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A detailed analysis of the condition of regularity as stated in [1] for the validity of the theorem on ε-capacity of decomposable channel is given, and some of its modifications are considered. One of the purposes of this paper is also to show how the regularity condition of [1] may be weakened.

Some facts used in Part I of [1] without explicit proofs are established in Lemmas I and II.

CONTENTS

1. INTRODUCTION

Throughout the entire paper let $(\mathscr{A}, \mathbf{A})$ be a given measurable space, where $\mathbf{A}$ is a σ-algebra of subsets of $\mathscr{A}$. If ξ is a probability measure on $\mathbf{A}$, and if V is a (finite) random variable on the probability space $(\mathscr{A}, \mathbf{A}, \xi)$, then the *lower* and *upper θ-quantiles* of V (with respect to ξ) are defined as the greatest lower bound (infimum) and the least upper bound (supremum)

$$(1.1)\qquad q(\theta, \xi; V) = \inf\,\{r : \xi\{V \leqq r\} \geqq \theta\}\,, \qquad 0 < \theta \leqq 1\,,$$
$$q'(\theta, \xi; V) = \sup\,\{r : \xi\{V \geqq r\} \geqq 1 - \theta\}\,, \quad 0 \leqq \theta < 1\,,$$

respectively.

Let us assume that we are given a family $W = \{W_z\}_{z \in R}$ of finite real-valued functions W_z defined on $\mathscr{A}$ and measurable relative to $\mathbf{A}$ (with parameters z lying in a nonempty set R); if ξ is a probability measure on the *basic space* $(\mathscr{A}, \mathbf{A})$, then the lower and upper *θ-quantiles of the family* W (with respect to ξ) will be defined as the supremal quantities

$$q_*(\theta, \xi) = \sup_{z\in R} q(\theta, \xi; W_z), \quad 0 < \theta \leqq 1, \tag{1.2}$$
$$q^*(\theta, \xi) = \sup_{z\in R} q'(\theta, \xi; W_z), \quad 0 \leqq \theta < 1.$$

The family $W = \{W_z\}_{z\in R}$ of random variables on the basic space is said to be *regular* (with respect to ξ) if the least upper bound V of the family W given by

$$V(\alpha) = \sup_{z\in R} W_z(\alpha), \quad \alpha \in \mathscr{A}, \tag{1.3}$$

is a finite random variable on the basic space (i.e. measurable relative to $\boldsymbol{A}$), satisfying the relation

$$q(\theta, \xi; V) = q_*(\theta, \xi) \quad \text{for all } \theta, \quad 0 < \theta \leqq 1. \tag{1.4}$$

The assumptions we shall make as to the family W in the sequel, will be as follows: for every $z \in R$, W_z is the random variable on the basic space defined by the equation

$$W_z(\alpha) = I(\nu^\alpha \mu_z) \quad \text{for} \quad \alpha \in \mathscr{A}, \tag{1.5}$$

where (1) $\{\nu^\alpha\}_{\alpha\in\mathscr{A}}$ is a measurable family of strongly stable ergodic channels (with a finite input alphabet B and a finite output alphabet A) such that (the duration of past history)

$$m(\nu^\alpha) \leqq m \quad \text{for all} \quad \alpha \in \mathscr{A}$$

and for some integer m (cf. [1], Part I, Sec. 1); (2) $R = R_B$ is the set of all regular points in the space B^I; (3) μ_z is the ergodic input associated with the regular point $z \in R$ (cf. (1.9) in [1]); (4) $\nu^\alpha\mu_z$ is the measure defined on $\boldsymbol{F}_{AB}$ for $\nu = \nu^\alpha$, $\mu = \mu_z$ by the relation (1.15) given in [1], and $I(\nu^\alpha\mu_z)$ is the information rate of the measure $\nu^\alpha\mu_z$ (cf. (1.12) in [1]).

Let us point out that, under the assumptions made, the measurability of W_z given by (1.5) was shown to hold in [1], Part II, Sec. 5. Moreover, the least upper bound V associated with the family W by (1.3) is exactly the capacity, viz.

$$V(\alpha) = \mathscr{C}(\nu^\alpha) \quad \text{for every} \quad \alpha \in \mathscr{A}, \tag{1.6}$$

so that V is a bounded nonnegative random variable on the basic space, as follows from relations (3.7), (1.22), (3.23), and (3.24), and from Sec. 5 of [1].

As in [1], the channel ν associated with a given probability ξ on the basic space by the definition

$$\nu = \int \nu^\alpha d\xi(\alpha) \tag{1.7}$$

(cf. (1.27) in [1]) is said to be *decomposable* into components ν^α ($\alpha \in \mathscr{A}$). The channel ν

is said to be *regularly decomposable* if the family $W = \{W_z\}_{z \in R}$, where W_z is defined by (1.5), is regular (with respect to ξ).

