, i. e. the set of all real numbers except 0, is a commutative group also with respect to the multiplication which we denote here by $\times$ in order to distinguish it from the repetitive addition mentioned in Section 2 and investigated in [2].

It means that it is possible to define, under certain conditions, the multiplicative convolution of fuzzy-subsets of R. As we shall need to distinguish in the following text between both convolutions, the additive and the multiplicative one, we shall modify the notation in the following way. If f and g are fuzzy-subsets of R then their *additive convolution* defined by (1.1) will be denoted by $f\langle * +\rangle g$, so that

$$\left[f\langle * +\rangle g \right](x) = \int f(x - y) \, g(y) \, \mathrm{d}y = \int f(y) \, g(x - y) \, \mathrm{d}y . \tag{4.1}$$

If f and g are fuzzy-subsets of $R - \{0\}$ then we may define the *multiplicative convolution* $f\langle * \times \rangle g$ by

$$(4.2) \qquad [f\langle * \times \rangle g](x) = \int_{R-\{0\}} f\left(\frac{x}{y}\right) g(y)\,\mathrm{d}y\,, \quad x \in R\,.$$

The general properties of multiplicative convolutions are not as plesant as the properties of the additive ones. It is caused by the fact that the Lebesgue measure λ is not Haar measure according to the multiplication $\times$ on R. It means that the real applicability of multiplicative convolutions on $(R, \mathscr{B}, \lambda)$ is very limited, as follows also from the next two sections.

If we consider another measure μ on $(R - \{0\}, \mathscr{B}_0)$, $\mathscr{B}_0 = \{B \in \mathscr{B} : 0 \notin B\}$ which would be a Haar measure according to multiplication, i.e.

$$M \in \mathscr{B}_0 \Leftrightarrow x \times M \in \mathscr{B}_0\,, \quad \mu(x \times M) = \mu(M)\,,$$
$$x \in R - \{0\},\ M \in \mathscr{B}_0,\ x \times M = \{y \in R - \{0\} : \exists(z \in M)\, y = x \times z\}\,,$$

then the multiplicative convolution according to μ has all the properties introduced in [2] and mentioned for additive convolution also in Section 2 of this paper.

Unfortunately, the class of Haar measures on $R - \{0\}$ according to the multiplication operation is very poor, so that the measure spaces with such measures are not very useful for our applications.

The presented work is subjected to rather different problems, it is to illustrate the properties of additive convolutions of fuzzy-sets which were investigated in [2], and the properties of the multiplicative ones are on the margin of our interest only. This fact, together with the disadvantages of multiplicative convolutions causes that in the next two sections we introduce only the main properties of multiplicative convolutions without any detailed investigation of their further features.

5. COMMUTATIVITY, ASSOCIATIVITY AND DISTRIBUTIVITY OF MULTIPLICATIVE CONVOLUTIONS

As mentioned in the previous section, multiplicative convolutions have not some of good properties, useful for their applicability. Especially, they are neither commutative nor associative, as follows from the next examples.

Example 5.1. Let f and g be fuzzy-subsets of $R - \{0\}$ such that

$$f(x) = \tfrac{1}{2} \quad \text{for} \quad 1 \leqq x < 2\,, \quad f(x) = 0 \quad \text{for} \quad x < 1\,, \quad x \geqq 2\,,$$
$$g(x) = \tfrac{2}{3} \quad \text{for} \quad 2 \leqq x < 3\,, \quad g(x) = 0 \quad \text{for} \quad x < 2\,, \quad x \geqq 3\,.$$

We can see in Table 1 that $f\langle * \times \rangle g \neq g\langle * \times \rangle f$.

Table 1.

x	$[f\langle * \times \rangle g](x)$	$[g\langle * \times \rangle f](x)$
$x \leqq 2$	0	0
$2 < x < 3$	$(x-2)/3$	$(x-2)/6$
$x = 3$	$1/3$	$1/6$
$3 < x < 4$	$1/3$	$x/18$
$x = 4$	$1/3$	$2/9$
$4 < x < 6$	$1 - x/6$	$(6-x)/9$
$x \geqq 6$	0	0

Example 5.2. Let us consider fuzzy-subsets f and g of $R - \{0\}$ identical with the ones from Example 5.1. It follows from Table 2 that

$$\left(f\langle * \times \rangle g\right)\langle * \times \rangle f \neq f\langle * \times \rangle \left(g\langle * \times \rangle f\right)$$

so that the multiplicative convolutions are not associative. Some elected values of $\left(f\langle * \times \rangle g\right)\langle * \times \rangle f$ and $f\langle * \times \rangle \left(g\langle * \times \rangle f\right)$ are presented in Table 3, together with their differences and with their relative percentual differences.

