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DA PRATO-ZABCZYK'S MAXIMAL INEQUALITY REVISITED I

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Summary. Existence, uniqueness and regularity of mild solutions to semilinear non-autonomous stochastic parabolic equations with locally lipschitzian nonlinear terms is investigated. The adopted approach is based on the factorization method due to Da Prato, Kwapień and Zabczyk.

Keywords: stochastic evolution equations, regularity properties, factorization method

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0. INTRODUCTION AND PRELIMINARIES

In their seminal paper [7] G. Da Prato and J. Zabczyk proved that any mild solution of a stochastic semilinear evolution equation

$$d\xi = A\xi \, dt + f(t,\xi) \, dt + g(t,\xi) \, dw, \quad \xi(0) = \xi_0$$

in a Hilbert space H has almost surely continuous sample paths in H under minimal restrictions: A is an infinitesimal generator of a strongly continuous semigroup, ξ_0 measurable and independent with an infinite-dimensional Brownian motion w, f and g Lipschitz continuous on H. The result is not only important by itself, providing an answer to a problem opened for a long time, but maybe even more significant is its proof, based on the factorization method, introduced in the paper [5] to solve the same problem for linear equations. In [5] the method was also modified for sectorial operators A and the continuity of paths in some interpolation spaces between H and $Dom(A)$ was established.

This text stemmed out from my effort to understand the strength of the factorization technique. It turned out that we can extend Da Prato and Zabczyk's results in three ways:

(i) Non-autonomous problems of the form

$$d\xi = A(t)\xi\, dt + f(t,\xi)\, dt + g(t,\xi)\, dw$$

will be considered; in particular, we replace semigroups in the definition of a mild solution by two-parameter evolution systems.

(ii) In the parabolic case, we will establish existence and regularity of solutions to equations whose nonlinear terms are defined and lipschitzian only on a Banach space E embedded continuously into H. Such a problem has been thouroughly investigated if the noise is additive, i.e. for the diffusion coefficient g constant, but we aim at treating the case of a state-dependent noise.

(iii) We will prove existence of (possibly exploding) solutions to equations the coefficients of which are Lipschitz only on bounded sets in E.

The theorems presented here are anything but suprising and some of them may be even known (at least in the autonomous case). I find it interesting, nonetheless, that all the results can be infered in a unified manner, based on the factorization method and various local uniqueness statements.

In order to be able to state our results let us introduce some notation and assumptions which will be used in the sequel.

(A) *Let* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$ *be a stochastic basis,* H, Y *real separable Hilbert spaces,* $w(t)$ *a* Y*-valued (possibly cylindrical) Wiener process with a covarince operator* $Q \in \mathcal{L}(Y)$, $Q \geqslant 0$.

Hereafter $\mathcal{L}(V, Z)$ denotes the space of all continuous linear operators from a Banach space V into a Banach space Z; if there is a danger of confusion, we will denote its norm by $||.||_{V\to Z}$. If $G \in \mathcal{L}(Y, H)$, then we set $||G||_Q^2 \equiv \mathrm{tr}\{GQG^*\}$, that is, $||G||_Q$ is the Hilbert-Schmidt norm of the operator $GQ^{1/2}$. If $(V, ||.||_*)$ is a Banach space, $r \geqslant 1$, then $||.||_{r,*}$ stands for the norm in the space $L^r(\Omega; V)$ of V-valued Bochner r-integrable functions. In the particular case $V = \mathbb{R}$, however, we will use the notation $|.|_r$. The space of λ-Hölder continuous V-valued functions on a metric space U will be denoted by $\mathcal{C}^{0,\lambda}(U; V)$, $\mathcal{C}(U; V)$ will stand for the space of continuous V-valued functions.