Remark. The assumption that $m(v^\alpha) \leqq m$ for $\alpha \in \mathscr{A}$ may be replaced by the requirement that $m(v^\alpha) \leqq m$ a.s. $[\xi]$ for $\alpha \in \mathscr{A}$ (cf. Sec. 5 in [1]).

Now we shall state some theorems which yield various conditions, each equivalent to that of regularity, in terms of distribution functions; the proofs will be given in Section 4 below. In the statements of the theorems mentioned we shall make use of the following notations:

(1.8) F is a distribution function of V, e.g. the d.f. given by

$$F(t) = \xi\{V \leqq t\}\,, \quad t \text{ real}$$

(cf. (1.3) and (1.6) above);

(1.9) F_z is a distribution function of $W_z (z \in R)$

(cf. (1.5) above); by F_* we shall designate the function defined by

$$(1.10) \qquad F_*(t) = \inf_{z \in R} F_z(t)\,, \quad t \text{ real}\,.$$

It is easy to see that, under the assumptions made, F represents a distribution function, i.e. a monotonically increasing function such that $F_*(-\infty) = 0$, $F_*(+\infty) = 1$.

Theorem 1. *The channel v given by (1.7) is regularly decomposable if and only if the equality $F_*(r) = F(r)$ holds at every continuity point r of the distribution function F.*

In what follows we shall denote by D the set of all discontinuity points of the distribution function F, i.e.

$$(1.11) \qquad D = \{t : F(t + 0) - F(t - 0) > 0\}\,.$$

We shall say that the distribution function F is *increasing* at a point r *from the left* if $F(t) < F(r - 0)$ for any $t < r$. The set of those real points r at which F is continuous and increasing from the left, will be denoted by S; it may easily be seen that the set S may be expressed in the form (cf. (1.1))

$$(1.12) \qquad S = \{r : r \notin D,\ F(r - 0) > 0,\ q(F(r - 0), \xi; V) = r\}\,.$$

On the other hand, the set of those real points at which F is both discontinuous and not increasing from the left, will be designated by D_0; in symbols:

$$(1.13) \qquad D_0 = \{r : r \in D,\ \text{if } F(r - 0) > 0 \text{ then } q(F(r - 0), \xi; V) < r\}\,.$$

Theorem 2. *The channel ν given by (1.7) is regularly decomposable if and only if there is a subset Q of the set S which is dense in S and such that the inequality $F_*(r-0) \leqq F(r-0)$ holds for all r lying in Q or in D_0.*

If r and t are any real numbers, and if

$$\mathrm{C}(r, t): \qquad \text{either there is } z \in R \text{ such that } F_z(t+0) < F(r-0), \text{ or}$$

$$\xi\{W_z \leqq t,\ V < r\} \geqq F(r-0) \quad \text{for all} \quad z \in R\,,$$

then we shall say that condition $\mathrm{C}(r, t)$ is valid for the r and t.

Theorem 3. *The channel ν given by (1.7) is regularly decomposable if and only if (1) the inequality $F_*(r-0) \leqq F(r-0)$ holds for every $r \in D_0$, and (2) there is a subset Q of the set S which is dense in S and such that condition $\mathrm{C}(r, t)$ is valid for every $r \in Q$ and for any real t.*

It is easy to verify that by using the notations given above the regularity condition of [1] (cf. Sec. 1, Part I, and also (3.7) in [1]) may be restated as follows: both (the given measure) ξ and the channel ν associated with ξ by (1.7) are called *regular* if (1) the inequality $F_*(r-0) \leqq F(r-0)$ holds for every $r \in D_0$, and (2) there is a countable subset Q of the set S which is dense in S and such that condition $\mathrm{C}(r, t)$ is valid for every $r \in Q \cup (D - D_0)$ and for any real t. Consequently, we may state the following corollary to Theorem 3 showing that the condition of regular decomposability and that of regularity coincide.

