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HYPERREFLEXIVE OPERATORS ON FINITE DIMENSIONAL HILBERT SPACES

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Summary. In this paper a complete characterization of hyperreflexive operators on finite dimensional Hilbert spaces is given.

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0. INTRODUCTION

Let H be a complex separable Hilbert space, $\mathscr{B}(H)$ the algebra of all continuous linear operators on H and $T \in \mathscr{B}(H)$. We denote by $\{T\}'$ the commutant of T ($X \in \{T\}'$ if and only if $XT = TX$) and by $\{T\}'' = \bigcap\{\{X\}'\colon XT = TX\}$ the double commutant of T. A closed linear subspace $M \subset H$ is called invariant for an algebra $\mathscr{A}$ if it is invariant for every $X \in \mathscr{A}$. If M is invariant for the algebra $\{T\}'$ then M is said to be hyperinvariant for T. The lattice of all invariant subspaces of $\mathscr{A}$ is denoted by $\operatorname{Lat}\mathscr{A}$.

Let $\mathscr{M}$ be a set of subspaces of H. We denote by $\operatorname{Alg}\mathscr{M}$ the algebra of all $X \in \mathscr{B}(H)$ for which every $M \in \mathscr{M}$ is invariant. The algebra $\mathscr{A} \subset \mathscr{B}(H)$ is called reflexive if $\mathscr{A} = \operatorname{Alg}\operatorname{Lat}\mathscr{A}$. An operator $T \in \mathscr{B}(H)$ is called reflexive if $\operatorname{Alg} T$—the weakly closed algebra generated by T and the identity I—is reflexive. We shall write $\operatorname{Lat} T$ instead of $\operatorname{Lat}\operatorname{Alg} T$. In [3] a characterization of reflexive operators on finite dimensional spaces was given. If the commutant of the operator T is reflexive then T is called hyperreflexive. In this paper hyperreflexive operators on finite dimensional spaces are characterized.

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In this paper $\oplus$ means the direct (not necessarily orthogonal) sum.

1. The nilpotent case

First we consider the nilpotent operators on a finite dimensional space H. Let $N \in \mathscr{B}(H)$ be a nilpotent operator of order n (i.e. $N^n = 0$, $N^{n-1} \neq 0$). Since similarity preserves hyperreflexivity (see [2] for a little more general result) we may assume that the matrix representation of N is

$$N = J(k_1) \oplus J(k_2) \oplus \ldots \oplus J(k_m),\ n = k_1 \geqslant k_2 \geqslant \ldots \geqslant k_m. \tag{1}$$

Let the corresponding decomposition of H be

$$H = H_1 \oplus H_2 \oplus \ldots \oplus H_m. \tag{2}$$

Here $J(k)$ means the $k \times k$ Jordan cell (i.e. each entry on the first subdiagonal is 1, and all other entries are 0). We shall use the following descriptions of $\{N\}'$ and $\mathrm{Lat}\{N\}'$ [4, p. 128]:

$$\begin{gathered}\mathscr{L} \in \mathrm{Lat}\{N\}' \Longleftrightarrow \mathscr{L} = \bigoplus_{j=1}^{m} \ker J(k_j)^{r_j} \text{ for an } m\text{-tuple of integers,}\\ r_1, \ldots, r_m,\quad r_1 \geqslant \ldots \geqslant r_m \geqslant 0,\quad k_1 - r_1 \geqslant \ldots \geqslant k_m - r_m \geqslant 0.\end{gathered} \tag{3}$$

Let $A \in \mathscr{B}(H)$ have a block decomposition (corresponding to the decomposition (2) of H) $A = (A_{ij})$. Then

$$A \in \{N\}' \Longleftrightarrow \begin{cases} A_{ii} \in \{J(k_i)\}' \text{ for all } i; \\ \text{for } i < j,\ A_{ij} = \begin{pmatrix} 0 \\ X \end{pmatrix} \text{ with } X \in \{J(k_j)\}'; \\ \text{for } i > j,\ A_{ij} = (Y\ 0) \text{ with } Y \in \{J(k_i)\}'. \end{cases} \tag{4}$$