Now let us make precise the meaning of the notion of an evolution system. Setting $\Delta(T) = \{(s,t) \in [0,T], s \leqslant t\}$, we say that an operator-valued function U on $\Delta(T)$ is an evolution system (or an abstract fundamental solution) provided the following hypothesis is fulfilled:

(E) *Let* $U\colon \Delta(T) \longrightarrow \mathcal{L}(H)$ *be a mapping such that*

(i) $\forall s \in [0,T] \quad U(s,s) = I$;

(ii) $\forall s,t,u \in [0,T],\ s \leqslant u \leqslant t, \quad U(t,u)U(u,s) = U(t,s)$;

(iii) $\forall x \in H \quad U(.,.)x\colon \Delta(T) \longrightarrow H$ *is continuous.*

Occasionally we will use the notation $\|U\|_\infty \equiv \sup\{\|U(t,s)\|, (s,t) \in \Delta(T)\}$. Let $A(t)\colon Dom(A(t)) \longrightarrow H$ be closed linear operators in H. If

$$D_1U(t,s)v = A(t)U(t,s)v, \; D_2U(t,s)v = -U(t,s)A(s)v$$

for $0 \leqslant s < t \leqslant T$ and any $v \in Z$, where Z is a linear subspace contained in each $Dom(A(t))$, then we say that U is an evolution system for $\{A(t), 0 \leqslant t \leqslant T\}$ (with a regularity subspace Z). We denote by D_1U, D_2U the derivatives of $U(.,.)$ with respect to the first and second variable, respectively. Let us note that any C_0-semigroup is a particular case of an evolution system satisfying (E).

If one supposes that some "parabolicity" assumptions are fulfilled for the operators $\{A(t)\}_{t\in[0,T]}$, then much more regular evolution systems can be constructed. We summarize the assumptions we will use in the following hypothesis.

(P) 1) $A(t)\colon Dom(A(t)) \longrightarrow H$, $0 \leqslant t \leqslant T$, are closed densely defined operators and $Dom(A(t)) = Dom(A(0))$, $0 \leqslant t \leqslant T$.

2) $\exists M \geqslant 1 \; \forall \lambda \in \mathbb{C}, \operatorname{Re}\lambda \geqslant 0, \forall t \in [0,T] \quad \lambda I - A(t)$ is invertible and

$$\|(\lambda I - A(t))^{-1}\| \leqslant \frac{M}{1+|\lambda|}.$$

3) $\exists L^* \geqslant 0 \quad \exists \varrho \in]0,1] \quad \forall s,t,\tau \in [0,T]$

$$\|[A(t) - A(s)]A(\tau)^{-1}\| \leqslant L^*|t-s|^\varrho.$$

4) $\forall \alpha \in]0,1[\quad \exists L_\alpha > 0 \quad \forall t \in [0,T] \quad \forall x \in Dom((-A(0))^\alpha)$

$$Dom((-A(t))^\alpha) = Dom((-A(0))^\alpha),$$

and

$$L_\alpha^{-1}\|(-A(0))^\alpha x\| \leqslant \|(-A(t))^\alpha x\| \leqslant L_\alpha \|(-A(0))^\alpha x\|.$$

As is well known (see e.g. [27], Th. 5.6.1), under the assumptions (P1)–(P3) there exists a unique evolution system $U\colon \Delta(T) \longrightarrow \mathcal{L}(H)$ such that

$$\|U(t,s)\| \leqslant C, \quad 0 \leqslant s \leqslant t \leqslant T; \tag{0.1}$$

$$U(.,s) \in C^1(]s,T];\mathcal{L}(H)), \quad 0 \leqslant s < T, \quad D_1U(t,s) = A(t)U(t,s), \text{ and}$$

$$\|A(t)U(t,s)\| \leqslant \frac{C}{t-s}, \quad 0 \leqslant s < t \leqslant T; \tag{0.2}$$

$$U(t,.)x \in C^1([0,t[;H), \quad 0 < t \leqslant T, \ x \in Dom(A(0)), \text{ and}$$

$$(0.3) \qquad D_2U(t,s)x = -U(t,s)A(s)x, \quad 0 \leqslant s < t \leqslant T;$$

$$(0.4) \qquad \|A(t)U(t,s)A(s)^{-1}\| \leqslant C, \quad 0 \leqslant s \leqslant t \leqslant T,$$

