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LINEAR TIME-OPTIMAL CONTROL PROBLEM WITH INCOMPLETE INFORMATION ABOUT STATE. OPTIMIZATION OF GUARANTEED RESULT

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In the paper we study one problem related to two-point time-optimal control problem called the "worst case design problem". The initial state of a linear dynamical system may be an arbitrary point of a given ball radius ε and may depend on a random factor. Therefore our problem is formulated and solved as a game against the Nature at first in pure, next in mixed strategies. If the elements of the matrix $A(\cdot)$ are measures our problem may have no solution.

1. INTRODUCTION

Optimal control problems under uncertainty or conflict are very interesting. The literature on this subject is very extensive — let us mention only [1, 2 — 7, 9, 10]; the greater bibliography may be found in [2, 5, 9]. In these papers uncertain factors are, in general, random terms in the right-hand sides of differential equations of dynamical systems. There are, in general, ordinary differential equations, linear [2, 3, 5, 6], nonlinear [2, 5, 6] and with delayes [7]. Optimization problems are formulated and solved as differential games in which the first player's problem is solved and his guaranteed result is optimized. Sometimes, as in [6] all funnel of trajectories starting from a given set is controlled. There are problems with fixed time duration and with the cost functionals of the Mayer or Lagrange type. In [3] the linear-quadratic problem is solved.

In this paper we assume that the initial state (which may be fatally selected) of a linear dynamical system may be an arbitrary point of a ball radius ε with a given center $\bar{x}$ and the random factor may act in a most unfavourable way. The performance index is the first instant when the trajectory attains a ball radius ε with a given center x_1 and the guaranteed value of this time is minimized. We meet such a problem in practice when we solve the classical time-optimal control problem with initial state $\bar{x}$ and final state x_1 when the state x of the system may be observed with the accuracy ε only but not exactly. This problem is formulated and solved here as a two-person, zero-sum game against the Nature, at first — in pure, next — in mixed strategies (the first player is called the Engineer).

2. FORMULATION OF THE PROBLEM

Let us consider the following dynamical system:

$$\dot{x} = A(t)\,x + B(t)\,u\,, \tag{1}$$

$$x(t_0) \in K \triangleq \{x: \|x - \bar{x}\| \leqq \varepsilon\} \tag{2}$$

where: $A(\cdot) \in L^1[t_0, T]$ and $B(\cdot) \in L^2[t_0, T]$ are given $n \times n$ and $n \times m$ matrices, $\bar{x} \in \mathbb{R}^n$ is a given vector, $\varepsilon > 0$ is a given real number and $T \leqq \infty$. The admissible controls

$$u(\cdot) \in \mathscr{U} = \{u(\cdot): [t_0, T] \to \mathbb{R}^m;\ |u_j(t)| \leqq c_j,\ c_j > 0,\ j = 1, \ldots, m\,,\quad t \in [t_0, T] \text{ and } u(\cdot) \in L^2[t_0, T]\}\,,$$

c_j are some given numbers. The target set is

$$M \triangleq \{x: \|x - x_1\| \leqq \varepsilon\} \tag{3}$$

where x_1 is a given vector.

The Engineer is interested in the shortest time to attain the set M. For the fixed initial state this is a usual time-optimal control problem with the target M. But we assume that the initial state may depend on random factors and may be an arbitrary point of the set K. For the given $x_0 \in K$ and $u(\cdot) \in \mathscr{U}$ let $t_1(u(\cdot), x_0)$ denotes the first instant t_1 such that the trajectory $x(\cdot)$ of the system (1) with $u = u(\cdot)$ and with the initial state $x(t_0) = x_0$ satisfies the condition

$$x(t_1) \in M \quad (\text{i.e. } x(t) \notin M \quad \text{for all} \quad t < t_1)\,;$$

$$\text{if such } t_1 \text{ does not exist, we put} \quad t_1(u(\cdot), x_0) = +\infty\,.$$

The Engineer tends to minimize the function $t_1(\cdot, \cdot)$ in $u(\cdot) \in \mathscr{U}$, but he has no possibilities to select $x(t_0)$. Therefore we formulate our problem as the following game:

$$\Gamma = \{(\text{Eng.}, \text{Nat.})\,,\quad (\mathscr{U}, K), t_1\}$$

where: the first player, the Engineer, has $\mathscr{U}$ as the set of pure strategies, the second one, which will be the Nature, has K as the set of pure strategies and $t_1 = t_1(u(\cdot), x_0)$ is the payoff for Engineer if he applies his strategy $u(\cdot)$ against x_0 of the Nature; this function was defined above. The Engineer tends to minimization of the payoff function while the Nature have none interest in maximization of this function. Hence, this is not a strictly antagonistic game.