Recall that $\{J(k)\}'$ consists of polynomials in $J(k)$ and thus of lower-triangular matrices with equal entries on each subdiagonal ($a_{i+1,j+1} = a_{ij}$, $1 \leqslant i, j \leqslant k$). Now, we are able to describe $\mathrm{Alg\,Lat}\{N\}'$:

Theorem 1. *Let $N \in \mathscr{B}(H)$ be a nilpotent operator of the form (1). Let $A \in \mathscr{B}(H)$ have a block decomposition (corresponding to the decomposition (2) of H) $A = (A_{ij})$. Then A belongs to $\mathrm{Alg\,Lat}\{N\}'$ if and only if it has the following form:*

$$A_{ij} = \begin{cases} \text{a lower-triangular matrix if } i = j; \\ \begin{pmatrix} 0 \\ X \end{pmatrix} \text{ with lower-triangular } X \text{ if } i < j; \\ (Y\ 0) \text{ with lower-triangular } Y \text{ if } i > j. \end{cases} \tag{5}$$

Proof. Let us recall that for a pair of integers k, r; $0 \leqslant r \leqslant k$, the space $\ker J(k)^r$ consists of all vectors $x = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ for which $\lambda_1 = \lambda_2 = \ldots = \lambda_{k-r} = 0$. If $\mathcal{L}$ is a subspace of the form (3) and if $x = x_1 \oplus x_2 \oplus \ldots \oplus x_m \in \mathcal{L}$, then for every A with a block decomposition satisfying (5) and for all pairs i, j; $1 \leqslant i, j \leqslant m$, $A_{ij}x_j \in \ker J(k_i)^{r_i}$. This shows that $A \in \operatorname{Alg}\operatorname{Lat}\{N\}'$.

Now, suppose that $A = (A_{ij}) \in \operatorname{Alg}\operatorname{Lat}\{N\}'$. Let the integers j and s satisfy $1 \leqslant j \leqslant m$, $0 \leqslant s \leqslant k_j$. If

$$r_i = \begin{cases} s \text{ if } s \leqslant k_i, \\ k_i \text{ if } s > k_i. \end{cases}$$

then $\mathcal{L} = \bigoplus_{i=1}^{m} \ker J(k_i)^{r_i} \in \operatorname{Lat}\{N\}'$.

$$A\mathcal{L} \subset \mathcal{L} \Rightarrow A_{ij} \ker J(k_j)^s \subset \ker J(k_i)^s, \qquad s = 0, 1, \ldots, k_j,$$

for all pairs i, j satisfying $1 \leqslant i \leqslant j$. It follows that A_{ii} is lower-triangular for all i and if $i < j$ then $A_{ij} = \begin{pmatrix} 0 \\ Y \end{pmatrix}$ with a lower-triangular $k_j \times k_j$ matrix Y.

Let the integers i and s satisfy $1 \leqslant i \leqslant m$, $0 \leqslant s \leqslant k_i$. Setting $r_j = \max\{s + k_j - k_i, 0\}$ for $j = 1, 2, \ldots, m$ we obtain $\mathcal{L} = \bigoplus_{j=1}^{m} \ker J(k_j)^{r_j} \in \operatorname{Lat}\{N\}'$, and this yields

$$A_{ij} \ker J(k_j)^{s+k_j-k_i} \subset \ker J(k_i)^s$$

for $j < i$ and every s; $0 \leqslant s \leqslant k_i$. It follows that $A_{ij} = (X, 0)$ with a lower-triangular $k_i \times k_i$ matrix X. This completes the proof. □

Corollary. *A nilpotent operator $N \in \mathcal{B}(H)$ is hyperreflexive if and only if $N = 0$.*

Proof. This is an obvious consequence of the descriptions (4) and (5) of the commutant of N and of the algebra $\operatorname{Alg}\operatorname{Lat}\{N\}'$, respectively. □

We shall need the following simple result:

Lemma 2. *The double commutant of every nilpotent operator N consists of polynomials in N.*