for some constant C (which may differ in various estimates). The assumption (P2) also implies that the fractional powers $(-A(t))^\alpha$, $\alpha \in \mathbf{R}$, are well defined, hence (P4) makes sense. Set $H_\alpha(t) \equiv Dom((-A(t))^\alpha)$ and endow $H_\alpha(t)$ with the graph norm. The assumption (P4) is weaker than the supposition

$$(0.5) \qquad [H_\alpha(t), H_\beta(t)]_\varrho = H_{\alpha(1-\varrho)+\beta\varrho}(t), \ 0 \leqslant \alpha < \beta \leqslant 1, \ \varrho \in]0,1[, \ t \in [0,T]$$

where $[.,.]_\varrho$ denotes the usual complex interpolation functor. (Indeed, $H_1(t) = H_1(0)$ by (P1), $H_0(t) = H, 0 \leqslant t \leqslant T$, and (P3) implies the existence of a constant $L_1 > 0$ such that

$$L_1^{-1}\|A(0)x\| \leqslant \|A(t)x\| \leqslant L_1\|A(0)x\|$$

for any $x \in H_1(0)$ and $t \in [0,T]$.) Theorem 3.3 in [1] states that (0.5) is valid (for any $\alpha, \beta \in \mathbf{R}, \alpha < \beta$) provided

$$(0.6) \qquad \forall t \in [0,T] \ \ \exists \varepsilon > 0 \quad \sup_{\tau \in [-\varepsilon,\varepsilon]} \|(-A(t))^{i\tau}\|_{H \to H} < \infty.$$

The estimate (0.6) holds, in particular, for self-adjoint operators $A(t)$; in that case, however, (P4) can be proved directly (cf. [13], Th. 1.3). Other conditions for the validity of (0.6) will be mentioned in Example 6.1.

It should be remarked that if an operator A: $Dom(A) \longrightarrow H$ gives rise to an analytic semigroup on H, then the operator $A - \beta I$ satisfies (P) for some $\beta \in \mathbf{R}$.

For simplicity we set $H_\alpha \equiv H_\alpha(0)$, $\|x\|_\alpha \equiv \|(-A(0))^\alpha x\|$. (Sometimes we will use the symbols $H_0, \|.\|_0$ for the original space H and its norm, respectively.)

In [31], Th. 2, it is established that under (P1)–(P3) one has

$$\|U(t,s)\|_{H_\beta(s) \to H_\alpha(t)} \leqslant C|t-s|^{\beta-\alpha}, \ 0 \leqslant \beta \leqslant \alpha \leqslant 1,$$

$$\|U(t+v,s) - U(t,s)\|_{H_\beta(s) \to H_\alpha(0)} \leqslant Cv^{\beta-\alpha}, \ 0 \leqslant \alpha \leqslant \beta \leqslant 1.$$

Since we assume in (P4) that the spaces H_α are independent of t we get three useful estimates (which are well-known for analytic semigroups, cf. e.g. [27], Th. 2.6.13 or [14], Th. 1.4.3):

$$(0.7) \qquad \|U(t,s)\|_{H_\alpha \to H_\gamma} \leqslant \frac{C_\gamma}{(t-s)^{\gamma-\alpha}}$$

for $0 \leqslant s < t \leqslant T$, $0 \leqslant \alpha < \gamma \leqslant 1$;

$$\|U(t,s) - I\|_{H_{\alpha+\delta} \to H_\delta} \leqslant \hat{C}(t-s)^\alpha \tag{0.8}$$

for $\delta \in [0,1[$, $\alpha \in]0, 1-\delta[$; and

$$\sup\{\|U(t,s)\|_{H_\delta \to H_\delta}; (s,t) \in \Delta(T)\} \leqslant C'_\delta < \infty, \tag{0.9}$$

$0 \leqslant \delta \leqslant 1$. If (0.5) holds, then these estimates can be easily derived by interpolation. Indeed, let us realize that (0.2) means $\|U(t,s)\|_{H_0 \to H_1} \leqslant C(t-s)^{-1}$, which together with (0.1) implies

$$\|U(t,s)\|_{[H_0,H_0]_\mu \to [H_0,H_1]_\mu} = \|U(t,s)\|_{H_0 \to H_\mu} \leqslant C(t-s)^{-\mu},$$