3. SOLUTION OF THE FIRST PLAYER'S PROBLEM

In the case when all decisions (i.e. selection of the "best" initial state and optimal control) falls within Engineer's cognizance, the corresponding problem of minimization of $t_1(\cdot, \cdot)$ under (1)–(3) jointly in $u(\cdot)$ and x_0 was solved in [1, 8]. However for our game-theoretical problem those methods are useless.

Definition 1. The *guaranteed result* for the first player who applies his strategy $u(\cdot)$ is

$$T(u(\cdot)) \triangleq \max_{x_0 \in K} t_1(u(\cdot), x_0).$$

Hence the first player's problem (called the minimax problem) may be formulated as the problem of minimization of the guaranteed result $T(u(\cdot))$ in $u(\cdot) \in \mathcal{U}$. To find the most safe strategy for Engineer, assume that the system (1) satisfies the following hypothesis (related to controllability assumption).

Hypothesis H.

$$\underset{\bar{u}(\cdot)}{\exists} \ \underset{x_0 \in K}{\forall} \ \underset{\bar{t} > t_0}{\exists} \quad (\bar{x}(\bar{t}) \in M)$$

where $\bar{x}(\cdot)$ is the trajectory of the system (1) with initial state x_0 which corresponds to the control $\bar{u}(\cdot)$.

Let us denote by $\mathcal{U}_1$ the set of all $\bar{u}(\cdot) \in \mathcal{U}$ for which Hypothesis H is true and by $X_u(\bar{t}, K)$ the "attainable set"

$$X_u(\bar{t}, K) = \{x\colon x = x(\bar{t}) = \phi(\bar{t})\, x_0 + \phi(\bar{t}) \textstyle\int_{t_0}^{\bar{t}} \phi^{-1}(s)\, B(s)\, u(s)\, \mathrm{d}s\,,\ x_0 \in K\}$$

where $\phi(\cdot)$ is the fundamental matrix of the system $\dot{x} = A(t)\, x$, normed at t_0 (it suffices to consider this set only for $u(\cdot) \in \mathcal{U}_1$).

To solve the minimax problem, let us fix a $\bar{u}(\cdot) \in \mathcal{U}_1$ and solve the following problem:

Find $x_0 \in K$ such that the trajectory $x(\cdot)$ of the system (1) with initial state $x(t_0) = x_0$ and with the control $\bar{u}(\cdot)$ satisfies the inclusion $x(t_1) \in M$ (and $x(t) \notin M$ for $t < t_1$) with maximal as possible (with respect to all $x(t_0) \in K$) time t_1.

This way we obtain the value $T(\bar{u}(\cdot))$ and the corresponding $x_0(\bar{u})$. This time $T(\bar{u}(\cdot))$ is well defined because of continuous dependence of the "attainable set" $X_u(\bar{t}, K)$ on time $\bar{t}$ and on K in the Hausdorff metric and by compactness of the ball K. Moreover, $x(t_1) \in \partial M$, the boundary of M, and the "worst" initial state x_0 may not be unique. The function $\bar{u}(\cdot) \to T(\bar{u}(\cdot))$ is a continuous one and the set $\mathcal{U}_1$ is a closed subset of the compact in $L^2[t_0, T]$ (by the Riesz theorem – see [2]) set $\mathcal{U}$, so $\mathcal{U}_1$ is also compact.

Therefore there is a control $u'(\cdot) \in \mathcal{U}_1$ such that

$$T(u'(\cdot)) = \min_{u(\cdot) \in \mathcal{U}_1} T(u(\cdot)).$$

Then the control $u'(\cdot)$, the corresponding initial state $x_0 = x_0(u')$ and the corresponding time

$$T(u'(\cdot)) = \min_{u(\cdot) \in \mathcal{U}} \ \max_{x_0 \in K} t_1(u(\cdot), x_0)$$

gives the solution of the first player's problem in the game Γ.

Remark 1. Hypothesis H is fulfilled, for example, if there is a control $\bar{u}(\cdot) \in \mathcal{U}$ which steers $\bar{x}$ to x_1 and if the fundamental matrix $\phi(\cdot)$ has the property: $\|\phi(t)\| \leqq 1$ for all $t > t_0$.

