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OPTIMAL CONTROL CHARACTERISTICS OF A QUEUEING SYSTEM WITH BATCH SERVICES

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A batch service queueing system with a finite and infinite service capacity and Poisson input is considered. Costs are charged for serving the customers and for holding them in the system. Two variants of this model are examined. In variant 1, the waiting cost is charged for all the customers in the system and in variant 2, the waiting cost is charged only for the customers waiting in the queue. Although it is well known that the optimal policy for this system is a control limit policy, the explicit results concerning the optimal control limits, α-discounted costs and the long-run average costs per unit time are known only in some special cases. The objective is to find the explicit formulae for these costs under any i-policy, (serve if and only if there are at least i customers waiting in the queue), examine their properties and derive inequalities for finding the optimal control limits in the general case.

1. INTRODUCTION

We consider a batch service queueing system with Poisson input, service times are independent random variables with the same distribution $B(\cdot)$ independent of the batch size. The cost of serving i customers is $K + ci$ where K and c are any nonnegative constants. This cost is charged at the beginning of a service. The waiting cost of i customers per unit time is $h(i)$ where $h(\cdot)$ is any nonnegative real function. The system is reviewed at the times when either a service has just been completed or the server is free and a customer arrives. At each of the review times one of the following actions is taken: (a) no customers are served, or (b) a batch consisting of all or a portion of the waiting customers is served and the next service can be initiated only when the server becomes available, i.e., after the completion of the previous service. Thus each batch size and its time of service are subject to control and the number of customers served in a batch cannot exceed service capacity $Q \leqq +\infty$.

A wide range of applicability and relative simplicity of the foregoing model attracted attention of many researchers (see e.g. [1], [3], [4], [6], [8] p. 164 and

[9]). The serving of people by elevators, shuttles and other mass transportation systems, charter airline flights, the accumulation of freight cars for a given destination in a marshalling yard until there are enough to constitute a train, transhipment of mail and military supplies, the processing of computer programs and the deterioration models are some of the practical applications (see the references for other application areas). We examine two variants of this model. In variant 1, the waiting cost is charged for all the customers in the system and in variant 2, the waiting cost is charged only for the customers waiting in the queue. Variant 2 is useful in some transportation applications like the serving of people by shuttles and other mass transportation systems where it is natural to charge the waiting cost only for the customers waiting in the queue (cf. also [2] and [9]).

Deb and Serfozo [1] proved that the optimal control policy for variant 1 is a control limit policy, i.e., if i denotes the number of customers waiting in the queue then service begings if and only if the server is free and $i \geqq i^*$ where i^* is some control limit. The optimal batch size is $\min\{i, Q\}$. It can be expected that the optimal policy for variant 2 is again a control limit policy. This result for the discounted cost case is stated in Theorem 1 and for the average cost case in Theorem 2 of this paper.

Although the form of the optimal control policy is well known, the computational results concerning this model were obtained only in some special cases. Deb and Serfozo [1] were able to find the explicit results only in case of the linear waiting cost and exponential service times. Their computational method of α-discounted cost cannot be used in the general case. In the average cost case, they found the most general case intractable (see [1], p. 356). For the system with zero service times, Ross [7] derived the explicit expression for α-discounted cost in case the initial number of waiting customers is zero. Finally, for variant 2 and an infinite capacity server, the average cost case was treated by Weiss [9] for the linear waiting cost and by this author [4] for the general cost structure.

The aim of this paper is to find in the general case the explicit formulae for α-discounted cost and for the long-run average cost per unit time under any i-policy, (serve if an only if there are at least i customers in the system), examine the properties of these costs and find the optimal control limits for both variants. In Section 2, we give some preliminaries. Results for the discounted cost case are presented in Section 3 and for the average cost case in Section 4. It is shown that in the class of i-policies both costs are unimodal functions in i and the optimal control limits can be computed easily from the derived inequalities.

2. PRELIMINARIES

Let the input process $\{X_t,\ t \geqq 0\}$ be a Poisson process with intensity $\lambda > 0$, N be the set of all nonnegative integers and K, c, $h(\cdot)$, $B(\cdot)$, Q as in Section 1. Denote

$$a = \lambda/(\alpha + \lambda)\,, \quad b = \int_0^{+\infty} t \,\mathrm{d}B(t)\,, \quad b_\alpha = \int_0^{+\infty} \mathrm{e}^{-\alpha t}\,\mathrm{d}B(t)\,,$$

$$p_n = \int_0^{+\infty} \frac{[(\alpha + \lambda)\,t]^n}{n!}\,\mathrm{e}^{-(\alpha+\lambda)t}\,\mathrm{d}B(t)\,, \quad q_n(t) = \frac{(\lambda t)^n}{n!}\,\mathrm{e}^{-\lambda t}\,,$$

$$H(n) = h(n)/(\alpha + \lambda)\,, \quad x(n) = \sum_{i=1}^{+\infty} p_i \sum_{j=0}^{i-1} a^j H(n + j)$$

for all $n \in N$. Suppose that $h(\cdot)$ satisfies for variant 1

$$h(n + 1) - h(n) > \alpha b_\alpha^{-1}(c + \gamma) \tag{2.1}$$

and for variant 2

$$h(n + 1) - h(n) > \alpha(c + \gamma) \tag{2.2}$$

for all $n \in N$, $\alpha > 0$, γ is a positive real number if $Q = +\infty$ and $\gamma = K/Q$ if $Q < +\infty$.

