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# ON STRATIFIABLE FUZZY TOPOLOGICAL SPACES

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*Summary.* The aim of the paper is to extend the notion of stratifiability from the category Top of topological spaces to the category CFT of [Chang] fuzzy topological spaces and to develop the corresponding theory.

*Keywords*: fuzzy topology, stratifiability, stratifiable fuzzy space

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## 0. INTRODUCTION

The notion of a stratifiable *topological* space first appeared (implicitly) in Ceder's paper [4] and later was defined explicitly and studied carefully by Borges [1]. The class of stratifiable topological spaces has many valuable properties. In particular, it is countably multiplicative and hereditary; it is preserved by closed mappings; the adjunction space of two stratifiable spaces is stratifiable itself. The class of stratifiable spaces "does not differ too much" from the class of metrizable spaces and is one of the most important classes of the so called generalized metric spaces (see e.g. [4], [2], [19], [13], [15]). In addition, stratifiable spaces are convenient for the use of tools and methods of homotopy topology (see e.g. [3], [20]). Finally, we should like to emphasize that the definition of a stratifiable topological space does not essentially rely on the notion of a point (unlike the definition of a metric space), which is an important advantage as regards the prospects of extending this concept to the fuzzy case and to study it in fuzzy setting.

The aim of the paper is to extend the notion of stratifiability from the category Top of topological spaces to the category CFT of [Chang] fuzzy topological spaces [5] and to develop the corresponding theory. As is shown in Section 2 this extension is good in Lowen's sense. The obtained theory of stratifiable fuzzy topological spaces is more or less similar to its classical prototype. However, the technique developed here in the proofs of the results differs rather essentially from the one which was applied in the crisp case.

We use here the standard terminology and notation accepted in fuzzy topology (see e.g. [5], [14], [18], [24]). In particular, the term "a fuzzy (topological) space" is always used in Chang's sense (see e.g. [5]).

If $A$ is a fuzzy subset of a set $X$ (i.e. $A \in I^X$, where $I := [0,1]$), then $A^C := 1 - A$ is its complement; if $\mathcal{A}$ is a family of fuzzy subsets of $X$ (i.e. $\mathcal{A} \subset I^X$), then $\mathcal{A}^C := \{A^C : A \in \mathcal{A}\}$. We identify a crisp subset $A$ of $X$ with its characteristic function $A: X \to \mathcal{Z}$ ($:= \{0,1\}$). The intersection $A \wedge U$ where $A \subset X$ and $U \in I^X$ will be often denoted by $U^A$.

The closure of a fuzzy set $M \in I^X$ in a fuzzy topological space $(X, \mathcal{T})$ is denoted by $\overline{M}$. We usually use letters $U$, $V$ and $W$ to denote open fuzzy sets in the corresponding fuzzy space $(X, \mathcal{T})$. (When speaking about fuzzy topological spaces we often omit the word "topological").

## 1. Definition and basic properties

(1.1) **Definition.** A fuzzy topological space $(X, \mathcal{T})$ is called *stratifiable* if to every $U \in \mathcal{T}$ one can assign a sequence of fuzzy sets $U_n \in \mathcal{T}$ (i.e. $U \to (U_n)_{n \in N}$) in such a way that

(1) $$\overline{U_n} \leqslant U \text{ for all } n \in \mathbf{N};$$

(2) $$\bigvee_n U_n = U;$$

(3) $$U \leqslant V \; (V \in \mathcal{T}) \text{ implies } U_n \leqslant V_n \quad \text{for all } n \in N.$$

(1.2) Remark. Since $\overline{A \vee B} = \overline{A} \vee \overline{B}$ for arbitrary $A$, $B \in I^X$, without loss of generality one can assume that in a stratifiable fuzzy space $(X, \mathcal{T})$ the stratification $U \to (U_n)_{n \in \mathbf{N}}$ is increasing, i.e. the following additional condition is fulfilled:

(4) $$U_n \leqslant U_{n+1} \quad \text{for all } n \in \mathbf{N}.$$

(1.3) Examples.

(a) Let $X$ be a set and $\mathcal{T} = I^X$. Then the fuzzy space $(X, \mathcal{T})$ is stratifiable.

(b) Let $X$ be a set and $\mathcal{T} = (0,1]^X \cup \{0,1\}$. Noticing that all fuzzy sets from $(0,1)^X$ are clopen, it is easy to conclude that the fuzzy space $(X, \mathcal{T})$ is stratifiable.

(c) If $(X, T)$ is a stratifiable topological space, then the fuzzy space $(X, \omega T)$. where $\omega$ is Lowen's functor [12], is stratifiable, too (see Section 2 for the details).

It is easy to establish the following.

(1.4) **Proposition.** *Let $(X, \mathcal{T})$ be a stratifiable fuzzy space and $(A, \mathcal{T}^A)$ its subspace (where $A \subset X$). Then $(A, \mathcal{T}^A)$ is stratifiable, too.*

When studiyng the properties of fuzzy stratifiable spaces it is sometimes convenient to use their alternative characterization contained in Proposition 1.6 below (cf. [4]). However, to formulate it we need

(1.5) **Definition.** A collection $\mathcal{N}$ of ordered pairs $\tilde{V} = (V^1, V^2) \in \mathcal{T} \times I^X$ is called a *fuzzy pair-base* of a fuzzy space $(X, \mathcal{T})$ if for each $U \in \mathcal{T}$ there exists a subcollection $\mathcal{N}' \subset \mathcal{N}$ such that $U = \bigvee\{V^1 : \tilde{V} \in \mathcal{N}'\} = \bigvee\{V^2 : \tilde{V} \in \mathcal{N}'\}$. A collection $\mathcal{N} \subset \mathcal{T} \times I^X$ is called *cushioned* if $\bigvee\{V^1 : \tilde{V} \in \mathcal{N}'\} \leqslant \bigvee\{V^2 : \tilde{V} \in \mathcal{N}'\}$ for every $\mathcal{N}' \subset \mathcal{N}$. A pair-base $\mathcal{N}$ which is a union of a countable number of cushioned collections is called *$\sigma$-cushioned.*

(1.6) **Proposition.** *A fuzzy space $(X, \mathcal{T})$ is stratifiable iff it has a $\sigma$-cushioned fuzzy pair-base.*

Proof. Let $\mathcal{N} = \bigcup\limits_n \mathcal{N}_n$ be a $\sigma$-cushioned pair-base of a fuzzy space $(X, \mathcal{T})$. For every $U \in \mathcal{T}$ let $U_n = \bigvee\{V^1 : \tilde{V} \in \mathcal{N}_n, V^2 \leqslant U\}$. It is easy to notice that $U \to (U_n)_{n \in \mathbb{N}}$ is really a stratification in $(X, \mathcal{T})$.