Table 2.

x	$F(x) = [(f\langle * \times \rangle g)\langle * \times \rangle f](x)$	$G(x) = [f\langle * \times \rangle (g\langle * \times \rangle f)](x)$
$x \leqq 2$	0	0
$2 < x \leqq 3$	$\frac{1}{6}x \ln\left(\frac{x}{2} - 1\right) + \frac{1}{13}$	$\frac{1}{24}x^2 - \frac{1}{6}x + \frac{1}{6}$
$3 < x \leqq 4$	$\frac{1}{6}x \ln \frac{3}{2} - \frac{1}{6}$	$\frac{1}{72}x^2 - \frac{1}{12}$
$4 < x \leqq 6$	$\frac{1}{4}x - \frac{7}{6} - \frac{1}{12}x \ln \frac{x}{4} - \frac{1}{6}x \ln \frac{x}{6}$	$-\frac{11}{288}x^2 + \frac{5}{12}x - \frac{11}{12}$
$6 < x \leqq 8$	$\frac{1}{3} - \frac{1}{12}x \ln \frac{3}{2}$	$-\frac{1}{288}x2 + \frac{1}{3}$
$8 < x \leqq 12$	$1 - \frac{1}{12}x(1 + \ln 12) + \frac{1}{12}x \ln x$	$\frac{1}{144}x^2 - \frac{1}{6}x + 1$
$x > 12$	0	0

The simultaneous application of additive and multiplicative convolutions does not fulfil the property of distributivity. It means that the "multiplication" of fuzzy-quantities represented by multiplicative convolutions of fuzzy-sets is not distributive.

Table 3.

x	$F(x)$	$G(x)$	$G(x) - F(x)$	$\frac{100(G(x) - F(x))}{F(x)}$
2·5	0·009643	0·010416	0·000773	8·02
3	0·036066	0·041666	0·005601	15·53
3·5	0·069855	0·093750	0·023895	24·21
4	0·103643	0·138889	0·035245	34·01
5	0·142291	0·211805	0·069513	48·85
6	0·130601	0·208333	0·077733	59·52
7	0·096812	0·163194	0·066382	68·57
8	0·063023	0·111111	0·048088	76·30
9	0·034238	0·062500	0·028262	82·54
10	0·014732	0·027778	0·013046	88·55

The reason of it was explained already in the previous section – the Lebesgue measure λ is not Haar measure according to multiplication.

Example 5.3. Let us consider fuzzy-subsets f, g and h of $R - \{0\}$ such that

$$f(x) = \tfrac{1}{3} \text{ for } 0 < x \leqq 1\,, \quad f(x) = 0 \text{ for } x \leqq 0\,, \quad x > 1\,,$$

$$g(x) = \tfrac{1}{2} \text{ for } 1 < x \leqq 2\,, \quad g(x) = 0 \text{ for } x \leqq 1\,, \quad x > 2\,,$$

$$h(x) = \tfrac{2}{3} \text{ for } 1 \leqq x < 2\,, \quad h(x) = 0 \text{ for } x < 1\,, \quad x \geqq 2\,.$$

Table 4.

x	$f(x)$	$g(x)$	$h(x)$	$[f\langle * + \rangle g]\,(x)$	$[f\langle * \times \rangle h]\,(x)$	$[g\langle * \times \rangle h]\,(x)$
$x \leqq 0$	0	0	0	0	0	0
$0 < x < 1$	1/3	0	0	0	1/3	0
$x = 1$	1/3	0	2/3	0	1/3	0
$1 < x < 2$	0	1/2	2/3	$(x - 1)/4$	$(2 - x)/3$	$(x - 1)/3$
$x = 2$	0	1/2	0	1/4	0	1/3
$2 < x < 3$	0	0	0	$(3 - x)/4$	0	$(4 - x)/6$
$x = 3$	0	0	0	0	0	1/6
$3 < x \leqq 4$	0	0	0	0	0	$(4 - x)/6$
$x > 4$	0	0	0	0	0	0

Table 5.

