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Conjugate Gradient Algorithm for Optimal Control Problems with Parameters

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This paper presents an extension of a conjugate gradient algorithm for the solution of non-linear optimal control problems with parameters. Several numerical examples are treated to study the properties of the proposed algorithm.

1. INTRODUCTION

The basic conjugate gradient method for unconstrained optimal control problems was proposed by Lasdon et al. in [1]. The penalty function approach to the solution of inequality constrained optimal control problems has been considered in [2] and the clipping-off technique that solves optimal control problems with magnitude constraint on the control inputs is described in [3, 4]. The proposed algorithm is more simple and easier to apply than an alternative sequential conjugate-gradient-restoration algorithm of [5]. On the other hand, the algorithm of [5] need not to use penalty function to deal with optimal control problems with terminal constraints. Also the robust conjugate-gradient algorithm in [6] can compute optimal controls, however without parameter optimization.

2. OPTIMAL CONTROL PROBLEM

The aim is to minimize the cost functional

$$J = \varphi[x, \pi]_{t_f} + \int_{t_0}^{t_f} L(x, u, \pi, t)\, \mathrm{d}t \tag{1}$$

subject to the differential equation constraint

$$\dot{x}(t) = f(x, u, \pi, t)\,, \quad x(t_0) = x_0\,, \tag{2}$$

magnitude constraints imposed on the control inputs

(3) $$a_j \leqq u_j(t) \leqq b_j\,, \quad j = 1, 2, \ldots, r\,,$$

and on the parameters

(4) $$c_k \leqq \pi_k \leqq d_k\,, \quad k = 1, 2, \ldots, q\,.$$

Here f denotes a non-linear n-dimensional vector function of n-dimensional state vector x, r-dimensional control input vector u, and q-dimensional parameter vector π. Further t_0 is the given initial time, t_f is the terminal time, which is assumed to be given, and φ and $L$are scalar functions. It is assumed that, given a control $\boldsymbol{u}$, a parameter π, (2) can be solved for a unique $x = x(u, \pi)$. Thus $J = J(u, \pi)$ is a unique function of $\boldsymbol{u}$ and π. Finally, it is assumed the existence of the gradient of $J(\boldsymbol{u}, \pi)$ and that $J(\boldsymbol{u}, \pi)$ is bounded below.

As usual (see [7]), let H denote the Hamiltonian function given by

(5) $$H(x, \lambda, u, \pi, t) = \lambda^{\mathrm{T}} f(x, u, \pi, t) + L(x, u, \pi, t)\,,$$

where $\lambda(t)$ is an n-dimensional vector function, T denotes the transposition. Then necessary optimality conditions are for the case without magnitude constraints (3), (4) as follows – see for instance [3] and [8],

(6) $$\dot{\lambda} = -\frac{\partial H}{\partial x}\,,$$

(7) $$\lambda(t_f) - \left(\frac{\partial \varphi}{\partial x}\right)_{t=t_f} = 0\,,$$

(8) $$\frac{\partial H}{\partial u} = 0\,,$$

(9) $$\left(\frac{\partial \varphi}{\partial \pi}\right)_{t=t_f} + \int_{t_0}^{t_f} \frac{\partial H}{\partial \pi}\,\mathrm{d}t = 0\,.$$

It is assumed that $\partial H/\partial u$ is continuous in u and $\partial H/\partial \pi$ and $\partial \varphi/\partial \pi$ in π. All expressions are evaluated along the optimal solution x^*, u^* and π^*. All vectors, including also gradient of various functions, are supposed to be column-vectors.

Conditions (8), (9) yield (see for instance [9]), that the variations per unit gradient stepsize of u and π can be taken as

(10) $$g(t) = \frac{\partial H}{\partial u}\,,$$

and

$$h = \left(\frac{\partial \varphi}{\partial \pi}\right)_{t=t_f} + \int_{t_0}^{t_f} \frac{\partial H}{\partial \pi} \, dt \,. \tag{11}$$

3. CONJUGATE GRADIENT ALGORITHM

For the sake of simplicity in the explanations that follows, a scalar control input and parameter will be considered only. The extension to multi-control inputs and multi-parameters is rather straightforward.