Corollary. *A necessary and sufficient condition for the channel ν given by (1.7) to be regularly decomposable is that it be regular.*

Let us remark that the assertion of Lemma 2.2 stated in [1], Part I, is equivalent to the regular decomposability of the channel ν, as follows from Lemma 3.7 proved in Section 3 below and from the equalities

$$\begin{aligned}
\tilde{c}(\theta, \nu) &= q(\theta, \xi; V)\,, & 0 < \theta \leqq 1\,,\\
\tilde{c}'(\theta, \nu) &= q'(\theta, \xi; V)\,, & 0 \leqq \theta < 1\,,\\
c_*(\theta, \nu) &= q_*(\theta, \xi)\,, & 0 < \theta \leqq 1\,,\\
c^*(\theta, \nu) &= q^*(\theta, \xi)\,, & 0 \leqq \theta < 1\,;
\end{aligned} \tag{1.14}$$

the latter relations may be obtained by an immediate confrontation of the definitions given here and those given in [1] (cf. (1.28), (1.24), (2.5), (2.2), (3.26), (3.19), and (5.8) in [1]); this means that the channel ν defined by (1.7) is regularly decomposable if and only if $c_*(\theta, \nu) = \tilde{c}(\theta, \nu)$ for $0 < \theta \leqq 1$.

It follows from the equivalence mentioned in the preceding paragraph that the Corollary implies Lemma 2.2 of [1], and that the proof of Theorem 3 given in Section 4 below yields another method how to show the validity of the assertion of Lemma 2.2. Moreover, the conditions of regular decomposability stated in Theorem 3 also show that the regularity condition of [1] may be weakened.

Let us define for a probability measure ξ on the basic space and for a real number r such that

$$r > r_{\min} = \text{ess. inf}\,\{V(\alpha) : \alpha \in \mathscr{A}[\xi]\} \tag{1.15}$$

the probability measure $\xi_{(r)}$ by the relation

$$\xi_{(r)}(M) = \frac{\xi(M \cap \{\alpha : V(\alpha) < r\})}{\xi\{\alpha : V(\alpha) < r\}}, \quad M \in \mathbf{A}\,; \tag{1.16}$$

it is evident that the assumption that r is greater than the essential infimum guarantees the positivity of $\xi\{V < r\}$. The original proof of Lemma 2.2 in [1] is based upon the fact (used in [1] without an explicit proof) stated in the following lemma (which will be proved in Section 4 below):

Lemma I. *If ξ is regular, and if r belongs to S or to $D - D_0$, then* (cf. (1.7))

$$\lim_{\theta \to 1} c^*(\theta, \nu_{(r)}) = \lim_{\theta \to 1} c^*(\theta \,.\, \gamma, \nu)\,,$$

where $\gamma = \xi\{V < r\}$, and $\nu_{(r)} = \int \nu^{\alpha} d\xi_{(r)}(\alpha)$, provided that $c^*(\theta, \nu)$ is continuous at $\theta = \gamma$.

Let us associate with a probability measure ξ on the basic space and with any real number

$$r < r_{\max} = \text{ess. sup}\,\{V(\alpha) : \alpha \in \mathscr{A}[\xi]\} \tag{1.17}$$

the measure $\xi^{[r]}$ by the definition

$$\xi^{[r]}(M) = \frac{\xi(M \cap \{\alpha : V(\alpha) \geqq r\})}{\xi\{\alpha : V(\alpha) \geqq r\}}, \quad M \in \mathbf{A}\,; \tag{1.18}$$

evidently, assumption (1.17) guarantees that $\xi\{V \geqq r\}$ is positive.

If the channel $\nu^{[r]}$ is defined by

$$\nu^{[r]} = \int \nu^{\alpha} d\xi^{[r]}(\alpha)\,, \tag{1.19}$$

and if

A(r): there is a countable subset E_r of the open interval (0, 1) such that

$$c(\varepsilon, \nu^{[r]}) = \bar{c}(\varepsilon, \nu^{[r]}) = \tilde{c}(\varepsilon, \nu^{[r]}) \quad \text{for all} \quad \varepsilon \notin E_r (0 < \varepsilon < 1)\,,$$

we shall say that assumption A(r) is valid for the r. If assumption A(r) is valid for every real $r < r_{\max}$, we shall say that *all subchannels* of the decomposable channel ν (given by (1.7)) *possess quantilized ε-capacity* (the notations used in the statement of the condition A(r) are taken from [1]; cf. (2.1) loc. cit. and (1.14) above).

Making use of the Corollary, we may restate the main theorem of [1], viz. the theorem on the existence of ε-capacity, in terms of regular decomposability:

Theorem on ε-capacity. *All subchannels of a decomposable channel ν possess quantilized ε-capacity if and only if the channel ν is regularly decomposable.*

Theorems 1 – 3 enable various modifications in the statement of the latter theorem.

In the proof of Theorem on ε-capacity as given in Part I of [1] use was made of the following facts (not explicitly proved there) stated in

Lemma II. *If ξ is regular, or if all subchannels of the channel ν associated with ξ by (1.7) possess quantilized ε-capacity, then the equalities*

$$\tilde{c}(\theta, \nu^{[r]}) = \tilde{c}(1 - \gamma + \theta \,.\, \gamma, \nu)\,,$$
$$c^*(\theta, \nu^{[r]}) = c^*(1 - \gamma + \theta \,.\, \gamma, \nu)\,, \quad \gamma = \xi\{V \geqq r\} \;\; (0 < \theta < 1)\,,$$

hold for all $r < r_{\max}$.