x	$F(x) = [(f\langle * + \rangle g) \langle * \times \rangle h] (x)$	$G(x) =$ $= [(f\langle * \times \rangle h) \langle * + \rangle (g\langle * \times \rangle h)] (x)$
$x \leqq 1$	0	0
$1 < x \leqq 2$	$\frac{x}{6} (\ln x - 1) + \frac{1}{6}$	$\frac{1}{18} (x - 1)^2$
$2 < x \leqq 3$	$\frac{1}{3} x - \frac{5}{6} + \frac{1}{2} x \ln 2 - \frac{1}{3} x \ln x$	$-\frac{1}{54} x^3 + \frac{1}{22} x^2 - \frac{7}{54}$
$3 < x \leqq 4$	$\frac{1}{6} x - \frac{1}{3} + \frac{1}{6} x \ln \frac{x}{3} - \frac{1}{3} x \ln \frac{x}{2}$	$\frac{1}{36} x^3 - \frac{1}{3} x^2 + \frac{5}{4} x - \frac{149}{108}$
$4 < x \leqq 5$	$1 - \frac{1}{6} x + \frac{1}{6} x \ln \frac{x}{6}$	$\frac{1}{36} x^2 - \frac{11}{36} x + \frac{91}{108}$
$5 < x \leqq 6$	$1 - \frac{1}{6} x + \frac{1}{6} x \ln \frac{x}{6}$	$-\frac{1}{108} x^3 + \frac{1}{6} x^2 - x + 2$
$x < 6$	0	0

Then the convolutions $f\langle * + \rangle g$, $f\langle * \times \rangle h$, $g\langle * \times \rangle h$ are described in Table 4, and the final convolutions

$$\left(f\langle * + \rangle g\right) \langle * \times \rangle h\,, \quad \left(f\langle * \times \rangle h\right) \langle * + \rangle \left(g\langle * \times \rangle h\right),$$

are described in Table 5. Some elected values of the final convolutions are, as well as

Table 6.

x	$F(x)$	$G(x)$	$F(x) - J(x)$	$\frac{100(F(x) - G(x))}{F(x)}$
1·25	0·028929	0·003472	0·025457	87·99
1·5	0·018033	0·013889	0·004144	22·98
2	0·064382	0·055556	0·008827	13·71
2·5	0·102858	0·101852	0·001007	0·98
3	0·107775	0·120370	−0·012595	−11·69
3·5	0·091372	0·113426	−0·022054	−24·14
4	0·063023	0·064815	−0·001792	− 2·84
4·5	0·034238	0·030093	0·004146	12·11
5	0·014732	0·009259	0·005473	37·15
5·5	0·003573	0·001157	0·002415	67·61

their differences and the percentual values of relative differences, concentrated in Table 6.

6. MULTIPLICATIVE CONVOLUTIONS AND REPETITION OF ADDITIVE CONVOLUTIONS

In spite of the disadvantageous properties of multiplicative convolutions which were discussed and illustrated in Sections 4 and 5, we could consider them to be proper for the following application.

If $\boldsymbol{a}$ is a fuzzy-quantitity with values represented by a fuzzy-subset of R then the values of its repetitive addition

$$n\boldsymbol{a} = \boldsymbol{a} + \boldsymbol{a} + \ldots + \boldsymbol{a}\,, \quad n \in N\,,$$

were modelled by convolutions in (2.4) for determined n or in (2.5) for non-determined value of n. In fact, the fuzzy-quantity $n\boldsymbol{a}$ is a product and we may to calculate its values by means of multiplicative convolution of $f_{\boldsymbol{a}}$ with some fuzzy-set g_n representing the values of n. The problem is, how to define the fuzzy-set g_n in order to obtain acceptable results, and, of course, to decide which results we consider to be acceptable. The main condition which should be fulfilled is that for determined $n \in N$ the multiplicative convolution of $f_{\boldsymbol{a}}$ and g_n must give the same results as formula (2.4). It means that we have to find, for any natural number $n \geqq 2$ a fuzzy-set g_n such that its multiplicative convolution with $f_{\boldsymbol{a}}$ coincides with $f_{n\boldsymbol{a}}$ obtained by (2.5) or with $f_{\boldsymbol{a}} * \ldots * f_{\boldsymbol{a}}$ from the same formula, at least almost everywhere on R.

Generally, we may try to find g_n as a fuzzy-subset of R or of N. In the first case we may calculate $f_{\boldsymbol{a}}\langle * \times\rangle\, g_n$ by (4.2), in the second case we may use an analogy of combined "convolution" (3.1) and to use formula

$$\sum_{i \in N} f_{\boldsymbol{a}}\left(\frac{x}{i}\right) g_n(i) \tag{6.1}$$

where we suppose that $g_n(0) = 0$.