The conjugate gradient method computes the direction of search in the i-th iteration as follows: the conjugate gradient direction of u is determined by

$$p^i(t) = -g^i(t) + \beta^i \, p^{i-1}(t) \,, \tag{12}$$

where

$$\beta^i = \frac{\int_{t_0}^{t_f} g^i(t) \, g^i(t) \, dt}{\int_{t_0}^{t_f} g^{i-1}(t) \, g^{i-1}(t) \, dt} \tag{13}$$

and

$$g^i(t) = \left.\frac{\partial H}{\partial u}\right|_i \,, \tag{14}$$

provided that $\int_{t_0}^{t_f} g^{i-1}(t) \, g^{i-1}(t) \, dt \neq 0$ and $\beta^0 = 0$, the conjugate gradient direction of π is determined by

$$q^i = -h^i + \gamma^i q^{i-1} \,, \tag{15}$$

where

$$\gamma^i = \frac{h^i h^i}{h^{i-1} h^{i-1}} \,, \tag{16}$$

and

$$h^i = \left.\left(\frac{\partial \varphi}{\partial \pi}\right)_{t=t_f}\right|_i + \int_{t_0}^{t_f} \left.\frac{\partial H}{\partial \pi}\right|_i \, dt \,, \tag{17}$$

provided that $h^{i-1} h^{i-1} \neq 0$ and $\gamma^0 = 0$.

The new estimate of $u(t)$ is then

$$u^{i+1}(t) = u^i(t) + \alpha^i \, p^i(t) \tag{18}$$

and that of π

(19) $$\pi^{i+1} = \pi^i + \alpha^i q^i ,$$

where α^i is determined by one-dimensional search so as to minimize

(20) $$J(u^{i+1}, \pi^{i+1}) = \min_{\alpha^i} J(u^i + \alpha^i p^i, \pi^i + \alpha^i q^i) .$$

The proof of the convergence of the algorithm can be performed analogously as in [1].

4. CONSTRAINTS TREATMENT

Let us consider the case in which there is a control variable $u(t)$ and a parameter π which must satisfy magnitude saturation constraints. Generalization to more than one control variable and one parameter is straightforward. The necessary optimality conditions (8), (9) are replaced (see [8], [10], by

(21) $$H(x^*, \lambda, u^*, \pi^*, t) = \min_{u \in U} H(x^*, \lambda, u^*, \pi^*, t) ,$$

(22) $$\left[\left(\frac{\partial \varphi}{\partial \pi}\right)_{t=t_f} + \int_{t_0}^{t_f} \frac{\partial H}{\partial \pi} \, dt\right]^T \delta\pi \geqq 0 ,$$

where x^*, u^*, π^* are optimal solutions, U denotes the admissible control region – see (3), $\delta\pi$ is any feasible parameter change.

Before $J(u^{i+1}, \pi^{i+1})$ is computed in each trial of the α-search, $u^{i+1}(t)$ and π^{i+1} are truncated according to the upper and lower bounds (see [3]) in the following way:

(23) $$u^{i+1}(t) = \begin{cases} u^i(t) + \alpha^i p^i(t) , & \text{if } a \leqq u^{i+1}(t) \leqq b , \\ a , & \text{if } u^{i+1}(t) < a , \\ b , & \text{if } u^{i+1}(t) > b , \end{cases}$$

$$\pi^{i+1} = \begin{cases} \pi^i + \alpha^i q^i , & \text{if } c \leqq \pi^{i+1} \leqq d , \\ c , & \text{if } \pi^{i+1} < c , \\ d , & \text{if } \pi^{i+1} > c . \end{cases}$$

5. SUMMARY OF ALGORITHM

STEP 1. Set $i = 0$ and select the initial estimates u^0 and π^0.

STEP 2. Solve the state equations (2) forwards with $u = u^i$, $\pi = \pi^i$, and the adjoint equations (6), (7) backwards, and then compute p^i and q^i from (12), (15).

STEP 3. Set

$$u^{i+1}(t) = u^i(t) + \alpha^i p^i(t) \quad \text{and}$$

$$\pi^{i+1} = \pi^i + \alpha^i q^i .$$

Before $J(u^{i+1}, \pi^{i+1})$ is computed, $u^{i+1}(t)$ and π^{i+1} are truncated, if necessary, according to (23). Choose α^i to minimize (20).

STEP 4. Repeat steps 2 and 3 until

$$|J(u^{i+1}, \pi^{i+1}) - J(u^i, \pi^i)| < \delta |J(u^i, \pi^i)|$$

where δ is a specified positive number (here $\delta = 10^{-4}$).

6. EXAMPLES

The described algorithm was implemented on the IBM 370/135 computer in single-precision arithmetic and was programmed in PL/1. The integration was performed using the IBM DERE subroutine. The definite integrals were computed using IBM QSF subroutine.