Remark. Lemma II implies that the channel ν is regular (or, equivalently, regularly decomposable) if and only if all subchannels $\nu^{[r]}$ are regular(ly decomposable); the sufficiency of the latter condition trivially follows from the relation $\nu^{[r]} = \nu$ for $r \leqq r_{\min}$.

Theorem 5 in [1] shows that there are regularly decomposable channels. On the other hand, Example 1 in Section 6 of [2] shows that there are decomposable channels which are not regular.

2. QUANTILES

First we shall state some elementary properties of quantiles which are needed in the subsequent analysis of the problem of regularity.

If ξ is a probability measure on the basic space $(\mathscr{A}, \mathbf{A})$, let us assume that A is a given measurable set in the space $\mathscr{A}$, i.e. $A \in \mathbf{A}$, such that $\gamma = \xi(A) > 0$. Then if the measure $\tilde{\xi}$ is defined on the basic space by

$$\tilde{\xi}(M) = \gamma^{-1} \xi(M \cap A) \quad \text{for} \quad M \in \mathbf{A}\,, \tag{2.1}$$

the following inequalities are valid for quantiles with respect to $\tilde{\xi}$ of a random variable V:

$$\begin{aligned} q(\theta, \tilde{\xi}; V) &\geqq q(\theta \,.\, \gamma, \xi; V) && \text{for} \quad 0 < \theta \leqq 1\,, \\ q'(\theta, \tilde{\xi}; V) &\leqq q'(1 - \gamma + \theta \,.\, \gamma, \xi; V) && \text{for} \quad 0 \leqq \theta < 1\,. \end{aligned} \tag{2.2}$$

Proof. Since according to (2.1)

$$q(\theta, \tilde{\xi}; V) = \inf \{t : \xi(\{V \leqq t\} \cap \Lambda) \geqq \theta \cdot \gamma\},$$

the inequality $\xi\{V \leqq t\} \geqq \xi(\{V \leqq t\} \cap \Lambda)$ together with the definition (1.1) imply the first inequality in (2.2). The proof of the second is dual: because of the relation

$$q'(\theta, \tilde{\xi}; V) = \sup \{t : \xi(\{V \geqq t\} \cap \Lambda) \geqq (1 - \theta) \gamma\}$$

a similar reasoning yields the inequality

$$q'(1 - (1 - \theta) \gamma, \xi; V) \geqq q'(\theta, \tilde{\xi}; V),$$

Q.E.D.

Throughout the remainder of this section we shall assume that we are given a (finite) random variable V on the basic space; the lower and upper θ-quantiles with respect to a probability measure ξ of the random variable V will be designated in a simpler way by $q(\theta, \xi)$, $q'(\theta, \xi)$.

If $\xi^{[r]}$ is the measure associated with a probability ξ by (1.18) for some $r < r_{max}$ (cf. (1.17)), i.e. $\xi^{[r]} = \tilde{\xi}$ associated with $\Lambda = \{\alpha : V(\alpha) \geqq r\}$ by (2.1), then the relations (2.2) may be improved because the quantiles taken with respect to $\xi^{[r]}$ may be expressed by those taken with respect to ξ directly, viz.

$$(2.3) \qquad \begin{aligned} q(\theta, \xi^{[r]}) &= q(1 - \gamma + \theta \cdot \gamma, \xi), \quad 0 < \theta \leqq 1, \\ q'(\theta, \xi^{[r]}) &= q'(1 - \gamma + \theta \cdot \gamma, \xi), \quad 0 \leqq \theta < 1, \end{aligned}$$

where $\gamma = \xi\{V \geqq r\}$. The same is true for the measure $\xi_{(r)}$ associated with ξ by (1.16), provided that $r > r_{min}$ (cf. (1.15); i.e. $\xi_{(r)} = \tilde{\xi}$ associated with $\Lambda = \{\alpha : V(\alpha) < r\}$ by (2.1)), viz.

$$(2.4) \qquad \begin{aligned} q(\theta, \xi_{(r)}) &= q(\theta \cdot \gamma, \xi), \quad 0 < \theta \leqq 1, \\ q'(\theta, \xi_{(r)}) &= q'(\theta \cdot \gamma, \xi), \quad 0 \leqq \theta < 1, \end{aligned}$$

where $\gamma = \xi\{V < r\}$.