For variant 1, the dynamic programming equation for the optimal α-discounted cost is of the form

$$V_{(\alpha,1)}(n) = \min\left\{\mathsf{E}\left[\int_0^T \mathrm{e}^{-\alpha t}\,h(n)\,\mathrm{d}t + \mathrm{e}^{-\alpha T}\,V_{(\alpha,1)}(n + 1)\right],\right. \tag{2.3}$$

$$\left.\min_{0 \leqq m \leqq n \wedge Q} \mathsf{E}\left[K + cm + \int_0^B \mathrm{e}^{-\alpha t}\,h(n + X_t)\,\mathrm{d}t + V_{(\alpha,1)}(n + X_B - m)\,\mathrm{e}^{-\alpha B}\right]\right\}$$

for all $n \in N$, $n \wedge Q = \min\{n, Q\}$. The first set of brackets contains the cost of not serving and holding the n waiting customers for a time T until the next arrival. The second set of brackets contains the cost of serving m customers and holding the customers in the system during a service time B after which another action is taken with the system in state $n + X_B - m$. It was proved in [1] that under assumption (2.1) the expected value of the cost function in the second set of brackets in (2.3) is strictly decreasing in m for a fixed n. Hence, after some computations, (2.3) can be written as

$$V_{(\alpha,1)}(n) = \min\{H(n) + a\,V_{(\alpha,1)}(n + 1),\ K + c(n \wedge Q) + \tag{2.4}$$

$$+\ x(n) + \sum_{k=0}^{+\infty} a^k p_k\,V_{(\alpha,1)}(n - n \wedge Q + k)\}$$

for all $n \in N$. It is easy to see that for variant 2 under assumption (2.2)

$$V_{(\alpha,2)}(n) = \min\{H(n) + a\,V_{(\alpha,2)}(n + 1),\ K + c(n \wedge Q) + \tag{2.5}$$

$$+\ x(n - n \wedge Q) + \sum_{k=0}^{+\infty} a^k p_k\,V_{(\alpha,2)}(n - n \wedge Q + k)\}$$

for all $n \in N$. In case both terms in brackets are equal we write $V_\alpha(n) = H(n) + a\, V_\alpha(n + 1)$ for both variants. Similarly to Proposition 3.4 and Lemma 3.6 in [1] one can prove the following lemmas.

Lemma 1. Let (2.2) hold. Then for all $n \in N$

$$V_{(\alpha,2)}(n + 1) - V_{(\alpha,2)}(n) \geqq c\,. \tag{2.6}$$

Lemma 2. Let $Q < +\infty$ and (2.2) hold. Then for all $n \geqq Q$

$$V_{(\alpha,2)}(n) = K + cQ + x(n - Q) + \sum_{k=0}^{+\infty} a^k p_k\, V_{(\alpha,2)}(n - Q + k)\,. \tag{2.7}$$

The structure of optimal policies for the discounted cost case is given by the following theorem.

Theorem 1. Let (2.2) hold. Then the optimal discounted cost policy for variant 2 is a control limit policy and the optimal control limit $i_2^*(\alpha)$ is given by

$$i_2^*(\alpha) = \min \{n \in N\colon K + cn + x(0) + \sum_{k=0}^{+\infty} a^k p_k\, V_{(\alpha,2)}(k) < \\ < H(n) + a\, V_{(\alpha,2)}(n + 1)\} \tag{2.8}$$

for any $\alpha > 0$.

Proof. The proof of this theorem is based on Lemma 1 and Lemma 2 and is the same as the proof of Theorem 3.1 in [1]. □

If (2.1) holds then the optimal control limit for variant 1 is given by (cf. [1])

$$i_1^*(\alpha) = \min \{n \in N\colon K + cn + x(n) + \sum_{k=0}^{+\infty} a^k p_k\, V_{(\alpha,1)}(k) < \\ < H(n) + a\, V_{(\alpha,1)}(n + 1)\}\,. \tag{2.9}$$

The structure of the optimal average cost policies is given for both variants by Theorem 5.3 in [1] which we state here as Theorem 2.

Theorem 2. Let $\lambda b < Q$ and for some $\delta > 0$

$$h(n + 1) - h(n) > \delta \quad \text{for all} \quad n \in N\,.$$

Then the optimal average cost policies for both variants are control limit policies. Both optimal control limits are finite and, in case $Q < +\infty$, do not exceed Q.

In the next section, we examine the discounted case.

3. OPTIMAL CONTROL LIMITS IN THE DISCOUNTED COST CASE

Throughout this section we suppose that service capacity $Q = +\infty$ and $h(\cdot)$ satisfies condition (2.1) for variant 1 and condition (2.2) for variant 2. We shall find explicit formulae for α-discounted costs $V_{(\alpha,1)}(i, n)$ and $V_{(\alpha,2)}(i, n)$ for any i-policy and any initial state $n \in N$. Denote $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$.

Consider variant 1. Let $\tilde{\tau}$ be the time of the beginning of the first service and $\{\tau_k, k \geqq 1\}$ the length of the kth period, i.e. the length of time between the beginning of the kth and the $(k+1)$st service. Further let $\{B_k, k \geqq 1\}$ denote service times and $\{X_s, s \geqq 0\}$ be a Poisson process with intensity $\lambda > 0$. Let $\{X'_s, s \geqq 0\}$ be another Poisson process with the same intensity λ, independent of $\{X_s, s \geqq 0\}$. Define

$$Z_j = \exp(-\alpha\tau_j)(K + cM_{\tau_j}) \tag{3.1}$$

$$Y_j = \exp(-\alpha\tau_j)\left\{\int_0^{B_{j+1}} \exp(-\alpha s)\, h(M_{\tau_j} + X'_s)\,\mathrm{d}s + \right. \tag{3.2}$$

$$\left. + \exp(-\alpha B_{j+1}) \int_0^{\tau_{j+1} - B_{j+1}} \exp(-\alpha s)\, h(X'_{B_{j+1}} + X_s)\,\mathrm{d}s\right\}$$