4. SOLUTION OF THE SECOND PLAYER'S PROBLEM

Now we solve the problem of the second player which is essential in the case of an antagonistic game.

Definition 2. The *guaranteed result* for the second player who applies his strategy x_0 is

$$S(x_0) \triangleq \min_{u(\cdot)\in\mathcal{U}} t_1(u(\cdot), x_0) .$$

Hence the problem of the second player (called the maximin problem) may be formulated as the problem of maximization of the guaranteed result $S(x_0)$ in $x_0 \in K$. To find the best strategy for the Nature assume that the following hypothesis (weaker than Hypothesis H) is satisfied.

Hypothesis G.

$$\forall_{x_0\in K} \ \exists_{u(\cdot)\in\mathcal{U}} \ \exists_{t_1 > t_0} \ (x(t_1) \in M)$$

is fulfilled, where $x(\cdot)$ is the solution of (1) with the initial state x_0 which corresponds to the control $u(\cdot)$.

Let us fix the initial state $x_0 \in K$ and solve the time-optimal control problem for the system (1) with initial state $x(t_0) = x_0$ and with the target M. This problem may be solved by using the moments problem or by the maximum principle (see [1, 6]). We obtain the optimal control $u_{x_0}(\cdot)$ and the corresponding optimal time $S(x_0)$.

From continuity of the function $x_0 \to S(x_0)$ (see [1], p. 185) and from compactness of the set K it follows that there exists an $x' \in K$ such that

$$S(x') = \max_{x_0\in K} S(x_0) ,$$

so the initial state x', the corresponding control $u_{x'}(\cdot)$ and the corresponding time

$$S(x') = \max_{x_0\in K} \min_{u(\cdot)\in\mathcal{U}} t_1(u(\cdot), x_0)$$

give the solution of the second player's problem in the game Γ.

5. EXAMPLE

In general, the solutions of both problems discussed in Sections 3 and 4 are different what we illustrate by the following example. Let us consider the following one-dimensional system

$$\dot{x} = x + u , \quad \bar{x} = 0 , \quad x_1 = 3 , \quad |u| \leqq 1 , \quad \varepsilon > 0 , \quad t \geqq 0 .$$

For fixed $x_0 \in [-\varepsilon, \varepsilon]$ the emission zone of the point x_0 is bounded from below by the curve $x(t) = e^t(x_0 - 1) + 1$ and from above by $x(t) = e^t(x_0 + 1) - 1$ while the emission zone of the whole initial interval $[-\varepsilon, \varepsilon]$ is bounded from below by the curve $x(t) = 1 - e^t(1 + \varepsilon)$ and from above by $x(t) = e^t(1 + \varepsilon) - 1$.

The *first* player's problem.

From the fact that the target is $x_1 = 3$ we fix a positive control $u(\cdot)$, i.e. $u(t) \geqq 0$ for all t. Then the "attainable set"

$$X_u(\bar{t}, K) = \left[e^{\bar{t}} \int_0^{\bar{t}} e^{-s} u(s) \, ds - \varepsilon e^{\bar{t}}, \; e^{\bar{t}} \int_0^{\bar{t}} e^{-s} u(s) \, ds + \varepsilon e^{\bar{t}}\right].$$

The worst for the Engineer is $x_0 = \varepsilon$. Then the guaranteed time $T(u(\cdot))$ is the solution $\bar{t}$ of the equation

$$e^{\bar{t}} \int_0^{\bar{t}} e^{-s} u(s) \, ds + \varepsilon e^{\bar{t}} = 3 - \varepsilon . \tag{4}$$

To minimize this time we must select $\bar{u}(t) \equiv 1$ and from (4) we obtain that the minimal value of the guaranteed time is

$$\bar{t}_{\text{opt}} = \ln \frac{4 - \varepsilon}{1 + \varepsilon}.$$

The *second* player's problem.

If the Nature fixed an $x_0 \in K$ then the corresponding attainable set at time t is

$$\mathscr{K}(t, x_0, \mathscr{U}) = \left[e^t(x_0 - 1) + 1, \; e^t(x_0 + 1) - 1\right].$$

The guaranteed value of the time for Nature we compute if the Engineer selects $u(t) \equiv 1$ and the corresponding time $S(x_0)$ is the solution $\hat{t}$ of the equation

$$e^{\hat{t}}(x_0 + 1) - 1 = 3 - \varepsilon ,$$

i.e.

$$S(x_0) = \ln \frac{4 - \varepsilon}{1 + x_0}.$$

The maximal value of this guaranteed time we obtain if the Nature selects $x_0 = -\varepsilon$ and the corresponding optimal time is

$$\hat{t}_{\text{opt}} = \ln \frac{4 - \varepsilon}{1 - \varepsilon}.$$

It is shown that $\hat{t}_{\text{opt}} > \bar{t}_{\text{opt}}$.