Conversely, assume that $(X, \mathcal{T})$ is stratifiable and let $U \to (U_n)_{n \in \mathbb{N}}$ be a stratification. For every $N \in \mathbb{N}$ let $\mathcal{N}_n = \{\tilde{U} = (U_n, U) : U \in \mathcal{T}\}$. It is easy to verify that for each $n \in \mathbb{N}$ the family $\mathcal{N}_n$ is cushioned, and hence $\bigcup \mathcal{N}_n = \mathcal{N}$ is a $\sigma$-cushioned pair-base of $(X, \mathcal{T})$. □

(1.7) **Theorem.** *The product $X$ of a countable number of stratifiable fuzzy spaces $X_i$, $i \in \mathbb{N}$, is stratifiable, too.*

Proof. For every $i \in \mathbb{N}$ fix a $\sigma$-cushioned pair-base $\mathcal{N}_i = \bigcup\limits_k \mathcal{N}_i^k$ in $X_i$ where each $\mathcal{N}_i^k$ is cushioned. Since the union of two cushioned families is obviously cushionded itself, without loss of generality we may assume that $\mathcal{N}_i^k \subset \mathcal{N}_i^{k+1}$ for all $i$, $k \in \mathbb{N}$. Consider the family of pairs of fuzzy subsets of $X$ defined as follows: $\mathcal{N}^k = \{\tilde{V} = (V^1 = p_1^{-1}(U_1^{1,k}) \wedge \ldots \wedge p_m^{-1}(U_m^{1,k}), V^2 = p_1^{-1}(U_1^{2,k}) \wedge \ldots \wedge p_m^{-1}(U_m^{2,k})) : (U_j^{1,k}, U_j^{2,k}) \in \mathcal{N}_j^k,\ j = 1, \ldots, m;\ m \in \mathbb{N}\}$, and let $\mathcal{N} = \bigcup \mathcal{N}^k$. A routine verification shows that $\mathcal{N}$ is a fuzzy pair-base in the product space $X$. Besides, if $\mathcal{M}^k \subset \mathcal{N}^k$, then

$$\overline{\bigvee\{V^1 : \tilde{V} \in \mathcal{M}^k\}} = \overline{\bigvee\{p_1^{-1}(U_1^{1,k}) \wedge \ldots \wedge p_m^{-1}(U_m^{1,k}) : \tilde{V} \in \mathcal{M}^k\}}$$
$$= \overline{(\bigvee\{p_1^{-1}(U_1^{1,k}) : \tilde{V} \in \mathcal{M}^k\}) \wedge \ldots \wedge (\bigvee\{p_m^{-1}(U_m^{1,k}) : \tilde{V} \in \mathcal{M}^k\})}$$
$$\leqslant (\overline{\bigvee\{p_1^{-1}(U_1^{1,k}) : \tilde{V} \in \mathcal{M}^k\}}) \wedge \ldots \wedge (\overline{\bigvee\{p_m^{-1}(U_m^{1,k}) : \tilde{V} \in \mathcal{M}^k\}})$$
$$\leqslant (\bigvee\{p_1^{-1}(U_1^{2,k}) : \tilde{V} \in \mathcal{M}^k\}) \wedge \ldots \wedge (\bigvee\{p_m^{-1}(U_m^{2,k}) : \tilde{V} \in \mathcal{M}^k\})$$
$$= \bigvee\{V^2 : \tilde{V} \in \mathcal{M}^k\},$$

i.e. every $\mathcal{N}^k$ is cushioned. Thus $\mathcal{N} = \bigcup_k \mathcal{N}^k$ is a $\sigma$-cushioned pair-base and hence the fuzzy space $X$ is stratifiable. □

In the rest of this section we establish some properties of stratifiable fuzzy spaces; these properties will be repeatedly used in the sequel.

**(1.8) Proposition.** *Let $(X, \mathcal{T})$ be a stratifiable fuzzy space, $U \in \mathcal{T}$ and $A \in \mathcal{T}^c$. Then there exists $U_A \in \mathcal{T}$ such that $U_A \leqslant U$ and $A \wedge U \leqslant U_A \leqslant \overline{U}_A \leqslant U \vee A$. Moreover, the sets $U_A$ can be chosen in such a way that if $U \leqslant V$ and $A \leqslant B$, then $U_A \leqslant V_B$ (here $V \in \mathcal{T}$, $B \in \mathcal{T}^c$).*

Proof. Let $U_A = \bigvee_n (U_n \wedge (1 - (\overline{A^c})_n)$. It is obvious that $U_A \leqslant U$, $U_A \in \mathcal{T}$ and if $U \leqslant V$, $A \leqslant B \in \mathcal{T}^c$, then $U_A \leqslant V_B$.

To show the inequality $A \wedge U \leqslant U_A$, assume that $(A \vee U)(x) > U_A(x)$ for some $x \in X$. Then one can choose $\varepsilon < 0$ satisfying $U(x) - \varepsilon > U_A(x)$ and $n \in \mathbf{N}$ such that $U_n(x) > U(x) - \varepsilon$. Since the inequality $(\overline{A^c})_n \leqslant A^c$ implies $1 - (\overline{A^c})_n \geqslant A$, we obtain $U_n \wedge (1 - (\overline{A^c})_n)(x) > U_A(x)$ and this contradicts the definition of $U_A$.

To show the inequality $\overline{U}_A \leqslant U \vee A$ assume that $\overline{U}_A(x) > (U \vee A)(x)$ for some $x \in X$. Then one can choose $\varepsilon > 0$ satisfying

$$\overline{U}_A(x) > A(x) + \varepsilon \tag{1}$$

and $n_0 \in N$ such that $A^c(x) < (A^c)_{n_0}(x) + \varepsilon$ and hence

$$A(x) + \varepsilon > 1 - (A^c)_{n_0}(x). \tag{2}$$

Since without loss of generality we may assume that the stratification is increasing, the inequality $1 - (\overline{A^c})_n \leqslant 1 - (A^c)_{n_0}$ holds for every $n \geqslant n_0$ and hence $\bigvee_{n \geqslant n_0} (U_n \wedge (1 - (\overline{A^c})_n) \leqslant 1 - (A^c)_{n_0}$. Therefore, applying (2) we conclude that

$$\overline{\bigvee_{n \geqslant n_0} U_n \wedge (1 - (\overline{A^c})_n)} \leqslant 1 - (A^c)_{n_0} < A(x) + \varepsilon. \tag{3}$$

On the other hand,

$$(4)\qquad \overline{\bigvee_{n\leqslant n_0} U_n \wedge (1-(\overline{A^c})_n)} = \bigvee_{n\leqslant n_0} \overline{(U_n \wedge (1-(\overline{A^c})_n)}.$$

Combining the inequalities (1), (3) and (4) we get

$$\begin{aligned}\overline{U}_A(x) &= \overline{\bigvee_n U_n \wedge (1-(\overline{A^c})_n)}(x)\\ &= \overline{\bigvee_{n\leqslant n_0} U_n \wedge (1-(\overline{A^c})_n)}(x) \vee \overline{(\bigvee_{n\geqslant n_0} U \wedge (1-(\overline{A^c})_n)}x\\ &\leqslant (U \vee (A+\varepsilon))(x) < \overline{U}(x).\end{aligned}$$

The contradiction completes the proof. □

(1.9) **Corollary.** *Let $(X,\mathcal{T})$ be a stratifiable fuzzy space, $U \in \mathcal{T}$, $A \in \mathcal{T}^c$ and $A \leqslant U$. Then there exists $U_A \in \mathcal{T}$ such that $A \leqslant U_A \leqslant \overline{U}_A \leqslant U$. Moreover, the sets $U_A$ can be chosen in such a way that if $U \leqslant V$ and $A \leqslant B$, then $U_A \leqslant V_B$ (here $V \in \mathcal{T}$, $B \in \mathcal{T}^c$, $B \leqslant V$).*

(1.10) **Proposition.** *In a stratifiable space $(X,\mathcal{T})$ there exists a stratification $U \to (U_n)_{n\in\mathbf{N}}$ such that $\overline{U}_n \leqslant U_{n+1}$ for all $U \in \mathcal{T}$ and $n \in \mathbf{N}$.*

Proof. Let $U \to (U_n)_{n\in N}$ be a stratification in $(X,\mathcal{T})$. By induction we shall define a new stratification $U \to (U'_n)_{n\in\mathbf{N}}$ with the required properties.