for $j \geqq 1$ where M_t is a Poisson random variable with parameter λt independent of $\{X_s, s \geqq 0\}$ and $\{X'_s, s \geqq 0\}$. The random variable $Z_j/\exp(-\alpha\tau_j)$ denotes the service cost incurred in the $(j+1)$st period and $Y_j/\exp(-\alpha\tau_j)$ is the discounted waiting cost in that period. Note that under any i-policy, random variables $\{\tau_k, k \geqq 1\}$ have the same distribution and this holds also for $\{Z_k, k \geqq 1\}$ and $\{Y_k, k \geqq 1\}$. Put for any $n \in N$, $i \in N$

$$U(i, n) = \int_0^{\tilde{\tau}} \mathrm{e}^{-\alpha s}\, h(n + X_s)\,\mathrm{d}s + \mathrm{e}^{-\alpha\tilde{\tau}}(K + c(i \vee n)) + \tag{3.3}$$

$$+ \mathrm{e}^{-\alpha\tilde{\tau}}\left\{\int_0^{B_1} \mathrm{e}^{-\alpha s}\, h(i \vee n + X'_s)\,\mathrm{d}s + \exp(-\alpha B_1)\,\cdot\right.$$

$$\left.\cdot \int_0^{\tau_1 - B_1} \mathrm{e}^{-\alpha s}\, h(X'_{B_1} + X_s)\,\mathrm{d}s\right\}.$$

Obviously, $U(i, n)$ is the total discounted cost under i-policy incurred in the time interval $\langle 0, \tilde{\tau} + \tau_1)$ assuming that the initial state was n.

Let $\mathsf{E}_{(i,n)}$ denote the expectation under i-policy for any initial state $n \in N$. Put

$$u(i, n) = \mathsf{E}_{(i,n)}\, U(i, n), \quad v(i, n) = \mathsf{E}_{(i,n)}\, \mathrm{e}^{-\alpha\tilde{\tau}},$$

$$v(i) = \mathsf{E}_{(i,n)} \exp(-\alpha\tau_j), \quad z(i) = \mathsf{E}_{(i,n)} Z_j,\ y(i) = \mathsf{E}_{(i,n)}\, Y_j$$

for any $i \in N$, $n \in N$, $j \geqq 1$. Since under any i-policy $\{\tau_j, B_j, Z_j, Y_j, j \geqq 1\}$ are independent of $\tilde{\tau}$, $\{Y_j, Z_j\}$ are independent of $\{\tau_k, k < j\}$ and from the independence of $\{\tau_k, k \geqq 1\}$ we get

$$(3.4) \qquad V_{(\alpha,1)}(i, n) = u(i, n) + v(i, n)\,(z(i) + y(i)) \sum_{k=0}^{+\infty} [v(i)]^k =$$

$$= u(i, n) + (a^{i-n} \wedge 1)\,(z(i) + y(i))/(1 - v(i))$$

for all $i \in N$, $n \in N$.

In the following lemmas, we find the form of $u(i, n)$ $z(i)$, $y(i)$ and $v(i)$ for an arbitrary function $h(\cdot)$ which satisfies (2.1) and for any service time distribution $B(\cdot)$. We use the same notation as in Section 2. Denote for $i \in N$, $n \in N$

$$(3.5) \qquad x_{i,n} = \sum_{k=0}^{i-1} a^k H(n + k)$$

$$(3.6) \qquad D_n = \int_0^B e^{-\alpha u} h(n + X'_u)\,du + e^{-\alpha B} \int_0^{\tau - B} e^{-\alpha u} h(X'_B + X_u)\,du$$

where τ is the length of time between the beginning of two successive services and B is the appropriate service time. The following lemma holds.

Lemma 3. For any initial state $n \in N$ and any i-policy

$$(3.7) \qquad u(i, n) = x_{i-n,n} + (a^{i-n} \wedge 1)\,\{x(i \vee n) + \sum_{k=0}^{i-1} a^k p_k x_{i-k,k} + K + c(i \vee n)\}$$

where we put $\sum_{j=0}^{-k} = 0$ for any $k \geqq 1$.

Proof. Let $\{T_k, k \geqq 1\}$ be the arrival times in Poisson process $\{X_s, s \geqq 0\}$, $T_0 = 0$, $T_{-k} = 0$ for $k \geqq 1$. We have

$$(3.8) \qquad \mathsf{E}_{(i,n)} \int_0^{\tilde{\tau}} e^{-\alpha u} h(n + X_u)\,du = \mathsf{E} \int_0^{T_{i-n}} e^{-\alpha u} h(n + X_u)\,du =$$

$$= \sum_{k=0}^{i-n-1} h(n + k)\, a^k (1 - a)/\alpha = x_{i-n,n}\,.$$

Further, conditioning on $B = t$, $X'_t = k$ yields for any $m \in N$

$$(3.9) \qquad \mathsf{E}_{(i,n)} D_m = \int_0^{+\infty} \left\{ \sum_{k=0}^{i-1} \left[\mathsf{E}\left(\int_0^t e^{-\alpha u} h(m + X'_u)\,du \mid X'_t = k \right) + \right.\right.$$

$$\left. + \mathsf{E}\left(e^{-\alpha t} \int_0^{T_{i-k}} e^{-\alpha u} h(k + X_u)\,du \right) \right] q_k(t) +$$

$$\left. + \sum_{k=i}^{+\infty} \mathsf{E}\left(\int_0^t e^{-\alpha u} h(m + X'_u)\,du \mid X'_t = k \right) q_k(t) \right\} dB(t) =$$

$$= \int_0^{+\infty} \mathsf{E}\left(\int_0^t e^{-\alpha u} h(m + X'_u)\,du \right) dB(t) + \sum_{k=0}^{i-1} \int_0^{+\infty} e^{-\alpha t} q_k(t)\,dB(t)\, x_{i-k,k} =$$

$$= x(m) + \sum_{k=0}^{i-1} a^k p_k x_{i-k,k}$$

and (3.7) follows from (3.8), (3.9) and from

$$\mathsf{E}_{(i,n)}\, \mathrm{e}^{-\alpha\bar{\tau}} = a^{i-n} \wedge 1\,. \tag{3.10}$$ □