6. MIXED EXTENSION OF THE GAME Γ

From the last example it follows that, in general, the game Γ has no saddle point in pure strategies. Hence let us consider the mixed extension of this game. From non-antagonicity it follows that the unique mixed strategy $\nu(\cdot)$ of the Nature is the uniform probability distribution on the set K. (If $\nu_1(\cdot)$ is another but known probability distribution of error of the state observation for system (1), then $v_1(\cdot)$ is the unique mixed strategy of Nature.)

The mixed strategies of the Engineer are all probability measures on the set $\mathscr{U}$. Let us denote the set of these measures by $\mathscr{M}$ and let $\mu(\cdot) \in \mathscr{M}$ be an arbitrary measure. Let $m(s)$ denotes the measure generated by μ on the Borel subsets of the parallelepiped $V = \bigtimes_{j=1}^{m} [-c_j, c_j]$ by the following formula:

$$\forall_{t_1 > t_0} \int_{t_0}^{t_1} \mathrm{d}t \int_V \mathrm{d}m(t) = \int_{\mathscr{U}|[t_0,t_1]} \mathrm{d}\mu(\cdot)$$

where $\mathrm{d}t$ is the usual Lebesgue measure on $[t_0, T]$ and $\mathscr{U}|_{[t_0,t_1]}$ denotes the set of restrictions of all admissible controls to the interval $[t_0, t_1]$.

Then the dynamics of our system will be described by the following measure-differential equation

$$\dot{x}(t) = A(t)\, x + \int_V B(t)\, u(s)\, \mathrm{d}m(s)\,. \tag{1'}$$

The initial condition for this equation will be now the mean value of the probability measure $v(\cdot)$ (resp. $v_1(\cdot)$):

$$x(t_0) = \int_K x\, \mathrm{d}v(x) \quad (\text{resp. } \int_K x\, \mathrm{d}v_1(x)) \tag{2'}$$

and from the fact that $v(\cdot)$ is the uniform probability distribution it follows that

$$x(t_0) = \bar{x}\,. \tag{2''}$$

The initial-value problem (1'), (2'') may be also written as the following integral equation

$$x(t) = \bar{x} + \int_{t_0}^{t} [A(s)\, x(s) + \int_V B(s)\, u(\tau)\, \mathrm{d}m(\tau)]\, \mathrm{d}s\,. \tag{5}$$

The payoff function for the Engineer is now the first instant t_1 when the trajectory of (5) attains the set M, so he must minimize in $\mu(\cdot) \in \mathscr{M}$ the function

$$t_1 = \int_{\mathscr{U}} t_1(u(\cdot), x)\, \mathrm{d}\mu(\cdot) \tag{6}$$

where the integrand was defined in Section 2.

This is a time-optimal control problem for the system (5) in the class of measures as admissible controls. By using the methods given in [11] we obtain the existence of an optimal measure, necessary conditions of optimality and we obtain also that the minimum time of (6) in the class $\mathscr{M}$ is smaller than that given by the minimax solution.

7. SPECIAL CASE

Finally let us consider one interesting case when the elements $a_{ij}(\cdot)$ of the matrix $A(\cdot)$ are measures (i.e. the Stieltjes measures generated by some right-continuous functions of bounded variation). As it is illustrated below, in this case the problem of minimization of the time under (1)–(3) may have no solution.

Let

$$A(t) = \hat{A}(t) + \sum_{k=1}^{\infty} C_k \delta(t - t_k)$$

where: $\hat{A}(\cdot)$ is the continuous part of the measure $A(\cdot)$, C_k are given $n \times n$ real matrices, the product $C_k \delta(t - t_k)$ is the matrix with elements $c^k_{ij} \delta(t - t_k)$, $\delta(\cdot)$ is the Dirac measure and the sequence $\{t_k\}$ is – by assumption – ordered: $t_0 < t_1 < \ldots < t_n < \ldots$ and such that the unique accumulation point of this sequence may be $+\infty$. Moreover, assume that $\det(E - C_k) \neq 0$ for $k = 1, 2, \ldots$ where E is the unit matrix.