$\mu \in]0,1[$. By (0.4), $\|U(t,s)\|_{H_1 \to H_1} \leqslant C$, hence

$$\|U(t,s)\|_{[H_0,H_1]_\alpha \to [H_\mu,H_1]_\alpha} = \|U(t,s)\|_{H_\alpha \to H_{(1-\alpha)\mu+\alpha}} \leqslant C(t-s)^{-\mu(1-\alpha)}$$

for $\alpha \in]0,1[$. If $\gamma \in]\alpha,1]$ then one can set $\mu = (\gamma-\alpha)/(1-\alpha)$ to obtain (0.7). Further, one obviously has $\|U(t,s) - I\|_{H \to H} \leqslant C$. Fix $\theta \in [0,1[$ and $x \in H_1$, then

$$\begin{aligned}\|U(t,s)x - x\|_\theta &= \left\| \int_s^t D_2U(t,r)x \,\mathrm{d}r \right\|_\theta = \left\| \int_s^t U(t,r)A(r)x \,\mathrm{d}r \right\|_\theta \\ &= \left\| \int_s^t (-A(0))^\theta U(t,r)A(r)x \,\mathrm{d}r \right\| \leqslant \int_s^t C_\theta (t-r)^{-\theta} \|A(r)x\| \,\mathrm{d}r \\ &\leqslant \text{const.}\,(t-s)^{1-\theta}\|x\|_1\end{aligned}$$

by (0.3) and (0.7), that is

$$\|U(t,s) - I\|_{H_1 \to H_\theta} \leqslant C(t-s)^{1-\theta}, \quad 0 \leqslant s \leqslant t \leqslant T.$$

Using again the complex interpolation, we obtain $(0 < \varkappa < 1)$

$$\|U(t,s) - I\|_{[H_0,H_1]_\varkappa \to [H_0,H_\theta]_\varkappa} = \|U(t,s) - I\|_{H_\varkappa \to H_{\theta\varkappa}} \leqslant C(t-s)^{(1-\theta)\varkappa}.$$

Now let $\delta \in [0,1[$, $\alpha \in]0, 1-\delta[$ be arbitrary, set $\varkappa = \alpha + \delta$, $\theta = \delta/(\alpha+\delta)$, and (0.8) follows. The formulae (0.1) and (0.4) show that $U(.,.)$ is bounded on $\Delta(T)$ as an $\mathcal{L}(H)$- and $\mathcal{L}(H_1)$-valued function, respectively, so (0.5) implies the validity of (0.9) as well.

Finally, let us note that by [2] (Th. 7.2 and Remark 9.1), $U(t,s)$ restricted to H_δ is an evolution system in H_δ if $0 \leqslant \delta < \varrho$, where ϱ is the constant which appears in (P3). Hence, in particular,

$$(0.10) \qquad \lim_{t \to s} \|U(t,s)x - x\|_\delta = 0, \quad x \in H_\delta, \quad \delta < \varrho.$$

The paper is organized as follows. In the next section, we state the main theorems. In Section 2 we quote four auxiliary propositions which will be needed frequently in the sequel. Section 3 is devoted to proofs of maximal inequalities for stochastic convolutions. Section 4 contains the proofs of theorems on mild solutions to semilinear stochastic evolution equations with globally Lipschitz continuous nonlinearities; in Section 5 the case of locally lipschitzian nonlinearities is treated. In the sixth section two illustrative examples are discussed.