Lemma 4. For any i-policy

$$z(i) = a^i(K + ci) \sum_{k=0}^{i-1} p_k + \sum_{k=i}^{+\infty} (K + ck)\, a^k p_k \tag{3.11}$$

$$v(i) = a^i \sum_{k=0}^{i-1} p_k + \sum_{k=i}^{+\infty} a^k p_k\,. \tag{3.12}$$

Proof. For any $j \geqq 1$ and any i-policy

$$\mathsf{E}_{(i,n)} Z_j = \int_0^{+\infty} \{\sum_{k=0}^{i-1} \mathrm{e}^{-\alpha t}\, \mathsf{E}(\exp(-\alpha T_{i-k}))\,(K + ci) + $$

$$+ \sum_{k=i}^{+\infty} (K + ck)\, \mathrm{e}^{-\alpha t}\}\; q_k(t)\, \mathrm{d}B(t) =$$

$$= \int_0^{+\infty} \mathrm{e}^{-\alpha t}\, \{\sum_{k=0}^{i-1} (K + ci)\, a^{i-k} + \sum_{k=i}^{+\infty} (K + ck)\}\; q_k(t)\, \mathrm{d}B(t) =$$

$$= (K + ci)\, a^i \sum_{k=0}^{i-1} p_k + \sum_{k=i}^{+\infty} (K + ck)\, a^k p_k$$

and (3.12) can be proved similarly. □

Lemma 5. For any $i \in N$

$$y(i) = a^i \sum_{k=0}^{i-1} p_k\, x(i) + \sum_{k=i}^{+\infty} a^k p_k\, x(k) + v(i) \sum_{k=0}^{i-1} a^k p_k x_{i-k,k}\,. \tag{3.13}$$

Proof. Similarly to the proof of Lemma 3, we get

$$y(i) = \sum_{k=0}^{i-1} a^{i-k} \int_0^{+\infty} \mathrm{e}^{-\alpha t}\, q_k(t)\, \mathrm{d}B(t)\, \mathsf{E}_{(i,n)}\, D_i + \tag{3.14}$$

$$+ \sum_{k=i}^{+\infty} \mathsf{E}_{(i,n)} D_k \int_0^{+\infty} \mathrm{e}^{-\alpha t}\, q_k(t)\, \mathrm{d}B(t) = \mathsf{E}_{(i,n)} D_i a^i \sum_{k=0}^{i-1} p_k + \sum_{k=i}^{+\infty} a^k p_k\, \mathsf{E}_{(i,n)} D_k$$

where D_k is defined by (3.6) and (3.13) follows from (3.9) and (3.14). □

Now we can prove the following theorem.

Theorem 3. Consider variant 1. For any i-policy and any initial state $n \in N$, α-discounted cost is given by

$$V_{(\alpha,1)}(i, n) = x_{i-n,n} + (a^{i-n} \wedge 1)(K + c(i \vee n) + x(i \vee n) + \tag{3.15}$$

$$+ (1 - v(i))^{-1} \{\sum_{k=0}^{i-1} p_k[a^i(K + ci + x(i)) + a^k x_{i-k,k}] +$$

$$+ \sum_{k=i}^{+\infty} a^k p_k[K + ck + x(k)]\})\,.$$

Proof. From Lemma 3 and Lemma 5, we have

$$(3.16)\qquad u(i,n) + (a^{i-n} \wedge 1)\, y(i)(1 - v(i))^{-1} = x_{i-n,n} + (a^{i-n} \wedge 1)\,. \\ .\,(x(i \vee n) + K + c(i \vee n) + (1 - v(i))^{-1}\,(\sum_{k=0}^{i-1} p_k(a^i x(i) + a^k x_{i-k,k}) + \\ + \sum_{k=i}^{+\infty} p_k a^k x(k)))$$

and (3.15) follows from (3.4), (3.16) and Lemma 4. □

Further, we examine variant 2. Put for $j \geqq 1$

$$(3.17)\qquad W_j = \int_0^{\tau_j} \exp(-\alpha s)\, h(X_s)\, ds\,.$$

Denote for $i \in N$

$$w(i) = \mathsf{E}_{(i,n)} W_j\,.$$

The following lemma holds.

Lemma 6. For any i-policy

$$(3.18)\qquad w(i) = x(0) + \sum_{k=0}^{i-1} a^k p_k x_{i-k,k}\,.$$

Proof. It follows immediately from (3.9). □

Similarly to (3.4), we can write

$$(3.19)\qquad V_{(\alpha,2)}(i,n) = \mathsf{E}_{(i,n)} \int_0^{\tilde{\tau}} e^{-\alpha u} h(n + X_u)\, du + \\ + (a^{i-n} \wedge 1)\{K + c(i \vee n) + (1 - v(i))^{-1}(z(i) + w(i))\}\,.$$

From (3.8), (3.19), Lemma 4 and Lemma 6, we have the following theorem.