For every $U \in \mathcal{T}$ let $U'_1 = U_1$, and let $U'_{n+1} = U_{A_n}$ where $A_n = \overline{U_{n+1} \vee U'_n}$. Notice first that $U'_{n+1} \leqslant (U_{n+1} \vee U'_n) \wedge U = A_n \wedge U \leqslant U$. Moreover, taking into account that $\overline{U'_1} = \overline{U}_1 \leqslant U$ and assuming that $\overline{U'_n} \leqslant U$ for some $n \in \mathbf{N}$ we conclude that $A_n \leqslant U$, and hence $\overline{U'}_{n+1} = \overline{U}_{A_n} \leqslant U$ by (1.9). Now it is easy to see that $\bigvee U_n = U$. Futhermore, according to (1.9) the inequality $A_n \leqslant U_{A_n} = U'_{n+1}$ holds and hence $\overline{U'_n} \leqslant U'_{n+1}$. To complete the proof one has to notice only that from (1.9) it follows also that $U \leqslant V$ implies $U'_n \leqslant V'_n$ for every $n \in \mathbf{N}$. □

(1.11) **Proposition.** *Let $(X,\mathcal{T})$ be a stratifiable fuzzy space, let $(A,\mathcal{T}^A)$ be its closed fuzzy subspace and $U^A \to (\varphi_n(U^A))_{n\in\mathbf{N}}$ a stratification in $(A,\mathcal{T}^A)$ (where $U \in \mathcal{T}$ and $U^A = U \wedge A$). Then there exists an extension of this stratification to $(X,\mathcal{T})$, i.e. a stratification $U \to (\varphi_n(U))_{n\in\mathbf{N}}$ in $(X,\mathcal{T})$ such that*

*(a) $\varphi_n(U) \wedge A = \varphi_n(U^A)$* and

*(b) $\overline{\varphi_n(U) \wedge A} = \overline{\varphi_n(U^A)}$.*

*Moreover, such extensions can be done consistently in the sense that* *(c) if $U \leqslant V$ and $\varphi_n(U^A) \leqslant \Psi_n(V^A)$ for all $n \in \mathbf{N}$, then $\varphi_n(U) \leqslant \Psi_n(V)$ for all $n \in \mathbf{N}$, too.*

*(Here $V \in \mathcal{T}$, $V^A = V \wedge A$, $V^A \to (\Psi_n(V^A))_{n \in \mathbf{N}}$ is a stratification in $(A, \mathcal{T}^A)$ and $V \to (\Psi_n(V))_{n \in \mathbf{N}}$ is its extension to $(X, \mathcal{T})$.)*

Proof. Take $U \in \mathcal{T}$ and let $U \to (U_n)_{n \in \mathbf{N}}$ be its stratification in $(X, \mathcal{T})$. One can easily check that $U^{A^c} \vee \varphi_n(U^A) \in \mathcal{T}$ and therefore we may define a fuzzy set $\varphi_n(U) \in \mathcal{T}$ as follows: $\varphi_n(U) = (U^{A^c})_n \vee (U^{A^c} \vee \varphi_n(U^A))_{\overline{\varphi_n(U^A)}}$. We shall show that $U \to (\varphi_n(U))_{n \in \mathbf{N}}$ is a stratification with the required properties.

The inequality $\overline{\varphi_n(U)} \leqslant U$ follows easily from (1.9). Applying (1.8) one gets the inequality $\varphi_n(U) \geqslant (U^{A^c} \vee \varphi_n(U^A))_{\overline{\varphi_n(U^A)}} \geqslant \varphi_n(U^A)$. It is easy to notice also that $U = \bigvee_n \varphi_n(U)$. In addition, Proposition (1.8) allows also to conclude that $U \leqslant V$ implies $\varphi_n(U) \leqslant \varphi_n(V)$, $n \in \mathbf{N}$, and hence $U \to (\varphi_n(U))_{n \in \mathbf{N}}$ is indeed a stratification. To check the property (a) notice first that by Proposition (1.8), $(\varphi_n(U))^A = (U^{A^c} \vee \varphi_n(U^A))_{\overline{\varphi_n(U^A)}} \wedge A \leqslant (U^{A^c} \vee \varphi_n(U^A)) \wedge A = \varphi_n(U^A) \leqslant (U^{A^c} \vee \varphi_n(U^A))_{\overline{\varphi_n(U^A)}} \leqslant (\varphi_n(U))^A$, and hence $(\varphi_n(U))^A = \varphi_n(U^A)$. Since $A$ is closed, $\overline{\varphi_n(U^A)} \leqslant \overline{\varphi_n(U)} \wedge A$. Applying (1.8) once again, we have

$$\overline{\varphi_n(U)} = \left((U^{A^c})_n \vee (U^{A^c} \vee \varphi_n(U^A))\right)_{\overline{\varphi_n(U^A)}} \leqslant U^{A^c} \vee \overline{\varphi_n(U^A)}$$

and therefore $\overline{\varphi_n(U)} \wedge A = \overline{\varphi_n(U^A)}$, i.e. (b) holds. The third condition follows immediately from the definition of $\varphi_n(U)$ and the last statement of Proposition (1.8) □

**(1.12) Corollary.** *Let $(X, \mathcal{T})$ be a stratifiable fuzzy space and $(A, \mathcal{T}^A)$ its closed subspace. If $U^A \to ((U^A)_n)_{n \in \mathbf{N}}$ is a stratification in $(A, \mathcal{T}^A)$ (where $U \in \mathcal{T}$, $U^A = U \wedge A$), then there exists a stratification $U \to (U_n)_{n \in \mathbf{N}}$ in $(X, \mathcal{T})$ such that for all $U \in \mathcal{T}$, $n \in \mathbf{N}$*

$$(a) \quad (U_n)^A = (U^A)_n,$$
$$(b) \quad (\overline{U}_n)^A = (\overline{U^A})_n.$$

Concluding this section let us notice that Corollary 1.9 implies also the following assertion:

**(1.13) Corollary.** *Every stratifiable fuzzy space is normal.*

It is obvious that a topological space $(X,T)$ is stratifiable (in Borges' sense [1]) iff $(X,T)$ considered as a fuzzy topological space is stratifiable (in the sense of Definition 1.1). Less evident is Theorem 2.3 stating the invariance of stratifiability under Lowen's functor $\omega$: Top $\rightarrow$ CFT [12]. However, before formulating this fact exactly we need to specify the use of the term "a stratifiable topological space".

By a stratifiable topological space we understand a topological space $(X,T)$ in which to every $U \in T$ a sequence $(U_n)_{n\in\mathbb{N}} \subset T$ can be assigned in such a way that (1) $\overline{U}_n \subset U$ for every $n \in \mathbb{N}$; (2) $\bigcup_n U_n = U$, and (3) if $U \subset V$ then $U_n \subset V_n$ for every $n \in \mathbb{N}$. This differs from the original Borges' definition [1] (see also [4]) in one respect. Namely, the original definition of a stratifiable topological space contains an additional requirement that $(X,T)$ is a $T_1$-space. Here we omit this requirement to make the definition consistent with our concept of stratifiability in fuzzy setting. Notice, however, that the condition $T_1$, being omitted in the definition of a stratifiable topological space, does not essentially influence the corresponding theory developed in [1], [2], [4] e.a.

In this section the closure operator in a topological space $(X,T)$ is denoted by "$-$" and the closure operator in the corresponding fuzzy topological space $(X,\omega T)$ by "$\tilde{}$". ($\omega T$ is the set of all lower semicontinuous functions from $(X,T)$ into I.) The following statement can be easily verified (and probably is well-known).