Proof. Relations (2.3) and (2.4) are immediate consequences of the equalities

$$\begin{aligned} q(\theta, \xi^{[r]}) &= \inf \{t : \xi\{V \leqq t\} \geqq \xi\{V < r\} + \theta \cdot \gamma\}, \\ q'(\theta, \xi^{[r]}) &= \sup \{t : \xi\{V \geqq t\} \geqq (1 - \theta) \gamma\}, \end{aligned}$$

and dual for $\xi_{(r)}$.

In the rest of this section a measure ξ on $\boldsymbol{A}$ is supposed to be kept fixed so that the θ-quantiles may be briefly denoted as $q(\theta)$, $q'(\theta)$. Making use of the notation (1.8), we may express the quantiles by means of the distribution function F in the form

$$(2.5) \qquad \begin{aligned} q(\theta) &= \sup \{r : F(r) < \theta\}, \quad 0 < \theta \leqq 1, \\ q'(\theta) &= \inf \{r : F(r) > \theta\}, \quad 0 \leqq \theta < 1. \end{aligned}$$

Lemma 2.1. *The quantiles q, q' are monotonically increasing functions, and $q(\theta) \leqq q'(\theta)$ for $0 < \theta < 1$; q is continuous from the left* (also at $\theta = 1$), *and q' is continuous from the right* (also at $\theta = 0$), *and*

$$q(\theta + 0) = q'(\theta) \quad \text{for} \quad \theta < 1\,, \qquad q'(\theta - 0) = q(\theta) \quad \text{for} \quad \theta > 0\,;$$

the equality $q(\theta) = q'(\theta)$ holds if and only if q is continuous at θ, or, equivalently, if q' is continuous at θ.

The elementary proof of the preceding lemma coincides with that of Lemma 3.3 given in [1], Part II, and makes use of (2.5).

Let us point out that for the sets D and S defined by (1.11) and (1.12), respectively, we easily obtain that

(2.6) $$\text{if} \quad r \in D \quad \text{then} \quad q(F(r - 0) + 0) = r\,, \quad \text{and}$$

(2.7) $$\text{if} \quad r \in S \quad \text{then} \quad q(F(r) + 0) \geqq q(F(r)) = r\,.$$

Lemma 2.2. *If $r \in S$ or $r \in D - D_0$ then there is a sequence r_n $(n = 1, 2, \ldots)$ strictly increasing to r and such that either* (1) *$r_n \in D_0$ for all n* (cf. (1.13)), *or* (2) *$r_n \in S$ for all n.*

Proof. Assume that there is an $r' < r$ such that the open interval (r', r) does not contain any point from S. The supposition that (r', r) does not contain any point from D leads to a contradiction: this enables us to construct a sequence having property (1) of the lemma. Otherwise, we may easily construct a sequence with property (2) which proves the lemma.

We shall say that the distribution function F is *increasing* at a point r *to the right* if $F(t) > F(r + 0)$ for any $t > r$. The set of those real points at which F is increasing to the right, will be denoted by S'; it is easy to find that the set S' may be expressed in the form

(2.8) $$S' = \{r : F(r + 0) < 1, q'(F(r + 0)) = r\}\,.$$

We shall make use of the preceding lemmas in the proofs of the propositions stated in the next section.

3. REGULAR FAMILIES

Throughout this section we shall assume that we are given a family $W = \{W_z\}_{z \in R}$ of random variables on the basic space (R nonempty). If ξ is a probability measure on the basic space, and if the measure $\tilde{\xi}$ is associated with ξ by the definition (2.1), where $\gamma = \xi(\Lambda) > 0$, $\Lambda \in \mathbf{A}$, then the relations (2.2) applied to W_z together with definitions (1.2) imply that

$$(3.1) \qquad q_*(\theta, \xi) \geqq q_*(\theta \,.\, \gamma, \xi) \qquad \text{for} \quad 0 < \theta \leqq 1\,,$$
$$q^*(\theta, \xi) \leqq q^*(1 - \gamma + \theta \,.\, \gamma, \xi) \quad \text{for} \quad 0 \leqq \theta < 1\,.$$

In the following lemmas $\xi^{[r]}$ is the measure associated with a probability ξ by (1.18) for $r < r_{\max}$ (cf. (1.17)), provided V is defined by (1.3), measurable relative to **A**, and finite.