Proof. We use the models (1), (2) and (4) of N and of its commutant, respectively. Since $\{N\}'' \subset \{N\}'$, $A \in \{N\}''$ has a block decomposition (A_{ij}) satisfying (4). For $p = 1, 2, \ldots, m$ let $B_p \in \{N\}'$ have the block $B_{pp} = I$ and all other blocks $B_{ij} = 0$.

$$AB_p = B_pA \Rightarrow A_{ij} = 0 \quad \text{for all } i \neq j.$$

There exist polynomials $p_1, p_2, \ldots, p_m$ such that $p_i(J(k_i)) = A_{ii}$ $(i = 1, 2, \ldots, m)$. Setting $C_j \in \{N\}'$ for $j \in \{2, 3, \ldots, m\}$ to be the matrix with the block $C_{1j} = \begin{pmatrix} 0 \\ I \end{pmatrix}$ and all other blocks 0 we obtain from $AC_j = C_j A$

$$p_j(J(k_j)) = p_1(J(k_j)), \quad j = 1, 2, \ldots, m$$

and so $A = p_1(N)$. □

2. General operators in finite dimensional spaces

The investigation of a general linear operator $T \in \mathscr{B}(H)$ can be reduced to the investigation of the nilpotent operators similarly as in [2] and [4].

Theorem 3. *Let $T \in \mathscr{B}(H)$ have the minimum polynomial $m_T(\lambda)$, $m_T(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i)^{m_i}$. Let $H_i = \ker(T - \lambda_i I)^{m_i}$.*

Then the following assertions hold:

(i) *$H_i \in \operatorname{Lat}\{T\}'$ for $i = 1, 2, \ldots, n$,*

(ii) *$T_i = T|H_i$ is hyperreflexive if and only if $m_i = 1$,*

(iii) *$H = H_1 \oplus H_2 \oplus \ldots \oplus H_n$,*

(iv) *T is hyperreflexive if and only if all operators T_i are hyperreflexive.*

Therefore T is hyperreflexive if and only if it is similar to a diagonal operator.

Proof. The assertion (i) is an easy consequence of the fact that

$$(T - \lambda_i I)^{m_i} \in \{T\}''.$$

To prove (ii) let us observe that $\{T_i\}' = \{T_i - \lambda_i I\}'$. The operator $T_i - \lambda_i$ is nilpotent and so the assertion (ii) follows from the corollary of Theorem 1.

The rest of this theorem is a consequence of the following lemma: □

Lemma 4. *Let H_1, H_2 be finite dimensional spaces. Let $H = H_1 \oplus H_2$ and let $X \in \mathscr{B}(H_1)$, $Y \in \mathscr{B}(H_2)$. Then the following assertions are equivalent:*

(1) *The minimum polynomials m_X, m_Y of X, Y are relatively prime.*

(2) $\operatorname{Alg}(X \oplus Y) = \operatorname{Alg} X \oplus \operatorname{Alg} Y$.

(3) $\operatorname{Lat}(X \oplus Y) = \operatorname{Lat} X \oplus \operatorname{Lat} Y$.

(4) $\operatorname{Alg}\operatorname{Lat}(X \oplus Y) = \operatorname{Alg}\operatorname{Lat} X \oplus \operatorname{Alg}\operatorname{Lat} Y$.

(5) $\{X \oplus Y\}' = \{X\}' \oplus \{Y\}'$.

(6) $\operatorname{Lat}\{X \oplus Y\}' = \operatorname{Lat}\{X\}' \oplus \operatorname{Lat}\{Y\}'$.

(7) $\operatorname{Alg}\operatorname{Lat}\{X \oplus Y\}' = \operatorname{Alg}\operatorname{Lat}\{X\}' \oplus \operatorname{Alg}\operatorname{Lat}\{Y\}'$.

(8) $\{X \oplus Y\}'' = \{X\}'' \oplus \{Y\}''$.

Proof. The proof of this theorem can be found e.g. in [4] and [6]. Let us recall the proof of (1) $\Leftrightarrow$ (2).