**(2.1) Assertion.** *Let $(X,T)$ be a topological space, $A \subset X$, $\alpha \in I$ and $U = \alpha A$. Then $\tilde{U} = \alpha\overline{A}$.*

**(2.2) Theorem.** *A topological space $(X,T)$ is stratifiable iff the associated fuzzy topological space $(X,\omega T)$ is stratifiable.*

Proof. Let a topological space $(X,T)$ be stratifiable; without loss of generality we may assume that its stratification $U \rightarrow (U_n)_{n\in\mathbb{N}}$ is increasing, i.e. $\overline{U}_n \subset U_{n+1}$ for each $U \in T$ and each $n \in \mathbb{N}$ [1]. Take $U \in \omega T$, and for every $k = 0, 1, \ldots, 2^m - 1$, $m \in \mathbb{N}$ consider an open in $(X,T)$ set $U^{k,m} = U^{-1}(k \cdot 2^{-m}, 1]$, and let $U^{k,m} \rightarrow (U_n^{k,m})n \in \mathbb{N}$ be its stratification in $(X,T)$ Denote $U_n^k = k \cdot 2^{-n} \cdot U_n^{k,n}$ and let $U_n = U_n^1 \vee \ldots \vee U_n^{2^n}$. It is obvious that $\bigvee_n U_n = U$. Applying (2.1) and noticing that $U^{k,n} = U^{2k,n+1}$, we get $\tilde{U}_n^k = k \cdot 2^{-n} \cdot \overline{U}_n^{k,n} \leqslant k \cdot 2^{-n} \cdot U_{n+1}^{k,n} \leqslant (2k) \cdot 2^{-(n+1)} \cdot U_{n+1}^{2k,n+1} = U_{n+1}^{2k} \leqslant U$ and therefore $\tilde{U}_n = \tilde{U}_n^1 \vee \ldots \vee \tilde{U}_n^{2^n} \leqslant U$. Besides, from the construction it is obvious that $U \leqslant V$ implies $U_n \leqslant V_n$ for every $n \in \mathbb{N}$. Thus $U \rightarrow (U_n)_{n\in\mathbb{N}}$ is indeed a stratification in $(X,\omega T)$.

Conversely, let a fuzzy space $(X, \omega T)$ be stratifiable. Take $U \in T$; then $U \in \omega T$ and therefore we can consider its stratification $U \to (U_n)_{n \in \mathbb{N}}$ in $(X, \omega T)$. We shall show that $U \to (U'_n)_{n \in \mathbb{N}}$, where $U'_n = U_n^{-1}(\frac{1}{2}, 1]$, is a stratification in $(X, T)$. Notice first that $\bigcup_n U'_n = \bigcup_n (U_n^{-1}(\frac{1}{2}, 1]) = (\bigvee_n U_n)^{-1}(\frac{1}{2}, 1] = U^{-1}(\frac{1}{2}, 1] = U$. In order to prove the inclusion $\overline{U'_n} \subset U$ consider the fuzzy set $V_n = (\frac{1}{2})U'_n$ and notice that by Assertion 2.1, $\tilde{V}_n = (\frac{1}{2}) \cdot \overline{U'_n}$. Since the set $\tilde{V}_n^{-1}(0, 1]$ is obviously closed in $(X, T)$ and $V_n \leqslant U_n$ we can conclude now that $\overline{U'_n} = \tilde{V}_n^{-1}(0, 1] \leqslant (\tilde{U}_n)^{-1}(0, 1] \subset U^{-1}(0, 1] = U$. To complete the proof notice that if $U \subset V (U, V \in T)$, then $U_n \leqslant V_n$ and hence $U'_n \subset V'_n$, too. □

(2.3) **Corollary.** *The definition of stratifiability of a fuzzy space is a good extension (in Lowen's sense [12]) of stratifiability of a topological space.*

## 3. Closed images of staratifiable fuzzy spaces

The aim of this section is to prove that the class of stratifiable fuzzy spaces is preserved by closed continuous mappings (Theorem 3.6). However, at the beginning we state some elementary (and probably well-known) facts about images and preimages:

(3.1) **Assertion.** *If $f: X \to Y$ is a surjection, then $f(f^{-1}(B)) = B$ for each $B \in I^Y$.*

(3.2) **Assertion.** *If $f: X \to Y$ is a surjection and $A = f^{-1}(B)$ for some $B \in I^Y$ (i.e. if $A$ is a full preimage), then $A = f^{-1}(f(A))$.*

(3.3) **Assertion.** *Let $f: X \to Y$ be a surjection and $A = f^{-1}(f(A))$. Then $f(A^c) = (f(A))^c$.*

(3.4) **Assertion.** *If $f: X \to Y$ is a mapping and $A \in I^X$, then $f(A^c) \geqslant (f(A))^c$.*

In accordance with the terminology of general topology a continuous mapping $f: X \to Y$ of fuzzy spaces will be called *closed* if for every closed fuzzy set $A$ of $X$ the image $f(A)$ is a closed fuzzy set in $Y$. It is easy to notice that a continuous mapping $f: X \to Y$ is closed iff $f(\overline{B}) = \overline{f(B)}$ for each $B \in I^X$.

(3.5) **Lemma.** *Let $X$, $Y$ be fuzzy spaces, let $f: X \to Y$ be a closed surjection and $A = f^{-1}(f(A)) \in I^X$. Then for every open neighborhood $U$ of $A$ in $X$ the fuzzy set $1 - \overline{1 - f(\overline{U})}$ is an open neighborhood of $f(A)$ in $Y$.*

Proof. By (3.3) we have $f(U^c) \leqslant f(A^c) = (f(A))^c$. On the other hand, (3.4) implies $f(U^c) \geqslant (f(U))^c$ and since $f$ is closed we get $\overline{(f(U))^c} \leqslant f(U^c)$. Combining the two inequalities we conclude that $f(A) \leqslant 1 - \overline{(f(U))^c}$. □

(3.6) **Theorem.** *If $f: X \to Y$ is a closed surjection, where $X, Y$ are fuzzy spaces and $X$ is stratifiable, then $Y$ is stratifiable, too.*

Proof. For each open fuzzy set $V$ of $Y$ define a stratification $V \to (V_n)_{n\in\mathbb{N}}$ as follows.

Let $U = f^{-1}(V)$ and let $U \to (U_n)_{n\in\mathbb{N}}$ be a stratification in $X$. Then $f^{-1}(f(\overline{U}_n)) = A_n$ is a closed fuzzy set in $X$. We shall show that $V \to (V_n)_{n\in\mathbb{N}}$ where $V_n = 1 - \overline{(f(U_{A_n}))^c}$ is a stratification in $X$.

Notice first that by (3.2), $\overline{V}_n = \overline{1 - \overline{(f(U_{A_n}))^c}} \leqslant \overline{1 - (f(U_{A_n}))^c} = \overline{f(U_{A_n})} = f(\overline{U}_{A_n}) \leqslant f(U) = V$, i.e. the condition (1) of Definition 1.1. is fulfilled. Since (3.5) implies that $f(A_n) \leqslant V_n$, to verify the second condition of Definition 1.1 it suffices to check that $V \leqslant \bigvee_n f(A_n)$. However, this is true, because applying (3.2) we get $\bigvee_n f(A_n) = f(\bigvee_n A_n) = f(\bigvee_n f^{-1}(f(\overline{U}_n))) \geqslant f(\bigvee_n \overline{U}_n) = f(U) = V$. Finally, if $V, V'$ are open fuzzy sets in $Y$ and $V \leqslant V'$, then obviously $U = f^{-1}(V) \leqslant U' = f^{-1}(V')$ and hence $U_n \leqslant U'_n$ and $A_n \leqslant A'_n$ for all $n \in \mathbb{N}$. Applying (1.9) we conclude that $U_{A_n} \leqslant U'_{A'_n}$ and hence $f(U_{A_n}) \leqslant f(U'_{A_n})$. This means that $V_n \leqslant V'_n$. □

## 4. Adjunction spaces

(4.1) **Definition.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be fuzzy topological spaces, $A$ a closed (crisp) subspace of $X$ and $f: A \to Y$ a continuous mapping. Let $\varrho$ be the equivalence relation on the discrete sum $X \oplus Y$ the only nontrivial classes of equivalence of which are $f^{-1}(y) \cup \{y\}$ where $y \in Y$. The quotient set $X \oplus Y/\varrho$ endowed with the quotient fuzzy topology (see e.g. [6] or [14]) will be denoted by $X \cup_f Y$ and called a (fuzzy) *adjunction space* (cf. [7]).