Theorem 4. Consider variant 2. For any i-policy and any initial state $n \in N$

$$(3.20)\qquad V_{(\alpha,2)}(i,n) = x_{i-n,n} + (a^{i-n} \wedge 1)\{K + c(i \vee n) + (1 - v(i))^{-1}\,. \\ .\,[x(0) + \sum_{k=0}^{i-1} a^k p_k x_{i-k,k} + a^i(K + ci) \sum_{k=0}^{i-1} p_k + \sum_{k=i}^{+\infty} (K + ck)\, a^k p_k]\}\,.$$

Note that for zero service times $p_0 = 1$, $p_k = 0$ for $k \geqq 1$, $V_{(\alpha,1)}(i,n) = V_{(\alpha,2)}(i,n)$ and their common value is given by

$$(3.21)\qquad V_{(\alpha,0)}(i,n) = x_{i-n,n} + (a^{i-n} \wedge 1)\{K + c(i \vee n) + \\ + (1 - a^i)^{-1}[x_{i,0} + a^i(K + ci)]\}\,.$$

Formula (3.21) was obtained for $n = 0$, $c = 0$ in [7].

Now we examine the properties of α-discounted costs for both variants. It follows from (2.8) and (2.9) that the optimal control limits do not depend on the initial state so it suffices to examine the properties of $V_{(\alpha,1)}(i, 0)$ and $V_{(\alpha,2)}(i, 0)$. We show that these costs are unimodal in i. Denote for $i \in N$

$$G(i) = \sum_{k=0}^{+\infty} a^k p_k H(i + k), \quad d(i) = 1 - \sum_{k=i}^{+\infty} a^k p_k,$$

$$t_1(i) = K + ci + \sum_{k=0}^{i-1} p_k(a^i x(i) + a^k x_{i-k,k}) + \sum_{k=i}^{+\infty} a^k p_k(x(k) + c(k - i)).$$

From (3.15), we get

$$V_{(\alpha,1)}(i, 0) = x_{i,0} + a^i x(i) + a^i t_1(i)(1 - v(i))^{-1} \tag{3.22}$$

and

$$V_{(\alpha,1)}(i + 1, 0) - V_{(\alpha,1)}(i, 0) = a^i[(1 - v(i))(1 - v(i + 1))]^{-1} . \tag{3.23}$$

$$. \{G(i)(1 - v(i + 1))(1 - v(i)) + a(1 - v(i)) t_1(i + 1) - (1 - v(i + 1)) t_1(i)\} =$$

$$= a^i d(i + 1)[(1 - v(i))(1 - v(i + 1))]^{-1} .$$

$$. [(1 - v(i))(G(i) + ac) - (1 - a) t_1(i)] .$$

We prove the following lemma.

Lemma 7. Let (2.1) hold. Define for $i \in N$

$$g_1(i) = (1 - v(i))(G(i) + ac) - (1 - a) t_1(i) . \tag{3.24}$$

The function $g_1(i)$ is increasing in i.

Proof. For any $i \in N$

$$t_1(i + 1) = t_1(i) + c\, d(i + 1) + a^i G(i) \sum_{k=0}^{i} p_k . \tag{3.25}$$

From (3.24) and (3.25), we get

$$g_1(i + 1) - g_1(i) = (1 - v(i + 1)) G(i + 1) - (1 - a) G(i) a^i \sum_{k=0}^{i} p_k -$$

$$- (1 - v(i)) G(i) + ac(v(i) - v(i + 1)) - c(1 - a) d(i + 1) =$$

$$= (1 - v(i + 1))[G(i + 1) - G(i) - c(1 - a)] .$$

From the definition of $G(i)$ and from (2.1), we have

$$G(i + 1) - G(i) = \sum_{k=0}^{+\infty} a^k p_k(H(i + 1 + k) - H(i + k)) \geqq$$

$$\geqq \sum_{k=0}^{+\infty} a^k p_k(1 - a)(c + \gamma) b_\alpha^{-1} = (1 - a)(c + \gamma) .$$

From this, we get

$$g_1(i + 1) - g_1(i) \geqq (1 - v(i + 1))[(1 - a)(c + \gamma) - (1 - a) c] =$$

$$= (1 - v(i + 1)) \gamma(1 - a) > 0 .$$

□

The optimal control limit is given by the following theorem.

Theorem 5. Consider variant 1 and let (2.1) hold. The optimal control limit is given by

$$i_1^*(\alpha) = \min \{ i \in N : g_1(i) \geqq 0 \} \tag{3.26}$$

where $g_1(i)$ is defined by (3.24).

For variant 2, put

$$t_2(i) = K + ci + c \sum_{k=i}^{+\infty} a^k \, p_k(k - i) + x(0) + \sum_{k=0}^{i-1} a^k p_k x_{i-k,k} \, . \tag{3.27}$$

Then for any i-policy

$$V_{(\alpha,2)}(i, 0) = x_{i,0} + a^i \, t_2(i) \, (1 - v(i))^{-1} \tag{3.28}$$

$$V_{(\alpha,2)}(i + 1, 0) - V_{(\alpha,2)}(i, 0) = a^i \{ H(i) + [(1 - v(i)) (1 - v(i + 1))]^{-1} \, . \tag{3.29}$$
$$. \, [a(1 - v(i)) \, t_2(i + 1) - (1 - v(i + 1)) \, t_2(i)] \} \, .$$

It is easy to see that for any $i \in N$

$$t_2(i + 1) = t_2(i) + c \, d(i + 1) + H(i) \, a^i \sum_{k=0}^{i} p_k \tag{3.30}$$

and

$$a(1 - v(i)) = 1 - v(i + 1) - (1 - a) \, d(i + 1) \, . \tag{3.31}$$

From (3.29)–(3.31), we have

$$V_{(\alpha,2)}(i + 1, 0) - V_{(\alpha,2)}(i, 0) = a^i \, d(i + 1) \, [(1 - v(i)) \, . \tag{3.32}$$
$$. \, (1 - v(i + 1))]^{-1} \, [(1 - v(i)) \, (H(i) + ac) - (1 - a) \, t_2(i)] \, .$$

Define for $i \in N$

$$g_2(i) = (1 - v(i)) \, (H(i) + ac) - (1 - a) \, t_2(i) \, . \tag{3.33}$$

Lemma 8. Let (2.2) hold. The function $g_2(i)$ is increasing in i.