The following statements imply that none of those possible procedures gives acceptable result. There exists a wide class of fuzzy-quantities for which no fuzzy-subset g_n with the desired property exists.

Statement 6.1. Let f be a fuzzy-subset of R such that the support-set S_f is a bounded and non-degenerated interval in R. If $n \in N$ $n \geqq 2$ then there is no fuzzy-subset g of R such that

$$\underbrace{f\langle * +\rangle \ldots \langle * +\rangle f}_{n\text{-}times} = f\langle * \times\rangle\, g$$

at least almost everywhere on R.

Proof. It follows from the assumptions about S_f that

$$\int f \, d\lambda > 0 ,$$

and that for any arbitrarily small $\varepsilon > 0$ there exist $r, s \in R$ such that $r \neq 0 \neq s$, and

$$f(x) = 0 , \quad x \leqq r - \varepsilon , \quad x \geqq s + \varepsilon ,$$
$$f(x) > 0 , \quad r \leqq x \leqq s .$$

If we denote

$$F(z) = [f\langle * + \rangle \ldots \langle * + \rangle f] (z)$$

then also

$$F(z) = 0 \quad \text{for} \quad z \leqq nr - n\varepsilon \quad \text{or} \quad z \geqq ns + n\varepsilon ,$$
$$F(z) > 0 \quad \text{for} \quad nr \leqq z \leqq ns .$$

Let us suppose that g is a fuzzy-subset of R such that $g(0) = 0$, and let us denote

$$G(z) = [f\langle * \times \rangle g] (z) = \int_{S_g} f\left(\frac{z}{y}\right) g(y) \, dy .$$

If for some $z \in R$ is $G(z) = 0$ then

$$\lambda\left(\left\{y : f\left(\frac{z}{y}\right) > 0 , \quad g(y) > 0\right\}\right) = \lambda\left(\left\{y : \frac{z}{s} \leqq y \leqq \frac{z}{r} , \quad g(y) > 0\right\}\right) = 0 .$$

Let us suppose that g was chosen so that $G(z) = F(z)$ almost everywhere on R. Then also $G(z) = 0$ almost everywhere for $z \leqq nr - n\varepsilon$ or $z \geqq ns + n\varepsilon$. It means that $g(y) = 0$ almost everywhere for

$$y \geqq n + \frac{n}{s} \varepsilon \quad \text{or} \quad y \leqq n - \frac{n}{r} \varepsilon ,$$

for $\varepsilon > 0$ arbitrarily small. But it means that $g(y) = 0$ almost everywhere on R, so that $G \equiv 0 \not\equiv F$, and there exists no g with the desired property. □

Statement 6.2. Let f be a fuzzy-subset of R such that its support-set S_f is a bounded and non-degenerated interval in R. Let $n \geqq 2$, $n \in N$, let us denote

$$F(x) = \underbrace{[f\langle * + \rangle \ldots \langle * + \rangle f]}_{n\text{-times}} (x) ,$$

and let at least one of the following conditions be fulfilled

(6.2) there exists $x < 0$, $x \in S_f$,

(6.3) there exists no $t \in R$ such that $F(x) = t\,f(x/n)$ almost everywhere on R.

Then there is no fuzzy-subset g of N such that

$$F(x) = \sum_{i \in N} f\left(\frac{x}{i}\right) g(i)$$

at least almost everywhere on R.

Proof. The assumptions about S_f imply that for any arbitrarily small $\varepsilon > 0$ there exist $r, s \in R$, $r \neq 0 \neq s$, such that

$$\begin{aligned} f(x) &= 0\,, \quad x \leqq r - \varepsilon\,, \quad x \geqq s + \varepsilon\,, \\ f(x) &> 0\,, \quad r \leqq x \leqq s\,, \\ F(x) &= 0\,, \quad x \leqq nr - n\varepsilon\,, \quad x \geqq ns + n\varepsilon\,, \\ F(x) &> 0\,, \quad nr \leqq x \leqq ns\,. \end{aligned}$$