Lemma 3.1. *If* $q^*(\theta, \xi^{[r]}) \geqq r$ *or* $q^*(1 - \gamma + \theta \,.\, \gamma, \xi) > r$, *where* $\gamma = \xi\{V \geqq r\}$ *then*

$$q^*(\theta, \xi^{[r]}) = q^*(1 - \gamma + \theta \,.\, \gamma, \xi) \quad (0 \leqq \theta < 1)\,.$$

Proof. Assume that on the contrary the latter equality does not hold for some θ satisfying the assumptions. Then it follows from (3.1) and from the assumptions of the lemma that

$$q^*(\theta, \xi^{[r]}) < t < q^*(1 - \gamma + \theta \,.\, \gamma, \xi)$$

for some $t \geqq r$. The latter inequalities imply that there is some $z \in R$ such that

$$\xi\{W_z \geqq t\} \geqq (1 - \theta)\,\gamma \quad \text{and} \quad \xi\{W_z \geqq t,\ V \geqq r\} < (1 - \theta)\,\gamma$$

which contradicts the set inclusions

$$\{W_z \geqq t\} \subset \{V \geqq t\} \subset \{V \geqq r\}\,.$$

Lemma 3.2. *If the family* W *is regular with respect to* ξ, *then*

$$q^*(\theta, \xi^{[r]}) = q^*(1 - \gamma + \theta \,.\, \gamma, \xi) \quad \text{for all } \theta\,, \quad 0 < \theta < 1\,,$$

where $\gamma = \xi\{V \geqq r\}$.

Proof. If $r \leqq r_{\min}$ (cf. (1.15)) then $\xi^{[r]} = \xi$, $\gamma = 1$ so that the assertion of the lemma holds. Let $r > r_{\min}$. If $r \notin D$ (cf. (1.11) then if $r \in S'$,

$$q^*(1 - \gamma + \theta \,.\, \gamma, \xi) \geqq q_*(1 - \gamma + \theta \,.\, \gamma, \xi) = q(1 - \gamma + \theta \,.\, \gamma, \xi) >$$
$$> q'(1 - \gamma, \xi) = q'(F(r), \xi) = r \quad \text{for} \quad 0 < \theta < 1\,,$$

and if $r \notin S'$, $q'(1 - \gamma, \xi) > r$, as follows from Lemma 3.5 given below, and from (1.4), Lemma 2.1, and (2.5); consequently, Lemma 3.1 may be applied which shows that the assertion ofthe lemma holds in the case considered.

Let us assume now that $r \in D$. If θ is such that $q^*(1 - \gamma + \theta \,.\, \gamma, \xi) > r$, then the assertion follows from Lemma 3.1. Suppose that the latter inequality does not hold; then

$$q^*(1 - \gamma + \theta \,.\, \gamma, \xi) = q_*(1 - \gamma + \theta \,.\, \gamma, \xi) = r\,.$$

On the other hand, there is a sequence r_n $(n = 1, 2, \ldots)$ increasing to r and such that $r_n \notin D$. Since the assertion of the lemma holds for all r_n, as shown above, we have by Lemma 3.5, and by (3.1) applied to $\xi^{[r]}$ with respect to $\xi^{[r_n]}$ that, for $0 < \theta' < \theta$,

$$q^*(1 - \gamma_n + \theta'\beta_n \cdot \gamma_n, \xi) = q^*(\theta'\beta_n, \xi^{[r_n]}) \leqq$$
$$\leqq q_*(\theta\beta_n, \xi^{[r_n]}) \leqq q_*(\theta, \xi^{[r]}) \leqq q^*(\theta, \xi^{[r]}),$$

where $\gamma_n = \xi\{V \geqq r_n\}$, $\beta_n = \gamma/\gamma_n$. From here and from (3.1) we obtain for $n \to \infty$ and $\theta' \to \theta$ the inequalities

$$q_*(1 - \gamma + \theta \cdot \gamma, \xi) \leqq q^*(\theta, \xi^{[r]}) \leqq q^*(1 - \gamma + \theta \cdot \gamma, \xi)$$

(cf. Lemma 3.5) which together with the above equality yields the desired result, Q.E.D.

In the subsequent lemmas $\xi_{(r)}$ is the measure associated with a probability measure ξ by (1.16) for $r > r_{\min}$ (cf. (1.15)), where V is supposed to be a random variable satisfying relation (1.3).

Lemma 3.3. *Given r, condition* C(r, t) [cf. Section 1] *is valid for the r and for all real t if and only if the equality*

$$q_*(1, \xi_{(r)}) = q_*(\gamma, \xi)$$

holds with $\gamma = \xi\{V < r\}$.