Proof. From (3.30) and (3.33), we have for any $i \in N$

$$g_2(i + 1) - g_2(i) = (1 - v(i + 1)) \, [H(i + 1) - H(i) - c(1 - a)]$$

and from (2.2)

$$H(i + 1) - H(i) \geqq \alpha(c + \gamma)/(\alpha + \lambda) = (1 - a) \, (c + \gamma)$$

which yields

$$g_2(i + 1) - g_2(i) \geqq (1 - v(i + 1)) \, \gamma(1 - a) > 0 \, .$$ □

Theorem 6. Consider variant 2 and let (2.2) hold. The optimal control limit is given by

$$i_2^*(\alpha) = \min \{ i \in N : g_2(i) \geqq 0 \} \tag{3.34}$$

where $g_2(i)$ is defined by (3.33).

Corollary 1. For zero service times, $i_1^*(\alpha) = i_2^*(\alpha)$ and their common value $i^*(\alpha)$ is given by

$$(3.35) \quad i^*(\alpha) = \min\{i \in N: \sum_{k=0}^{i-1} a^k(H(i) - H(k)) \geqq K + ci - c(1 - a^i)\lambda/\alpha\}.$$

Proof. For zero service times, $p_0 = 1$, $p_k = 0$ for $k \geqq 1$ and

$$t_1(i) = t_2(i) = K + ci + \sum_{k=0}^{i-1} a^k H(k), \; G(i) = H(i), \; 1 - v(i) = 1 - a^i.$$

Hence, we have

$$g_1(i) = g_2(i) = \{\sum_{k=0}^{i-1} a^k(H(i) - H(k)) + c(1 - a^i)\lambda/\alpha - K - ci\}(1 - a)$$

so that (3.35) holds. □

In the last section, we examine the average cost case.

4. OPTIMAL CONTROL LIMITS IN THE AVERAGE COST CASE

In this section, we consider both finite and infinite capacity and suppose that the assumptions of Theorem 2 are satisfied, i.e., let

$$(4.1) \quad \lambda b < Q, \quad b = \mathsf{E}(B)$$

$$(4.2) \quad h(n + 1) - h(n) > \delta \quad \text{for some} \quad \delta > 0 \quad \text{and all} \quad n \in N.$$

4.1. The infinite capacity server

We start with the infinite case ($Q = +\infty$) and use an approach similar to [4]. Let τ_j be the length of the jth cycle, i.e., the length of time between the beginning of the jth and the $(j + 1)$st service and let C_j be the cost of this cycle. Note that $\{C_j, j \geqq 2\}$ have the same distribution independent of the initial state. It follows from renewal theory that under any i-policy, the long run average cost per unit time is given by

$$(4.3) \quad R(i) = \mathsf{E}_i C_j / \mathsf{E}_i \tau_j$$

for any $j \geqq 2$. Consider variant 1. The cost of the jth cycle for any $j \geqq 2$ is

$$(4.4) \quad C_j = K + cM_{\tau_{j-1}} + \int_0^{B_j} h(M_{\tau_{j-1}} + X'_s)\,\mathrm{d}s + \int_0^{\tau_j - B_j} h(X'_{B_j} + X_s)\,\mathrm{d}s$$

where we used the same notation as in Section 3. Under any i-policy, the length of the jth cycle is

$$(4.5) \quad \tau_j = \max\{B_j, T_i\}$$

where T_i is the arrival time of the ith customer in the jth cycle. Denote

$$\tilde{q}_k = \int_0^{+\infty} q_k(t)\, \mathrm{d}B(t),\ \tilde{D}_k = \int_0^B h(k + X'_s)\, \mathrm{d}s + \int_0^{\tau - B} h(X'_B + X_s)\, \mathrm{d}s\,,$$

$$\tilde{x}_{m,n} = \sum_{k=0}^{m-1} h(n + k)/\lambda,\ \tilde{x}(m) = \sum_{k=1}^{+\infty} \tilde{q}_k \tilde{x}_{k,m}$$

for all $m \in N$, $n \in N$. The following lemma holds.

Lemma 9. For any i-policy and any $j \geqq 2$

$$\mathsf{E}_i C_j = \sum_{k=0}^{i-1} \tilde{q}_k(K + ci + \tilde{x}(i) + \tilde{x}_{i-k,k}) + \sum_{k=i}^{+\infty} \tilde{q}_k(K + ck + \tilde{x}(k)) \tag{4.6}$$

$$\mathsf{E}_i \tau_j = b + \sum_{k=0}^{i-1} (i - k)\, \tilde{q}_k/\lambda\,. \tag{4.7}$$

Proof. Similarly to the proof of Lemma 3, conditioning on $B_j = t$, $M_t = k$ yields for any $i \in N$, $j \geqq 2$

$$\mathsf{E}_i C_j = \sum_{k=0}^{i-1} \tilde{q}_k(K + ci + \mathsf{E}_i \tilde{D}_i) + \sum_{k=i}^{+\infty} \tilde{q}_k(K + ck + \mathsf{E}_i \tilde{D}_k)\,. \tag{4.8}$$