**Let $i: X \to X \oplus Y$, $e: Y \to X \oplus Y$ be the natural embeddings, $q: X \oplus Y \to X \cup_f Y$ the natural quotient mapping, and let $h := q \circ i: X \to X \cup_f Y$, $k := q \circ e: Y \to X \cup_f Y$. It is routine to prove the following lemma, the topological version of which is well-known:**

(4.2) **Lemma.** *A fuzzy set $V \in I^{X \cup_f Y}$ is open in $X \cup_f Y$ iff $h^{-1}(V) \in \mathcal{T}_X$ and $k^{-1}(V) \in \mathcal{T}_Y$. A fuzzy set $B \in I^{X \cup_f Y}$ is closed in $X \cup_f Y$ iff $h^{-1}(B) \in \mathcal{T}_X^c$ and $k^{-1}(B) \in \mathcal{T}_Y^c$.*

**(4.3) Lemma.** *If $V \in I^Y$ and $U \in I^X$, then*

(a) $h^{-1}(k(V)) = f^{-1}(V)$;

(b) $k^{-1}(h(U)) = f(U^A)$;

(c) $h^{-1}(h(U)) = U \vee (f^{-1}(f(U^A))$.

Proof. (a) If $x \in A$, then $h^{-1}(k(V))(x) = k(V)(h(x)) = Vf(x) = f^{-1}(V)(x)$; if $x \in X \backslash A$, then $h^{-1}(k(V))(x) = f^{-1}(V)(x) = 0$.

(b) If $y \in f(A)$, then $k^{-1}(h(V))(y) = h(V)(k(y)) = \sup\{U(x) \colon x \in h^{-1}(k(y))\} = \sup\{U(x) \colon x \in f^{-1}(y)\} = f(U^A)(y)$; if $y \in Y \backslash f(A)$, then $k^{-1}(h(U))(y) = f(U^A)(y) = 0$.

(c) $h^{-1}(h(x)) \neq \emptyset$ for every $x \in X$ and therefore $h^{-1}(h(U))(x) = h(U(h(x))) = \sup\{U(x') \colon x' \in h^{-1}(h(x))\} = U(x) \vee (\sup\{U^A(x') \colon x' \in f^{-1}(f(x))\}) = (U \vee f^{-1}(f(U^A)))(x)$. □

**(4.4) Theorem.** *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be stratifiable fuzzy spaces, $(A, \mathcal{T}^A)$ a closed subspace of $X$ and $f \colon A \to Y$ a continuous mapping. Then the adjunction fuzzy space $(X \cup_f Y, \sigma)$ is stratifiable, too.*

Proof. For a fuzzy set $W \in \sigma$ let $U = h^{-1}(W)$, $V = k^{-1}(W)$ and $U^A = U \wedge A$. Then according to (4.2) $U \in \mathcal{T}_X$, $V \in \mathcal{T}_Y$ and $U^A \in \mathcal{T}^A$. Consider the stratification $V \to (V_n)_{n \in \mathbb{N}}$ in $Y$ and let the fuzzy set $U_n^A \in I^X$ be defined by $(U_n^A)(x) = f^{-1}(V_n)(x)$ if $x \in A$ and $(U_n^A)(x) = 0$ otherwise. It is easy to see that $U^A \to (U_n^A)_{n \in \mathbb{N}}$ is a stratification in $(A, \mathcal{T}^A)$. Applying (1.12) we can extend this stratification to a stratification $U \to (U_n)_{n \in \mathbb{N}}$ in $(X, \mathcal{T}_X)$ with the following properties:

(a) $U_n \wedge A = U_n^A$;

(b) $\overline{U}_n \wedge A = \overline{U_n^A}$;

(c) if $W \leqslant W'$, then $U_n \leqslant U_n'$.

(Obviously we must distinguish here in notation between $U_n \wedge A$ and $U_n^A$.)

Let $W_n = h(U_n) \vee k(V_n)$. We shall show that $W \to (W_n)_{n \in \mathbb{N}}$ is indeed a stratification in $(X \cup_f Y, \sigma)$.

Notice first that according to Lemma 4.3 $h^{-1}(W_n) = h^{-1}(h(U_n)) \vee h^{-1}(k(V_n)) = U_n \vee f^{-1}(f(U_n \wedge A) \vee f^{-1}(V_n) = U_n \vee f^{-1}(f(U_n^A)) \vee U_n^A = U_n \in \mathcal{T}_X$ and $k^{-1}(W_n) = k^{-1}(h(U_n)) \vee k^{-1}(k(V_n)) = f(U_n^A) \vee V_n = V_n \in \mathcal{T}_Y$, and hence $W_n \in \sigma$.

To show that $\overline{W}_n \leqslant W$ let $P = h(\overline{U}_n) \vee k(\overline{V}_n)$; since, obviously, $W_n \leqslant P \leqslant W$, it is sufficient to show that $P$ is closed in $X \cup_f Y$, i.e. (according to (4.2)) that $h^{-1}(P) \in \mathcal{T}_X^c$ and $k^{-1}(P) \in \mathcal{T}_Y^c$. Applying Lemma 4.3 we get $h^{-1}(P) = h^{-1}(h(\overline{U}_n)) \vee h^{-1}(k(\overline{V}_n) = \overline{U}_n \vee f^{-1}(f(\overline{U_n^A})) \vee f^{-1}(\overline{V}_n) = \overline{U}_n \vee f^{-1}(\overline{V}_n) \in \mathcal{T}_X^c$ and $k^{-1}(P) = k^{-1}(h(\overline{U}_n)) \vee k^{-1}(k(\overline{V}_n)) = f(\overline{U_n^A}) \vee \overline{V}_n = \overline{V}_n \in \mathcal{T}_Y^c$.

To complete the proof it is sufficient to notice that $\bigvee_n W_n = W$ and that if $W \leqslant W'$, then $U \leqslant U'$ and $V \leqslant V'$ and therefore also $W_n = h(U_n) \vee k(V_n) \leqslant W'_n = h(U'_n) \vee k(V'_n)$. □

## 5. Fuzzy spaces dominated by stratifiable ones

Extending the notion of topological domination to the category of fuzzy spaces we come to

(5.1) **Definition.** Let $(X, \mathcal{T})$ be a fuzzy space and $\mathcal{S} = \{(X_\gamma, \mathcal{T}_\gamma) \colon \gamma \in \Gamma\}$ a family of its closed fuzzy subspaces such that $\bigcup_{\gamma \in \Gamma} X_\gamma = X$. The space $(X, \mathcal{T})$ is *dominated* by $\mathcal{S}$ if for every $\Gamma' \subset \Gamma$ satisfying $\bigcup_{\gamma \in \Gamma'} X_\gamma = X$ and every $U \in I^X$ we have $U \in \mathcal{T}$ iff $U \wedge X_\gamma \in \mathcal{T}_\gamma$ for all $\gamma \in \Gamma'$.