Proof. Since $F(r - 0) = \gamma$ (cf. (1.8)), the inequality $F_z(t + 0) \geqq F(r - 0)$ is equivalent to the relation $q(\gamma, \xi; W_z) \leqq t$ (cf. (1.9)), and the inequality

$$\xi\{W_z \leqq t,\ V < r\} \geqq F(r - 0)$$

is equivalent to the relation $q(1, \xi_{(r)}; W_z) \leqq t$. From the latter equivalences we conclude that condition C(r, t) is equivalent to the implication:

$$\text{if} \quad q_*(\gamma, \xi) \leqq t \quad \text{then} \quad q_*(1, \xi_{(r)}) \leqq t\,;$$

this together with (3.1) yields the desired equality, and conversely.

Lemma 3.4. *If Q is a subset of S* (cf. (1.12)) *which is dense in S, and if the equality*

$$q_*(1, \xi_{(r)}) = q_*(\gamma, \xi) \quad \text{with} \quad \gamma = F(r - 0)$$

(cf. (1.8)) *holds for every r in Q, then it holds for every r in S such that* $q_*(\theta, \xi)$ *is continuous at* $F(r - 0) = \gamma$.

Proof. If $r \in S - Q$ then there is either (1) a sequence $r_n \in Q$ $(n = 1, 2, \ldots)$ increasing to r, or (2) a sequence $s_n \in Q$ $(n = 1, 2, \ldots)$ decreasing to r. In the first case an immediate application of the first inequality in (3.1) for $\xi_{(r_n)}$ with respect to $\xi_{(r)}$ will yield (together with the above equality valid for r_n) the relation $q_*(1, \xi_{(r)}) \leqq \leqq q_*(\gamma, \xi)$ which compared with (3.1) guarantees the validity of the desired equality for r (cf. the proof of Lemma 3.2 where such an approximation method is described in some detail). The second case is treated similarly by using the second inequality in (3.1) for $\xi_{(r)}$ with respect to $\xi_{(s_n)}$ which gives the inequality

$$q_*(1, \xi_{(r)}) \leqq q_*(\gamma + 0, \xi) .$$

However, continuity assumption implies the validity of the equality $q_*(\gamma + 0, \xi) = = q_*(\gamma, \xi)$ which compared with (3.1) again proves the desired relation, Q.E.D.

In the remainder of this section we shall keep ξ fixed, and denote the quantiles of the given family W simply by $q_*(\theta)$, $q^*(\theta)$.

Lemma 3.5. *The quantiles q_*, q^* are monotonically increasing functions satisfying the inequality $q_*(\theta) \leqq q^*(\theta)$ for $0 < \theta < 1$; q is continuous from the left at every θ $(0 < \theta \leqq 1)$, and*

$$q^*(\theta - 0) = q_*(\theta) \quad \text{for} \quad 0 < \theta \leqq 1 ;$$

θ is a discontinuity point of q_* if and only if it is a discontinuity point of q^*; if θ is a point of continuity of q_* (and of q^*, respectively), then the equality $q_*(\theta) = = q^*(\theta)$ holds.

The lemma represents a more general formulation of Lemma 3.5 proved in [1], Part II.

As above we shall assume that V given by (1.3) is a (finite) random variable with a distribution function F. It follows from the definition (1.10) of the (distribution) function F_* that

$$F_*(t + 0) \geqq F(t + 0) , \quad t \text{ real} \tag{3.2}$$

because of the set inclusion $\{W_z \leqq t\} \supset \{V \leqq t\}$. An analogous fact for quantiles is established in the following

Lemma 3.6. *The quantiles satisfy relations*

$$q_*(\theta) \leqq q(\theta) \quad \text{for} \quad 0 < \theta \leqq 1 , \quad q^*(\theta) \leqq q'(\theta) \quad \text{for} \quad 0 \leqq \theta < 1 .$$

Proof. According to (1.3),

$$\xi\{W_z \leqq r\} \geqq \xi\{V \leqq r\} \geqq \theta$$

for $r = q(\theta)$, and for every $z \in R$. The latter inequality immediately implies that $q_*(\theta) \leqq r$ which yields the first inequality. The proof of the second is dual.

Let us remark that Lemma 3.6 together with Lemma 3.5 guarantee the validity of the assertion stated in Lemma 2.1 given in Part I of [1]. The subsequent lemma shows that the assertion of Lemma 2.2 as stated in [1] is equivalent to (1.4).

Lemma 3.7. *The family W is regular if and only if $q^*(\theta) = q(\theta)$ except a countable set of θ's.*

Proof. The lemma is an immediate consequence of condition (1.4) and Lemma 3.5.

Lemma 3.8. *Given a real number r such that $r \in D$ or $F(r - 0) < 1$* (cf. (1.11), (1.12)), *then $F_*(r - 0) \leqq F(r - 0)$ if and only if*

$$q_*(F(r - 0) + 0) \geqq r .$$

Proof. The relation $F_*(r - 0) \leqq F(r - 0)$ holds if and only if, for every $\theta > F(r - 0)$ and for every $t > r$, there is some $z \in R$ such that $F_z(t) \leqq \theta$; the latter condition is equivalent to the inequality $q_*(F(r - 0) + 0) \geqq r$; the lemma is valid.