Let us suppose that g is a fuzzy-subset of N such that $g(0) = 0$, and let us denote

$$G(z) = \sum_{i \in N} f\left(\frac{z}{i}\right) g(i)\,, \quad z \in R\,.$$

If for some $z \in R$ is $G(z) = 0$ then it means that $g(i) = 0$ for all $i \in N$ such that $f(z/i) > 0$, it means for all $i \in N$ such that

$$\frac{z}{s} \leqq i \leqq \frac{z}{r}\,.$$

Let us suppose that g has been constructed so that $G(z) = F(z)$ almost everywhere on R. Then $G(z) = 0$ for $z \leqq nr - n\varepsilon$ or $z \geqq ns + n\varepsilon$, so that

$$g(i) = 0 \quad \text{for} \quad i \in N\,, \quad i \leqq n - (n/r)\,\varepsilon \quad \text{or} \quad i \geqq n + (n/s)\,\varepsilon\,,$$

for $\varepsilon > 0$ arbitrarily small. If condition (6.2) is fulfilled then r may be chosen so that $r < 0$, and then

$$n - \frac{n}{r}\varepsilon > n$$

so that there is no $i \in N$ for which $g(i)$ may be different from 0. Then $G \equiv 0 \not\equiv F$ and the statement is proved. If $S_f \subset \{x \in R : x \geqq 0\}$ then $g(i) \neq 0$ only for $i = n$,

Table 7.

x	$f(x)$	$g_1(x)$	$g_2(x)$	$g_3(x)$	$g_4(x)$	$g_5(x)$	$g_6(x)$	$g_7(x)$
$x < 0$	0			0	0	0	0	0
$x = 0$	0	0	0	0	0	0	0	0
$0 < x < 1$	0			0	1	0	0	0
$x = 1$	1/2	0	1	1	1	0	1	0
$1 < x < 3/2$	1/2			1	1	0	1	1
$x = 3/2$	1/2			1	1	0	1	1
$3/2 < x < 2$	1/2			1	1	0	1	1
$x = 2$	2/3	1	1	1	1	1	1	1
$2 < x < 5/2$	2/3			0	0	1	1	1
$x = 5/2$	2/3			0	0	1	1	0
$5/2 < x < 3$	2/3			0	0	1	1	0
$x = 3$	0	0	0	0	0	1	1	0
$x > 3$	0			0	0	0	0	0

Table 8.

x	$[f\langle * + \rangle f]\,(x)$	$\left[\sum f\left(\frac{x}{i}\right) g(i)\right](x)$		$[f\langle * \times \rangle g]\,(x)$	
		$g = g_1$	$g = g_2$	$g = g_3$	$g = g_4$
$x \leqq 0$	0	0	0	0	0
$0 < x < 1$	0	0	0	0	$13x/36$
$x = 1$	0	0	1/2	0	13/36
$1 < x < 3/2$	0	0	1/2	$(x - 1)/2$	$13x/36$
$x = 3/2$	0	0	1/2	1/4	13/24
$3/2 < x < 2$	0	0	1/2	$(x - 1)/2$	$13x/36$
$x = 2$	0	1/2	7/6	1/2	13/18
$2 < x < 5/2$	$(x - 2)/4$	1/2	7/6	$(4 + x)/12$	$(36 - 5x)/36$
$x = 5/2$	1/18	1/2	7/6	13/24	47/72
$5/2 < x < 3$	$(x - 2)/4$	1/2	7/6	$(4 + x)/12$	$(36 - 5x)/36$
$x = 3$	1/4	1/2	1/2	7/12	7/12
$3 < x < 4$	$(5x - 12)/12$	1/2	1/2	$(36 - 5x/36$	$(36 - 5x)/36$
$x = 4$	2/3	2/3	2/3	4/9	4/9
$4 < x < 9/2$	$(14 - 2x)/9$	2/3	2/3	$(12 - 2x)/9$	$(12 - 2x)/9$
$x = 9/2$	5/9	2/3	2/3	1/3	1/3
$9/2 < x < 5$	$(14 - 2x)/9$	2/3	2/3	$(12 - 2x)/9$	$(12 - 2x)/9$
$x = 5$	4/9	2/3	2/3	2/9	2/9
$5 < x < 6$	$(24 - 4x)/9$	2/3	2/3	$(12 - 2x)/9$	$(12 - 2x)/9$
$x \geqq 6$	0	0	0	0	0

and then

$$G(x) = f\left(\frac{x}{n}\right) g(n) , \quad x \in R .$$

Condition (6.3) implies the validity of statement even in this case. □

It is useful to note that condition (6.3) is fulfilled for practically all fuzzy-subsets of R which may appear in applications except a few of very special types of them. The previous statements cover a sufficiently wide class of fuzzy-sets f, but they may be, in case of necessity, modified by using more complicated proofs, even for some further fuzzy-sets f with more general support-sets S_f. Nevertheless, the statements presented here are sufficient for showing the inconvenience of multiplicative convolutions for substituting the additive ones in the calculation of values of fuzzy-sets obtained as results of repetitive addition of fuzzy-quantities.