(5.2) Remark. It is obvious that a fuzzy space $(X, \mathcal{T})$ is dominated by a family of closed fuzzy subspaces $\mathcal{S} = \{(X_\gamma, \mathcal{T}_\gamma) \colon \gamma \in \Gamma\}$ such that $\bigcup_{\gamma \in \Gamma} X_\gamma = X$ if and only if for every $\Gamma' \subset \Gamma$ satisfying $\bigcup_{\gamma \in \Gamma'} X_\gamma = X$ and every $A \in I^X$ we have $A \in \mathcal{T}^c$ iff $A \wedge X_\gamma \in \mathcal{T}^c_\gamma$ for every $\gamma \in \Gamma'$.

The main result of this section is Theorem 5.7 stating that a fuzzy space dominated by a family of closed stratifiable fuzzy subspaces is stratifiable itself. We start with some auxiliary lemmas.

(5.3) **Lemma.** *Let a fuzzy space $(X, \mathcal{T})$ be dominated by a pair of its closed fuzzy subspaces $\{(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)\}$. Then for all $U_1 \in \mathcal{T}_1$ and $U_2 \in \mathcal{T}_2$ we have $U_1 \wedge U_2 \leqslant 1 - (\overline{1 - U_1 \vee U_2})$.*

Proof. Take $\tilde{U}_1, \tilde{U}_2 \in \mathcal{T}$ such that $U_1 = \tilde{U}_1 \wedge X_1$ and $U_2 = \tilde{U}_2 \wedge X_2$. To prove the lemma it is sufficient to show that $1 - U_1 \vee U_2 \leqslant 1 - \tilde{U}_1 \wedge \tilde{U}_2$ or, equivalently, that $\tilde{U}_1 \wedge \tilde{U}_2 \leqslant U_1 \vee U_2$. If $x \in X_1 \cap X_2$, then $(\tilde{U}_1 \wedge \tilde{U}_2)(x) = (U_1 \wedge U_2(x) \leqslant (U_1 \vee U_2)(x)$; if $x \in X_i \backslash X_j$, $i, j \in \{1, 2\}$, $i \neq j$, then $(\tilde{U}_i \wedge \tilde{U}_j)(x) \leqslant \tilde{U}_i(x) = U_i(x) = (U_i \vee U_j)(x)$. □

(5.4) **Lemma.** *Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be closed fuzzy subspaces of a fuzzy space $(X, \mathcal{T})$, $U_1 \in \mathcal{T}_1$ and $U_2 \in \mathcal{T}_2$. Then $U_1 \wedge (X \backslash X_2) \leqslant 1 - (\overline{1 - U_1 \vee U_2})$ and $U_2 \wedge (X \backslash X_1 \leqslant 1 - (\overline{1 - U_1 \vee U_2})$.*

Proof follows easily from the obvious fact that $U_1 \wedge (X \backslash X_2) \in \mathcal{T}_1$ and $U_2 \wedge (X \backslash X_1) \in \mathcal{T}_2$. □

**(5.5) Lemma.** *Let a fuzzy space $(X,\mathcal{T})$ be dominated by a pair of its closed fuzzy subspaces $\{(X_1,\mathcal{T}_1),(X_2,\mathcal{T}_2)\}$. If $(X_1,\mathcal{T}_1)$ and $X_2,\mathcal{T}_2)$ are stratifiable, then $(X,\mathcal{T})$ is stratifiable, too.*

Proof. Take a fuzzy set $U \in \mathcal{T}$ and let $U^j \to (U^j_n)_{n\in\mathbb{N}}$ be a stratification of $U^j = U \wedge X_j$ in $(X_j,\mathcal{T}_j)$, $j=1,2$. Let $U_n = 1-(\overline{1-U^1_n \vee U^2_n})$; we shall show that $U \to (U_n)_{n\in\mathbb{N}}$ thus obtained is a stratification in $(X,\mathcal{T})$.

Notice first that $1-(\overline{1-U^1_n \vee U^2_n}) \leqslant U^1_n \vee U^2_n$ and therefore $\overline{U}_n \leqslant \overline{U}^1_n \vee \overline{U}^2_n \leqslant U^1 \vee U^2 = U$. (Since $X_j$ $(j=1,2)$ is closed, the closures of $U^j_n$ in $X_j$ and in $X$ coincide.)

The inequality $\bigvee\limits_n U_n \leqslant U$ is obvious and hence to check the second condition of Definition 1.1 we have only to show that $\bigvee\limits_n U_n \geqslant U$. If $x \in X_1 \cap X_2$, then by Lemma 5.3 $(\bigvee\limits_n U_n)(x) \geqslant \bigvee\limits_n (U^1_n \wedge U^2_n)(x) = (U^1 \wedge U^2)(x) = U(x)$. If $x \in X_j \backslash X_i$, $j \neq i$; $i,j \in \{1,2\}$, then applying Lemma 5.4 we get $U^j_n(x) = U^j_n \wedge (X\backslash X_i)(x) \leqslant (1-(\overline{1-U^1_n \vee U^2_n}))(x) = U_n(x)$ and hence $(\bigvee\limits_n U_n)(x) \geqslant (\bigvee\limits_n U^j_n)(x) = U^j(x) = U(x)$.

Furthermore, it is obvious that $U \leqslant V$ implies $U_n \leqslant V_n$ and hence $U \to (U_n)_{n\in\mathbb{N}}$ is indeed a stratification in $(X,\mathcal{T})$. □

We shall need also the following statement the validity of which follows easily from (5.2):

**(5.6) Lemma.** *If a fuzzy space $(X,\mathcal{T})$ is diminated by a family of its closed subspaces $\{(X_\gamma,\mathcal{T}_\gamma)\colon \gamma\in\Gamma\}$ and $A^\gamma \in I^X$ for all $\gamma \in \Gamma$, then $\overline{\bigvee\limits_\gamma A^\gamma} = \bigvee\limits_\gamma \overline{A^\gamma}$.*

**(5.7) Theorem.** *A fuzzy space $(X,\mathcal{T})$ is stratifiable iff it is dominated by a family of its closed stratifiable fuzzy subspaces.*

Proof. The part "only if" is obvious. Conversely, assume that a fuzzy space $(X,\mathcal{T})$ is dominated by a family $\mathcal{S} = \{(X_\gamma,\mathcal{T}_\gamma)\colon \gamma\in\Gamma\}$ of its closed stratifiable subspaces. Consider a subfamily $\mathcal{C}_\alpha \subset \mathcal{S}$ and let $C_\alpha = \bigcup\{X_\gamma\colon (X_\gamma,\mathcal{T}_\gamma)\in\mathcal{C}_\alpha\}$. Let $\sigma_\alpha$ be the fuzzy topology on $C_\alpha$ induced by $\mathcal{T}$. Consider the set $\mathcal{E}$ of all pairs $(\mathcal{C}_\alpha, S_\alpha)$ such that $(C_\alpha,\sigma_\alpha)$ is stratifiable and $S_\alpha$ is one of its stratifications. We introduce the order on $\mathcal{E}$ by letting $(\mathcal{C}_\alpha,S_\alpha) < (\mathcal{C}_\beta,S_\beta)$ iff $\mathcal{C}_\alpha \subset \mathcal{C}_\beta$ and for all $U^\beta \in \sigma_\beta$, $n\in\mathbb{N}$, the foloowing two conditions hold:

$$(*)\qquad U^\beta_n \wedge C_\alpha = (U^\beta \wedge C_\alpha)^\alpha_n \quad (= U^\alpha_n),$$
$$(**)\qquad \overline{U}^\beta_n \wedge C_\alpha = \overline{(U^\beta \wedge C_\alpha)^\alpha_n} \quad (= \overline{U}^\alpha_n)$$

(when writing $V^\alpha \to (V^\alpha)^\alpha_n$ we mean a stratification of $V^\alpha$ in $(C_\alpha,\sigma_\alpha)$).