Under these assumptions (see [12–14]) for every $x_0 \in \mathbb{R}^n$ there exists the unique solution $x(\cdot)$ of equation (1) with the initial condition $x(t_0) = x_0$ which is a right-continuous, locally bounded variation function. At every instant t_k this solution has a jump equal to $\varepsilon_k = (E - C_k)^{-1} C_k x(t_k -)$. So, it is possible that the problem (1)–(3), min t_1, has no solution, what we illustrate by the following one-dimensional example.

Example. Let us consider the following one-dimensional system

$$\dot{x} = 0{\cdot}8\,\delta(t - 1)\,x + u\,, \quad \bar{x} = 1\,, \quad x_1 = 3 \quad \text{and let} \quad \varepsilon = 0{\cdot}1\,.$$

The admissible controls are all integrable functions defined on $[0, T]$ such that $u(t) \in [0, 1]$ for all t.

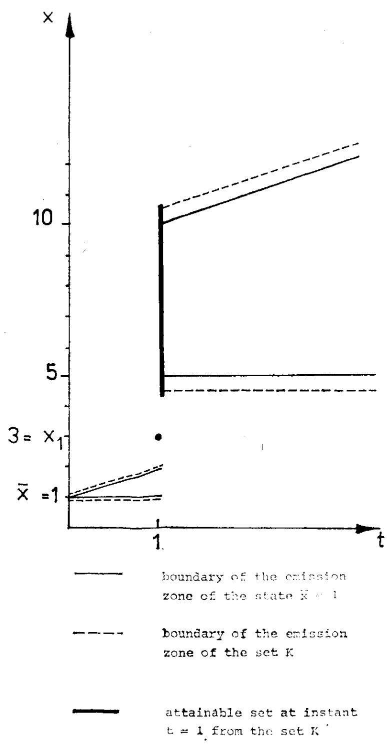

Fig. 1.

Then the emission zone of the point $\bar{x}$ is bounded from below (respectively from above) by the graph of the function

$$x(t) = \begin{cases} 1 & \text{for} \quad t \in [0, 1) \\ 5 & \text{for} \quad t \in [1, \infty) \end{cases} \quad \left(\text{resp. } x(t) = \begin{cases} t + 1 & \text{for} \quad t \in [0, 1\ ) \\ t + 9 & \text{for} \quad t \in [1, \infty) \end{cases}\right)$$

while the emission zone of the initial set $K = \{x: |x - 1| \leqq 0{\cdot}1\}$ is bounded by the curves:

$$x(t) = \begin{cases} 0{\cdot}9 & \text{for} \quad t \in [0, 1) \\ 4{\cdot}5 & \text{for} \quad t \in [1, \infty) \end{cases} \quad \left(\text{resp. } x(t) = \begin{cases} t + 1{\cdot}1 & \text{for} \quad t \in [0, 1\ ) \\ t + 9{\cdot}5 & \text{for} \quad t \in [1, \infty) \end{cases}\right)$$

from below and from above respectively (see Fig. 1 below). It is easy to see that Hypothesis H is not fulfilled and that the problem have no solution.

In the original problem such a situation was impossible because under the assumptions given in Section 2 all solutions of the equation (1) are continuous functions while in the present case the attainable set varies discontinuously in time in Hausdorff metric in the neighbourhood of every point t_k (see [13]).

8. CONCLUSIONS

In this paper the time-optimal control problem for non-autonomous linear system with fixed ends was studied. By assumption, the state of the system is known with a given accuracy $\varepsilon > 0$ only, but not exactly. Therefore the original problem was formulated as a non-antagonistic, two-person game Γ against the Nature in which the sets of pure strategies are: the set of all admissible controls for the time-optimal problem for the first player and the closed ε-neighbourhood of the initial state for the Nature.

The existence of solutions of the corresponding minimax and maximin problems was proved. One example illustrated the usual fact that the lower value of the game Γ is strictly smaller than the upper one is presented.

Next the mixed extension of the game Γ was studied. By non-antagonicity it reduces to the usual time-optimal control problem for some system described by an integral equation with probability measures as admissible controls.

At the end we consider the original problem in the case when the elements of the matrix $A(\cdot)$ of the state equation are measures, i.e. the solutions of this equations are piecewise-continuous functions. As it is illustrated by an example, in this case the problem may have no solution.

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