Table 9.

x	$[f\langle * + \rangle f](x)$	$[f\langle * \times \rangle g](x)$		
		$g = g_5$	$g = g_6$	$g = g_7$
$x \leq 1$	0	0	0	0
$1 < x < 3/2$	0	0	$(x-1)/2$	0
$x = 3/2$	0	0	$1/4$	0
$3/2 < x < 2$	0	0	$(x-1)/2$	$(2x-3)/4$
$x = 2$	0	0	$1/2$	$1/4$
$2 < x < 5/2$	$(x-2)/4$	$(x-2)/2$	$(7x-8)/12$	$(2x-3)/4$
$x = 5/2$	$1/8$	$1/4$	$19/24$	$1/2$
$5/2 < x < 3$	$(x-2)/4$	$(x-2)/2$	$(7x-8)/12$	$1/2$
$x = 3$	$1/4$	$1/2$	$13/12$	$1/2$
$3 < x < 4$	$(5x-12)/12$	$1/2$	$(54-5x)/36$	$(x+3)/12$
$x = 4$	$2/3$	$1/2$	$17/18$	$7/12$
$4 < x < 9/2$	$(14-2x)/9$	$(x+2)/12$	$(54-5x)/36$	$(x+3)/12$
$x = 9/2$	$5/9$	$13/24$	$7/8$	$5/8$
$9/2 < x < 5$	$(14-2x)/9$	$(x+2)/12$	$(54-5x)/36$	$(45-5x)/36$
$x = 5$	$4/9$	$7/12$	$23/36$	$5/9$
$5 < x < 6$	$(24-4x)/9$	$(x+2)/12$	$(54-5x)/36$	$(15-2x)/9$
$x = 6$	0	$2/3$	$2/3$	$1/3$
$6 < x < 7$	0	$(18-2x)/9$	$(18-2x)/9$	$(15-2x)/9$
$x = 7$	0	$4/9$	$4/9$	$1/9$
$7 < x < 15/2$	0	$(18-2x)/9$	$(18-2x)/9$	$(15-2x)/9$
$x = 15/2$	0	$1/3$	$1/3$	0
$15/2 < x < 8$	0	$(18-2x)/9$	$(18-2x)/9$	0
$x = 8$	0	$2/9$	$2/9$	0
$8 < x < 9$	0	$(18-2x)/9$	$(18-2x)/9$	0
$x \geq 9$	0	0	0	0

The following example illustrates the previous statements for $n = 2$. It presents the values of additive convolution $f\langle * +\rangle f$ and of multiplicative convolutions $f\langle * \times\rangle g$ for a few fuzzy-sets g which could be intuitively considered acceptable for the given purpose.

Example 6.1. Let us consider a fuzzy-subset f of R, fuzzy-subsets g_1, g_2 of N and fuzzy-subsets g_3, g_4, g_5, g_6, g_7 of R, which are defined by Table 7.
Then we may enumerate the additive convolution $f\langle * +\rangle f = \tilde{f}_{na}$ for $n = 2$, by means of (2.5). We may see in Table 8 and Table 9 that none of the multiplicative convolutions $f\langle * \times\rangle g_i$, $i = 1, \ldots, 7$, calculated by means of (6.1) in case of $i = 1, 2$ and by means of (4.2) in case of $i = 3, \ldots, 7$, is equal to $f\langle * +\rangle f = \tilde{f}_{2a}$. We see that all convolutions except the case of g_2 and g_6 are fuzzy-subsets of R. In case

$$\tilde{F}_2(x) = \sum_{i=1}^{2} f\left(\frac{x}{i}\right) g_2(i)\,, \quad \tilde{F}_6(x) = \int_1^3 f\left(\frac{x}{y}\right) g_6(y)\,\mathrm{d}y\,,$$

$\tilde{F}_2$ and $\tilde{F}_6$ are not fuzzy-subsets of R, as they are greater than 1 for $x \in \langle 2, 3)$ and $x \in \left(\frac{20}{7}, \frac{18}{5}\right)$, respectively. If necessary, we could transform them into fuzzy-sets $F_2(x)$ and $F_6(x)$ according to the schema

$$F_i(x) = \min\{1, \tilde{F}_i(x)\}, \quad x \in R\,, \quad i = 2, 6\,.$$

This procedure was mentioned in Section 2 of this paper and widely discussed in paper [2].

(Received June 30, 1976.)

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