The assumptions of the Zorn Lemma hold for $(\mathcal{E},<)$. Indeed, let $\{(\mathcal{C}_\alpha, S_\alpha)\colon \alpha \in A\}$ be a linearly ordered subset of $\mathcal{E}$ and let $\mathcal{C}_\beta = \bigcup_{\alpha\in A} \mathcal{C}_\alpha$. For $U^\beta \in \sigma_\beta$ let $U_n^\beta = \bigvee_{\alpha\in A} (U^\beta \wedge C_\alpha)_n^\alpha$. According to Lemma 5.6, $\overline{U}_n^\beta = \overline{\bigvee_{\alpha\in A} (U^\beta \wedge C_\alpha)_n^\alpha} = \bigvee_{\alpha\in A} \overline{(U^\beta \wedge C_\alpha)_n^\alpha} = \bigvee_{\alpha\in A} \overline{U}_n^\alpha \leqslant \bigvee_{\alpha\in A} U^\alpha = U^\beta$. Furthermore, $\bigvee_n U_n^\beta = \bigvee_n(\bigvee_{\alpha\in A} (U^\beta \wedge C_\alpha)_n^\alpha) = \bigvee_{\alpha\in A} (\bigvee(U_n^\alpha)) = \bigvee_{\alpha\in A} U^\alpha = U^\beta$. Finally, $U^\beta \leqslant V^\beta$ obviously implies $U_n^\beta \leqslant V_n^\beta$. Therefore $U^\beta \to (U_n^\beta)_{n\in\mathbb{N}}$ is a stratification in $(C_\beta, \sigma_\beta)$; we denote this stratification by $S_\beta$. Since every $C_\alpha$ is a closed fuzzy subset of $(X, \mathcal{T})$ (see (5.6)), it is easy to see that $S_\beta$ satisfies $(*)$ and $(**)$ and hence $(\mathcal{C}_\beta, S_\beta)$ is the least upper bound of $\{(\mathcal{C}_\alpha, S_\alpha)\colon \alpha \in A\}$.

According to the Zorn Lemma there exists a maximal element $(\mathcal{C}_0, S_0)$ in $\mathcal{E}$. To complete the proof we have only to show that $C_0 = X$, where $C_0 = \bigcup\{X_\gamma\colon (X_\gamma, \mathcal{T}_\gamma) \in \mathcal{C}_0\}$.

Suppose that $C_0 \neq X$. Then there exists $(X_\delta, \mathcal{T}_\delta) \in \mathcal{S}$ such that $X_\delta \not\subset C_0$. Let $C_1 = C_0 \cup X_\delta$ and let $\sigma_0$ and $\sigma_1$ be the induced fuzzy topologies on the sets $C_0$ and $C_1$, respectively. Obviously, $C_0$ and $X_\delta$ are closed subsets of $C_1$ and moreover, $(C_1, \sigma_1)$ is dominated by the family $\{(C_0, \sigma_0), (X_\delta, \mathcal{T}_\delta)\}$. Therefore by Lemma 5.5 the fuzzy space $(C_1, \sigma_1)$ is stratifiable and hence by (1.12) there exists a stratification $S_1(U^1 \to (U_n^1)_{n\in\mathbb{N}})$ in $(C_1, \sigma_1)$ such that $U_n^1 \wedge C_0 = (U^1 \wedge C_0)_n^0$ and $\overline{U}_n^1 \wedge C_0 = \overline{(U^1 \wedge C_0)_n^0}$. This means that $(\mathcal{C}_0, S_0) \prec (\mathcal{C}_1, S_1), (\mathcal{C}_0, S_0) \neq (\mathcal{C}_1, S_1)$, i.e. $(\mathcal{C}_0, S_0)$ is not maximal. This contradiction proves that $C_0 = X$ and hence $(X, \mathcal{T})$ is stratifiable. □

## 6. A FUNCTIONAL CHARACTERIZATION OF STRATIFIABLE FUZZY SPACES

The fuzzy unit interval $\mathcal{F}(I)$ was defined by B. Hutton in [9]; in the same paper a characterization of normal fuzzy spaces by means of mappings into $\mathcal{F}(I)$ was obtained. This result can be considered as the fuzzy version of the famous Urysohn Lemma. (In fact, Hutton [9] defines a more general concept of the $L$-fuzzy unit interval, where $L$ is a bounded lattice with order reversing involution, and uses notation $I(L)$ for it. The notation $\mathcal{F}(I)$ was first used in [16] and corresponds to Hutton's $I(I)$. For the terminology concerning the fuzzy unit interval the reader is refered to [9] (see also [8] or [16])). Later the fuzzy unit interval was used for the study of fuzzy uniform spaces [10], completely regular fuzzy spaces [10], [25], and some other concepts of fuzzy topology. It is the aim of this section to study the mappings of a stratifiable fuzzy space into $\mathcal{F}(I)$ and to get a characterization of stratifiable fuzzy spaces by means of mappings into $\mathcal{F}(I)$.

**(6.1) Theorem.** *Let $(X, \mathcal{T})$ be a stratifiable fuzzy space. Then with each pair $(A, U) \in \mathcal{T}^c \times \mathcal{T}$, where $A \leqslant U$, a continuous function $(f_{A,U} :=)\ f\colon X \to \mathcal{F}(I)$*

*can be associated such that* $A(x) \leqslant f(x)(1^-) \leqslant f(x)(0^+) \leqslant U(x)$ *for every* $x \in X$. *Moreover, this association can be done in such a coordinated way that if* $(B, V) \in \mathcal{T}^c \times \mathcal{T}$, $B \leqslant V$, $A \leqslant B$ *and* $U \leqslant V$, *then* $f_{B,V} \leqslant f_{A,U}$.

Proof. Let a pair $(A, U) \in \mathcal{T}^c \times \mathcal{T}$ such that $A \leqslant U$ be fixed. To construct the function $f(:= f_{A,U})$ let $U_0 = U$ and $U_1 = U_A$ where $U_A$ is defined according to (1.9). Applying (1.9) again let us set $U_{\frac{1}{2}} = (U_0)_{\overline{U}_1}$. As the result we have $U_1 \leqslant \overline{U}_1 \leqslant U_{\frac{1}{2}} \leqslant \overline{U}_{\frac{1}{2}} \leqslant U_0$. Further, applying (1.9) again, let us set $U_{\frac{1}{4}} = (U_0)_{\overline{U}_{\frac{1}{2}}}$ and $U_{\frac{3}{4}} = (U_{\frac{1}{2}})_{\overline{U}_1}$. Continuing this process by induction (which is analogous to that used in the proof of the Urysohn Lemma (see e.g. [11]; cf. also the proof ot the Fuzzy Urysohn Lemma [9])), for each dyadic rational number $r \in I$ a fuzzy set $U_r$ will be constructed in such a way that if $r_1 \leqslant r_2$, where $r_1, r_2 \in I$ are dyadic rational numbers, then $\overline{U}_{r_2} \leqslant U_{r_1}$.

Let $D$ denote the set of all dyadic rational numbers of $I$. For each $t \in I$ let $U_t = \sup\{U_s : s > t, s \in D\}$. It is clear that if $t, s \in I$ and $t < s$, then $\overline{U}_s \leqslant U_t$. Finally, let $U_t = 1$ for $t < 0$, and let $U_t = 0$ for $t > 1$.