We have for any $m \in N$

$$\mathsf{E}_i \tilde{D}_m = \mathsf{E}_i \left(\int_0^B h(m + X'_s)\, \mathrm{d}s + \int_0^{\tau - B} h(X'_B + X_s)\, \mathrm{d}s \right) = \tag{4.9}$$

$$= \sum_{k=0}^{i-1} \tilde{q}_k\, \mathsf{E} \int_0^{T_{i-k}} h(k + X_s)\, \mathrm{d}s + \int_0^{+\infty} \mathsf{E} \int_0^t h(m + X'_s)\, \mathrm{d}s\, \mathrm{d}B(t) = \sum_{k=0}^{i-1} \tilde{q}_k \tilde{x}_{i-k,k} + \tilde{x}(m)$$

and (4.6) follows from (4.8) and (4.9). Further, for any $j \geqq 1$

$$\mathsf{E}_i \tau_j = \sum_{k=0}^{i-1} \int_0^{+\infty} (\mathsf{E} T_{i-k} + t)\, q_k(t)\, \mathrm{d}B(t) + \sum_{k=i}^{+\infty} \int_0^{+\infty} t\, q_k(t)\, \mathrm{d}B(t) =$$

$$= \sum_{k=0}^{i-1} \frac{(i-k)}{\lambda} \int_0^{+\infty} q_k(t)\, \mathrm{d}B(t) + \int_0^{+\infty} t\, \mathrm{d}B(t) = \sum_{k=0}^{i-1} (i - k)\, \tilde{q}_k/\lambda + b\,. \qquad \square$$

The long run average cost per unit time is given by the following theorem.

Theorem 7. Consider variant 1. For any i-policy

$$R_1(i) = [\tilde{v}(i)]^{-1} \{K + \sum_{k=0}^{i-1} \tilde{q}_k(\tilde{x}(i) + \tilde{x}_{i-k,k}) + \sum_{k=i}^{+\infty} \tilde{q}_k\, \tilde{x}(k)\} + \lambda c \tag{4.10}$$

where

$$\tilde{v}(i) = b + \sum_{k=0}^{i-1} (i - k)\, \tilde{q}_k/\lambda\,. \tag{4.11}$$

The long run average cost for variant 2 was obtained in [4] and is stated here in the following theorem.

Theorem 8. For variant 2 and any i-policy, the long run average cost per unit time is given by

$$R_2(i) = [\tilde{v}(i)]^{-1} \{K + \tilde{x}(0) + \sum_{k=0}^{i-1} \tilde{q}_k \tilde{x}_{i-k,k}\} + \lambda c\,. \tag{4.12}$$

Further, we show that if (4.2) holds then both $R_1(i)$ and $R_2(i)$ attain their minima only once. Denote for $i \in N$

$$\beta_i = \tilde{v}(i) \sum_{j=0}^{+\infty} \tilde{q}_j\, h(i+j) - K - \sum_{k=0}^{i-1} \tilde{q}_k(\tilde{x}(i) + \tilde{x}_{i-k,k}) - \sum_{k=i}^{+\infty} \tilde{q}_k\, \tilde{x}(k)\,. \tag{4.13}$$

We have

$$\tilde{v}(i+1) = \tilde{v}(i) + \sum_{k=0}^{i} \tilde{q}_k/\lambda \tag{4.14}$$

$$\beta_{i+1} = \beta_i + \tilde{v}(i+1) \sum_{k=0}^{+\infty} \tilde{q}_k(h(i+k+1) - h(i+k))\,. \tag{4.15}$$

From (4.10), (4.13)–(4.15), we get

$$R_1(i+1) - R_1(i) = [\tilde{v}(i)\, \tilde{v}(i+1)]^{-1}\, \lambda^{-1} \sum_{k=0}^{i} \tilde{q}_k \beta_i\,. \tag{4.16}$$

The assumption (4.2) and (4.15) yield that β_i is increasing in i and from (4.16) we can conclude that $R_1(i)$ attains its minimum only once.

Theorem 9. Let (4.2) hold. Then both $-R_1(i)$ and $-R_2(i)$ are unimodal functions in i and the corresponding optimal control limits are given by

$$i_1^* = \min \{i \in N: \lambda c + \sum_{k=0}^{+\infty} \tilde{q}_k\, h(i+k) \geqq R_1(i)\} \tag{4.17}$$

$$i_2^* = \min \{i \in N: \lambda c + h(i) \geqq R_2(i)\} \tag{4.18}$$

where $R_1(i)$ and $R_2(i)$ are defined by (4.10) and (4.12) respectively.

The second part of this theorem concerning variant 2 was proved in [4].

Corollary 2. For zero service times, $R_1(i) = R_2(i)$ and their common value $R(i)$ is given by

$$R(i) = \lambda i^{-1}\{K + \sum_{k=0}^{i-1} h(k)/\lambda\} + \lambda c\,. \tag{4.19}$$

In this case, the optimal control limit is

$$i^* = \min \{i \in N: \sum_{k=0}^{i-1} (h(i) - h(k)) \geqq \lambda K\}\,. \tag{4.20}$$

4.2. The finite capacity server

It remains to examine the finite capacity case ($Q < +\infty$). We can use the approach in [1]. Consider any i-policy, $i \leqq Q$. Let η_n denote the time at which the nth service is completed, $\eta_0 = 0$ and let X_n be the number of customers waiting in the system at time η_n. If (4.1) holds then $\{X_n\}$ is a positive recurrent Markov chain and the limiting distribution $\{\psi_n, n \in N\}$ is the solution of the following equation (cf. [3], [6])

$$\psi_n = \sum_{j=0}^{Q-1} \psi_j \tilde{q}_n + \sum_{j=Q}^{Q+n} \psi_j \tilde{q}_{n-j+Q}, \quad n \in N. \tag{4.21}$$