Example 1. Minimize the cost functional

$$J = 1/2[x_1^2]_{1.5} + 1/2 \int_0^{1.5} u^2 \, dt$$

subject to the differential constraints

$$\dot{x}_1 = x_2 , \qquad x_1(0) = \pi ,$$

$$\dot{x}_2 = -x_1 + u + x_2(1 - x_1^2) , \quad x_2(0) = 1 .$$

To obtain the problem with fixed initial state, as required in (2), let us perform the substitution

$$y_1 = x_1 - \pi , \quad y_2 = x_2 .$$

In this way we obtain the following problem

$$J = 1/2[(y_1 + \pi)^2]_{1.5} + 1/2 \int_0^{1.5} u^2 \, dt ,$$

$$\dot{y}_1 = y_2 , \qquad y_1(0) = 0 ,$$

$$\dot{y}_2 = -(y_1 + \pi) + u + y_2[1 - (y_1 + \pi)^2] , \quad y_2(0) = 1 .$$

The initial estimates were chosen as $u = 0$, $\pi = 0$. The interval of integration was divided into 50 steps. The algorithm has converged in 3 iterations to the $J = 0{\cdot}30323$, $\pi = 1{\cdot}03902$. The converged solution for the Example 1 is given in Table 1.

Table 1.

t	y_1	y_2	u
0·00	0·00000E+00	1·00000E+00	−6·26581E−01
0·15	1·28989E−01	7·12482E−01	−6·10498E−01
0·30	2·12591E−01	4·01632E−01	−5·81324E−01
0·45	2·50107E−01	1·02659E−01	−5·39093E−01
0·60	2·44980E−01	−1·65173E−01	−4·83889E−01
0·75	2·02314E−01	−3·98320E−01	−4·16408E−01
0·90	1·26911E−01	−6·03270E−01	−3·38439E−01
1·05	2·22014E−02	−7·91139E−01	−2·53152E−01
1·20	−1·10193E−01	−9·74541E−01	−1·64951E−01
1·35	−2·70563E−01	−1·16624E+00	−7·89282E−02
1·50	−4·61071E−01	−1·37820E+00	0·00000E+00

In [11] the values $J = 0{\cdot}30549$ and $\pi = 1{\cdot}03820$ were reached within 7 iterations by the method of [12]. Comparing these results one observes faster convergence of the proposed algorithm for the solution of this type of problem.

With the control and the parameter constraints $u \geqq -0{\cdot}4$, $\pi \leqq 1$ was reached within 3 iterations $J = 0{\cdot}31655$, $\pi = 1$. The converged solution for the Example 1 with constraints is given in Table 2.

Table 2.

t	y_1	y_2	u
0·00	0·00000E+00	1·00000E+00	−4·00000E−01
0·15	1·32744E−01	7·61194E−01	−4·00000E−01
0·30	2·26583E−01	4·86913E−01	−4·00000E−01
0·45	2·78648E−01	2·08859E−01	−4·00000E−01
0·60	2·90137E−01	−5·17519E−02	−4·00000E−01
0·75	2·64383E−01	−2·87520E−01	−4·00000E−01
0·90	2·04958E−01	−5·01941E−01	−4·00000E−01
1·05	1·14440E−01	−7·02937E−01	−3·46228E−01
1·20	−5·43584E−03	−8·94921E−01	−2·62924E−01
1·35	−1·54115E−01	−1·08830E+00	−1·38924E−01
1·50	−3·32345E−01	−1·29047E+00	0·00000E+00

Example 2. Minimize the cost functional

$$J = \int_0^1 (x + \pi)^2 \, u^2 \, \mathrm{d}t - 2 \ln \left(x(1) + \pi \right),$$

subject to the differential constraint

$$\dot{x} = (x + \pi)^2 u, \quad x(0) = 0,$$

and parameter constraint $\pi \leqq 1$.

The initial estimates were chosen as $u = 1$, $\pi = 0$ and the interval of integration was divided into 100 steps. The algorithm has converged in 6 iterations with $J = -1{\cdot}00333$ and $\pi = 1$. The converged solution for Example 2 is given in Table 3.

Table 3.

t	x	u
0·0	0·00000E+00	8·29442E−01
0·1	8·92603E−02	8·06543E−01
0·2	1·92482E−01	7·79291E−01
0·3	3·12204E−01	7·46683E−01
0·4	4·51148E−01	7·07574E−01
0·5	6·11884E−01	6·60717E−01
0·6	7·96141E−01	6·05053E−01
0·7	1·00365E+00	5·40226E−01
0·8	1·23073E+00	4·67572E−01
0·9	1·46941E+00	3·91381E−01
1·0	1·70911E+00	3·19592E−01

The result is in a very good agreement with the analytical solution $J = -1$, $\pi = 1$, $x = e^t - 1$, $u = e^{-t}$.

7. CONCLUSIONS

The conjugate gradient algorithm for the solution of optimal control problems with parameters has been presented in this paper. The solved numerical examples show traditionally good convergence properties of conjugate gradient algorithms.

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