If (1) holds then there exist polynomials a, b such that $1 = am_X + bm_Y$. Therefore

$$I \oplus 0 = (b \cdot m_Y)(X \oplus Y) \in \mathrm{Alg}(X \oplus Y) \quad \text{and}$$
$$0 \oplus I = (a \cdot m_X)(X \oplus Y) \in \mathrm{Alg}(X \oplus Y).$$

This shows that (2) holds.

If (2) holds then there exists a polynomial q such that $q(X \oplus Y) = I \oplus 0$. Let us denote by p the least common divisor of m_X and m_Y, and let $f = m_X/p$. Since $q(Y) = 0$ p divides q, consequently $m_X = p \cdot f$ divides $q \cdot f$. It follows that $f(X) = I \cdot f(X) = q(X)f(X) = (q \cdot f)(X) = 0$ and so $p = 1$ and (1) holds.

By a simple computation [6] it can be also proved that (n) implies ($n+1$) for $n = 2, 3, \ldots, 7$. Using Lemma 2 we obtain easily that the assertions (2) and (8) are identical. This completes the proof of the lemma. □

3. Algebraic operators in separable spaces

The preceding results can be easily proved also for algebraic operators in infinite dimensional Hilbert spaces. To show this let us suppose that H is a complex separable (infinite dimensional) Hilbert space and that $T \in \mathscr{B}(H)$ is algebraic, i.e. there exists a polynomial p with complex coefficients such that $p(T) = 0$. In this case the notion of minimum polynomial makes sense and Lemma 3 remains true (with the same proof).

First we consider a nilpotent operator $T \in \mathscr{B}(H)$. Let $n \geqslant 1$ be an integer such that $T^n = 0$ and $T^{n-1} \neq 0$. We may assume that $\|T\| \leqslant 1$ (if $\|T\| > 1$, we replace T by $T/\|T\|$). So T is a contraction of class C_0 in the sense of [5, Chap. III.4] and we may use the theory of Jordan models of C_0-contractions [1, Theorem III.5.1].

Let H^2 denote the Hardy space of analytic functions in the unit circle. The minimal function of T is $m(\lambda) = \lambda^n$. By [2, Theorem B] T is hyperreflexive if and only if the operator $S(m) = P_m S|H(m)$ is hyperreflexive. Here $H(m) = (mH^2)^\perp$, P_m is the orthogonal projection onto $H(m)$ and $(Su)(\lambda) = \lambda u(\lambda)$ is the unilateral shift. In our case $m(\lambda) = \lambda^n$ and $S(m)$ is a nilpotent operator on the n-dimensional Hilbert space $H(m)$ with $S(m)^{n-1} \neq 0$. By the corollary of Theorem 1 this is possible only if $n = 1$, i.e. $T = 0$. Therefore the following analogue of Theorem 3 holds:

Theorem 5. *Let $T \in \mathscr{B}(H)$ be an algebraic operator having the minimum polynomial $m_T(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i)^{m_i}$. Let $H_i = \ker(T - \lambda_i I)^{m_i}$.*

Then the following assertions hold:

(i) $H_i \in \operatorname{Lat}\{T\}'$ *for* $i = 1, 2, \ldots, n$,

(ii) $T_i = T|H_i$ *is hyperreflexive if and only if* $m_i = 1$,

(iii) $H = H_1 \oplus H_2 \oplus \ldots \oplus H_n$,

(iv) T *is hyperreflexive if and only if all operators* T_i *are hyperreflexive.*

Remarks and open problems

1. In [2] a characterization of reflexive operators (for $\dim H < \infty$) was given. It follows from this characterization and from Theorem 3 that if $T \in \mathcal{B}(H)$ is hyperreflexive, then it is also reflexive. The other implication is not true, e.g. the operator $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is reflexive, but it is not hyperreflexive.

In general (i.e. for $\dim H = \infty$) it is an open problem whether each hyperreflexive operator must be reflexive.

2. The following question is also a natural open problem:

Is a hyperreflexive quasinilpotent operator necessarily equal to 0?

3. The assertion of Lemma 2 holds for every operator (in a finite dimensional space). This can be proved combining Lemma 2, Theorem 3 and Lemma 4. A slightly different proof can be found in a recent book of R.A. Horn and C.R. Johnson [7, Theorem 4.4.19].

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