The equation (4.21) can be solved e.g. by methods in [5]. Define for $i \leqq Q$, $n \in N$

$$\tilde{u}_1(i, n) = \mathsf{E}_{(i,n)} \left\{ \int_0^{\tilde{\tau}} h(n + X'_s) \, ds + \int_0^{B_1} h(i \vee n + X_s) \, ds \right\} + \tag{4.22}$$
$$+ K + c(i \vee (n \wedge Q)) = \tilde{x}_{i-n,n} + \tilde{x}(i \vee n) + K + c(i \vee (n \wedge Q))$$

$$\tilde{u}_2(i, n) = \mathsf{E}_{(i,n)} \left\{ \int_0^{\tilde{\tau}} h(n + X'_s) \, ds + \int_0^{B_1} h((n - Q) \vee 0 + X_s) \, ds \right\} + \tag{4.23}$$
$$+ K + c(i \vee (n \wedge Q)) = \tilde{x}_{i-n,n} + \tilde{x}((n - Q) \vee 0) + K + c(i \vee (n \wedge Q))$$

$$\tilde{v}(i, n) = \mathsf{E}_{(i,n)}(\tilde{\tau} + B_1) = b + \lambda^{-1}((i - n) \vee 0) \tag{4.24}$$

where we used the same notation as in Sections 3 and 4.1. Then by the strong law of large numbers for semi-Markov processes (cf. [1], [8], pp. 98, 104) the long run average costs under any i-policy, $i \leqq Q$ are given by

$$\tilde{R}_1(i) = \sum_{k=0}^{+\infty} \tilde{u}_1(i, k) \psi_k / \sum_{k=0}^{+\infty} \tilde{v}(i, k) \psi_k \tag{4.25}$$

$$\tilde{R}_2(i) = \sum_{k=0}^{+\infty} \tilde{u}_2(i, k) \psi_k / \sum_{k=0}^{+\infty} \tilde{v}(i, k) \psi_k. \tag{4.26}$$

We again prove that both $-\tilde{R}_1(i)$ and $-\tilde{R}_2(i)$ are unimodal and obtain inequalities for the optimal control limits in a general case. Denote

$$\tilde{u}_j(i) = \sum_{k=0}^{+\infty} \tilde{u}_j(i, k) \psi_k, \quad j = 1, 2, \quad f(i) = \sum_{k=0}^{+\infty} \tilde{v}(i, k) \psi_k,$$
$$\tilde{G}(i) = \sum_{n=0}^{+\infty} \tilde{q}_n h(i + n)/\lambda.$$

After some algebraic manipulations, we get for variant 1

$$\tilde{R}_1(i + 1) - \tilde{R}_1(i) = \tag{4.27}$$
$$= [f(i) f(i + 1)]^{-1} \sum_{k=0}^{i} \psi_k \{(c + \tilde{G}(i)) f(i) - \tilde{u}_1(i)/\lambda\}$$

and for variant 2

$$\tilde{R}_2(i + 1) - \tilde{R}_2(i) = [f(i) f(i + 1)]^{-1} \sum_{k=0}^{i} \psi_k \{(c + h(i)/\lambda) f(i) - \tilde{u}_2(i)/\lambda\}. \tag{4.28}$$

Put

$$f_1(i) = (c + \tilde{G}(i)) f(i) - \tilde{u}_1(i)/\lambda$$
$$f_2(i) = (c + h(i)/\lambda) f(i) - \tilde{u}_2(i)/\lambda.$$

We prove the following lemma.

Lemma 10. Let (4.2) hold. Then $f_1(i)$ and $f_2(i)$ are increasing in i, $i \leqq Q$.

Proof. For any $i < Q$

$$f_1(i+1) - f_1(i) = f(i+1)\,(\tilde{G}(i+1) - \tilde{G}(i))$$

$$f_2(i+1) - f_2(i) = f(i+1)\,(h(i+1) - h(i))/\lambda\,.$$

From (4.2), we get $\tilde{G}(i+1) - \tilde{G}(i) \geqq \delta/\lambda > 0$ and this completes the proof. □

The optimal control limits are given by the following theorem.

Theorem 10. Let (4.1), (4.2) hold and $Q < +\infty$. The optimal control limits for variant 1 and variant 2 are

$$\text{(4.29)} \quad \tilde{i}_1^* = \min\{i < Q: \lambda c + \lambda\,\tilde{G}(i) \geqq \tilde{R}_1(i)\}\,, \quad \text{if} \quad \lambda c + \lambda\,\tilde{G}(Q-1) \geqq \tilde{R}_1(Q-1)$$
$$= Q \quad \text{otherwise}\,,$$

$$\text{(4.30)} \quad \tilde{i}_2^* = \min\{i < Q: \lambda c + h(i) \geqq \tilde{R}_2(i)\}\,, \quad \text{if} \quad \lambda c + h(Q-1) \geqq \tilde{R}_2(Q-1)$$
$$= Q \quad \text{otherwise}$$

where $\tilde{R}_1(i)$ and $\tilde{R}_2(i)$ are given by (4.25) and (4.26) respectively.

Corollary 3. For zero service times $\psi_0 = 1$, $\psi_k = 0$ for $k \geqq 1$, $\tilde{R}_1(i) = \tilde{R}_2(i)$ and their common value $\tilde{R}(i)$ is given by

$$\text{(4.31)} \qquad \tilde{R}(i) = (\lambda i^{-1})\,(K + \sum_{k=0}^{i-1} h(k)/\lambda) + \lambda c \quad \text{for } i \leqq Q\,.$$

The optimal control limit in this case is

(4.32)
$$\tilde{i}^* = \min\{i < Q: \sum_{k=0}^{i-1}(h(i) - h(k)) \geqq \lambda K\}\,, \quad \text{if} \quad \sum_{k=0}^{Q-2}(h(Q-1) - h(k)) \geqq \lambda K$$
$$= Q \quad \text{otherwise}\,.$$

Note that there are some discrepancies in [1] on pages 357 and 358.

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