Define a function $f : X \to \mathcal{F}(I)$ by the equality $f(x)(t) = U_t(x)$. To show that this function has the required properties notice first that $A(x) \leqslant U_A(x) = U_1(x) = f(x)(1) = f(x)(1^-) \leqslant f(x)(0^+) \leqslant f(x)(0) = U_0(x) = U(x)$ for each $x \in X$. Further, to show the contininty of $f$ let $\varrho_a$, $\lambda_b$ be the elements of the standard subbase of the fuzzy topology on $\mathcal{F}(I)$ (i.e. $a, b \in \mathbf{R}$ and $\varrho_a(z) = z(a^+)$, $\lambda_b(z) = 1 - z(b^-)$, $z \in \mathcal{F}(I)$ (see [9])). If $a \in [0, 1)$, then $f^{-1}(\varrho_a)(x) = \varrho_a f(x) = f(x)(a^+) = \sup_{s>a} f(x)(s) = \bigvee_{s>a} U_s(x)$ and hence $f^{-1}(\varrho_a) \in \mathcal{T}$. If $a < 0$, then $f^{-1}(\varrho_a) = 1 \in \mathcal{T}$, and if $A \geqslant 1$, then $f^{-1}(\varrho_a) = 0 \in \mathcal{T}$. On the other hand, if $b \in (0, 1]$, then $f^{-1}(\lambda_b)(x) = \lambda_b f(x) = 1 - f(x)(b^-) = 1 - \inf_{s<b} f(x)(s) = 1 - \bigwedge_{s<b} U_s(x) = 1 - \bigwedge_{s<b} \overline{U}_s(x)$ and hence $f^{-1}(\lambda_b) \in \mathcal{T}$. Finally, if $b \leqslant 0$, then $f^{-1}(\lambda_b) = 0 \in \mathcal{T}$ and if $b > 1$, then $f^{-1}(\lambda_b) = 1 \in \mathcal{T}$. Thus the preimages of all elements of the standard subbase of $\mathcal{F}(I)$ are open in $X$ and hence $f$ is continuous.

The last statement of the theorem follows easily from the above construction and from (1.9) which asserts that if $(A, U), (B, V) \in \mathcal{T}^c \times \mathcal{T}$, $A \leqslant B$ and $U \leqslant V$, then $U_A \leqslant V_B$. □

**(6.2) Theorem.** *A fuzzy topological space* $(X, \mathcal{T})$ *is stratifiable iff to each* $U \in \mathcal{T}$ *a continuous mapping* $f_U : X \to \mathcal{F}(I)$ *satisfying* $f_U(x)(1^-) = U^c(x)$ $(x \in X)$ *can be assigned in such a coordinated way that if* $U \leqslant V \in \mathcal{T}$, *then* $f_U \leqslant f_V$.

Proof. Assuming that $(X, \mathcal{T})$ is stratifiable, take $U \in \mathcal{T}$ and let $U \to (U_n)_{n \in \mathbf{N}}$ be the corresponding stratification. Let $A = U^c$ and $W_n = (\overline{U}_n)^c$; then obviously $A = \bigwedge_n W_n$.

We define a family of open fuzzy sets $\{V_t : t \in [0,1)\}$ as folloows. Applying Proposition 1.8 and starting from the fuzzy set $V_1 = W_0$ we set $V_{\frac{1}{2}} = (V_0)_A \wedge W_1$, $V_{\frac{1}{4}} = (V_0)_{\overline{V}_{\frac{1}{2}}}$, $V_{\frac{3}{4}} = (V_{\frac{1}{2}})_A \wedge W_2$, $V_{\frac{1}{8}} = (V_0)_{\overline{V}_{\frac{1}{4}}}$, $V_{\frac{3}{8}} = (V_{\frac{1}{4}})_{\overline{V}_{\frac{1}{2}}}$, $W_{\frac{5}{8}} = (V_{\frac{1}{2}})_{\overline{V}_{\frac{3}{4}}}$, $V_{\frac{7}{8}} = (V_{\frac{3}{4}})_A \wedge W_3$, and so on. Continuing this process by induction (which is analogous to that used in the proof of the Urysohn Lemma [9]; see also the proof of the previous theorem), for each dyadic rational number $r \in [0,1)$ a fuzzy set $V_r \in \mathcal{T}$ will be constructed in such a way that if $r_1 < r_2$, then $\overline{V}_{r_2} \leqslant V_{r_1}$. Patterned after the proof of (6.1) a fuzzy set $V_t \in \mathcal{T}$ can be now defined for each $t \in [0,1)$ in such a way that $s < t$ $(s, t \in [0,1))$ implies $\overline{V}_t \leqslant V_s$. Define a function $f_U (=: f) : X \to \mathcal{F}(I)$ by the equality

$$f(x)(t) = \begin{cases} V_t(x), & \text{if } t \in [0,1); \\ 1, & \text{if } t < 0; \\ 0, & \text{if } t \geqslant 1. \end{cases}$$

To show that the function $f$ thus obtained has the required properties notice first that $A(x) = \inf\limits_{s<1} V_s(x) = \inf\limits_{s<1} f(x)(s) = f(x)(1^-)$ for each $x \in X$.

To verify the continuity of $f$, consider first $a \in [0,1)$ and $b \in (0,1]$. Then $f^{-1}(\varrho_a)(x) = \varrho_a(f(x)) = f(x)(a^+) = \sup\limits_{s>a} f(x)(s) = \bigvee\limits_{s>a} U_s(x)$ and $f^{-1}(\lambda_b)(x) = 1 - f(x)(b^-) = 1 - \inf\limits_{s<b} f(x)(s) = 1 - \bigwedge\limits_{s<b} U_s(x) = 1 - \bigwedge\limits_{s<b} \overline{U}_s(x)$. Further, if $a < 0$, then $f^{-1}(\varrho_a) = 1$, if $a \geqslant 1$, then $f^{-1}(\varrho_a) = 0$; if $b \leqslant 0$, then $f^{-1}(\lambda_b) = 0$ and if $b > 1$, then $f^{-1}(\lambda_b) = 1$. Thus $f^{-1}(\varrho_a) \in \mathcal{T}$ and $f^{-1}(\lambda_b) \in \mathcal{T}$ for all $a, b \in \mathbf{R}$ and hence $f$ is continuous. Finally from (1.8) and our construction it is clear that if $U \leqslant V (V \in \mathcal{T})$, then $f_U \leqslant f_V$.

Conversely, assume that $(X, \mathcal{T})$ is a fuzzy space and with each $U \in \mathcal{T}$ a continuous function $(f_U = :) f : X \to \mathcal{F}(I)$ satisfying $f(x)(1^-) = U^c(x)$ is associated in such a way that $U \leqslant V$ implies $f_U \leqslant f_V$.

For a fuzzy set $U \in \mathcal{T}$ let a sequence $(U_n)_{n\in\mathbf{N}}$ be defined where $U_n = f^{-1}(\lambda_{(1-\frac{1}{n})})$. To verify that $U \to (U_n)_{n\in\mathbf{N}}$ is indeed a stratification in $(X, \mathcal{T})$ notice that

(1)
$$\begin{aligned} \bigvee_n U_n(x) &= \bigvee_n f^{-1}(\lambda_{(1-\frac{1}{n})})(x) \\ &= \bigvee_n (1 - f(x))\left(\left(1 - \frac{1}{n}\right)^-\right) = 1 - f(x)(1^-) = U(x); \end{aligned}$$

(2)
$$\begin{aligned} U_n(x) &= f^{-1}(\lambda_{(1-\frac{1}{n})})(x) = 1 - f(x)\left(\left(1 - \frac{1}{n}\right)^-\right) \\ &\leqslant 1 - f(x)\left(\left(1 - \frac{1}{n}\right)^+\right) = 1 - f^{-1}(\varrho_{(1-\frac{1}{n})})(x) \\ &\leqslant 1 - f^{-1}(\lambda_{(1-\frac{1}{n+1})})(x) \end{aligned}$$

and hence $\overline{U}_n \leqslant U_{n+1} \leqslant U$ and, finally,

(3) since $U \leqslant V$ implies $f_U \leqslant f_V$, it is clear that $U_n \leqslant V_n$ for each $n \in \mathbb{N}$. □

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