Proposition 1. *The family W is regular if and only if the equality $F_*(r) = F(r)$ holds at every continuity point r of the distribution function F.*

Proposition 2. *The family W is regular if and only if there is a subset Q of S* (cf. (1.12)) *dense in S and such that the inequality $F_*(r - 0) \leqq F(r - 0)$ holds for all $r \in Q \cup D_0$* (cf. (1.13)).

Proof. I. Let us make the assumption that W is regular. We shall show that then

$$F_*(r + 0) \leqq F(r + 0)$$

for all r, r real. Assume that the contrary would be true, i.e. $F(r + 0) > F_*(r + 0) = \theta$. From the relations

$$\xi\{W_z \leqq r\} = F_z(r + 0) \geqq F_*(r + 0) = \theta$$

and from definition (1.2) we conclude that $q_*(\theta) \leqq r$; hence, according to (1.4), $q(\theta) = q_*(\theta) \leqq r$. On the other hand, $\xi\{V \leqq r\} = F(r + 0) > \theta$ implies that $q(\theta) > r$ which yields the desired contradiction.

By using (3.2) we obtain that the equality $F_*(r + 0) = F(r + 0)$ holds for all r which implies both the assertions stated in Proposition 1 and 2.

II. Assume that $F_*(r - 0) \leqq F(r - 0)$ for all $r \in Q \cup D_0$, where $Q \subset S$ dense in S. We shall prove that then the latter inequality must remain valid for all $r \in S \cup D$. This may easily be done by approximating from the left an $r \in (S \cup D - D_0)$ not lying in Q according to Lemma 2.2, making use, if necessary, of the density of Q in S. Then applying again Lemma 3.8 together with Lemma 3.6 we find that

$$q_*(F(r - 0) + 0) = q(F(r - 0) + 0) \quad \text{for all} \quad r \in (S \cap S') \cup D\,, \quad \text{i.e.}$$

$$q_*(\theta + 0) = q(\theta + 0) \quad \text{for all } \theta\,, \quad 0 < \theta < 1\,,$$

 except a countable set of Θ' s. From here and from Lemma 2.1 and Lemma 3.5 it immediately follows that the regularity condition (1.4) must be valid; hence the family W is regular. This proves Proposition 2.

Since the condition stated in Proposition 1 guarantees the validity of that given in Proposition 2, the latter considerations enable to conclude the validity of Proposition 1, Q.E.D.

4. EQUIVALENCE OF REGULARITY CONDITIONS

We shall start with two lemmas which establish the only facts that are needed in addition to those of the preceding section in proving the theorems and lemmas stated in Sec. 1 under the assumptions (1.5).

Lemma 4.1. *If condition* C(r, t) *is valid for every real* t *and for some* $r \in S \cup \cup (D - D_0)$, *then* $F_*(r - 0) \leqq F(r - 0)$.

The lemma follows from Lemma 3.3 given above and from Theorem 4 stated in [1] by the method used in the proof of Lemma 2.2 given in [1].

Lemma 4.2. *If condition* A(r) *is valid for some* r, $r < r_{\max}$, *then* $q^*(\theta, v^{[r]}) \geqq r$ *for* $0 < \theta < 1$.

The proof is based upon Theorem 2 of [1] applied to $v^{[r]}$.

Proof of Theorem 1. The theorem coincides with Proposition 1.

Proof of Theorem 2. The theorem is a version of Proposition 2.

Proof of Theorem 3. It follows from Lemma 4.1 and Theorem 2.

Proof of Lemma I. According to (1.14) and (1.16) the lemma coincides with Lemma 3.4 for the case considered. Let us mention that the assertion of Lemma I may be shown to be valid for any real r lying both in S and in $D - D_0$ by making use of the Corollary to Theorem 3 and of Theorem 4 of [1].

Proof of Lemma II. The assertion of the lemma follows from (1.14), (2.3), Lemma 4.2, and Lemma 3.2 together with Lemma 3.1.

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VÝTAH

K podmínce regularity pro rozložitelné sdělovací kanály

Karel Winkelbauer

Hlavním cílem práce je oslabit formulaci podmínky regularity, jež se ukázala v práci [1] jako nutná a postačující k platnosti věty o existenci ε-kapacity. Za tím účelem je provedena potřebná analýza této podmínky, přičemž jsou dokazovány některé vztahy, jichž bylo použito v práci [1] bez podrobnějšího důkazu.

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