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1. Main results

First, we can state the maximal inequality

Theorem 1.1. *Let the assumptions* (A) *and* (P) *be fulfilled. Let* $p > 2$, $\delta \in [0, \frac{1}{2} - \frac{1}{p}[$. *Let* $\psi\colon [0,T] \times \Omega \longrightarrow \mathcal{L}(Y,H)$ *be an* $(\mathcal{F}_t)$*-adapted measurable stochastic process such that*

$$\mathsf{E} \int_0^T \|\psi(s)\|_Q^p \, \mathrm{d}s < \infty.$$

Then

$$\mathsf{E} \sup_{0 \leqslant t \leqslant T} \left\| \int_0^t U(t,s)\psi(s) \, \mathrm{d}w(s) \right\|_\delta^p \leqslant C \mathsf{E} \int_0^T \|\psi(s)\|_Q^p \, \mathrm{d}s,$$

where C *depends only on* p, δ, T *and* U. *Moreover, the process*

$$\int_0^{\cdot} U(\,.\,,s)\psi(s) \, \mathrm{d}w(s)$$

has a modification with almost all sample paths λ*-Hölder continuous in* H_δ *with any* $\lambda \in [0, \frac{1}{2} - \frac{1}{p} - \delta[$.

Note that the assumptions of the theorem imply $\int_0^T \mathsf{E}\|\psi(s)\|_Q^2 \, \mathrm{d}s < \infty$, hence the stochastic integral investigated is well defined. The proof of Theorem 1.1 (which is presented in the third section) yields also the estimate

$$\mathsf{E}\Bigg(\sup_{0 \leqslant r,t \leqslant T} \frac{\| \int_0^t U(t,s)\psi(s) \, \mathrm{d}w(s) - \int_0^r U(r,s)\psi(s) \, \mathrm{d}w(s) \|_\delta}{|t-r|^\lambda} \Bigg)^p$$
$$\leqslant C(\lambda) \, \mathsf{E} \int_0^T \|\psi(s)\|_Q^p \, \mathrm{d}s$$

for any $p > 2$, $\delta \in [0, \frac{1}{2} - \frac{1}{p}[$, $\lambda \in]0, \frac{1}{2} - \frac{1}{p} - \delta[$, with a constant $C(\lambda)$ dependent only on p, T, δ, λ and U.

Further, tracing the proof of Theorem 1.1 one can check easily that it is not necessary to use the consequences of (P) if $\delta = 0$, hence we have also

Theorem 1.2. *Let the assumptions* (A) *and* (E) *be fulfilled. Let $p > 2$. Let ψ: $[0,T] \times \Omega \to \mathcal{L}(Y, H)$ be an $(\mathcal{F}_t)$-adapted measurablem stochastic process such that*

$$\mathsf{E} \int_0^T \|\psi(s)\|_Q^p \, \mathrm{d}s < \infty.$$

Then

$$\mathsf{E} \sup_{0 \leqslant t \leqslant T} \left\| \int_0^t U(t,s)\psi(s) \, \mathrm{d}w(s) \right\|^p \leqslant C \mathsf{E} \int_0^T \|\psi(s)\|_Q^p \, \mathrm{d}s,$$

where C depends only on p, T and $\|U\|_\infty$. Moreover, the process

$$\int_0^{\cdot} U(\cdot\,, s)\psi(s) \, \mathrm{d}w(s)$$

has a modification with almost all sample paths continuous in H.

Now, let us turn to the problem of solvability of the semilinear stochastic evolution equation

$$\mathrm{d}x(t) = [A(t)x(t) + f(t, x(t))] \, \mathrm{d}t + \sigma(t, x(t)) \, \mathrm{d}w(t), \tag{1.2}$$

$$x(0) = \varphi, \tag{1.3}$$

in H, where the operators $A(t)$ give rise to an evolution system U in H satisfying (E), and the nonlinear terms f, σ are defined on appropriate subspaces of H and are in a sense regular compared with $A(t)$. About the initial condition φ let us assume

(I) *Let $\varphi \colon \Omega \longrightarrow H$ be an $\mathcal{F}_0$-measurable random variable.*

Definition 1.1. An H-valued $(\mathcal{F}_t)$-adapted measurable stochastic process $\{x(t), 0 \leqslant t \leqslant T\}$ is called a *mild solution* to the problem (1.2), (1.3) on $[0,T]$, if

(i) almost surely, $f(t, x(t))$ and $\sigma(t, x(t))$ are well defined for almost all $t \in [0,T]$ and

$$\int_0^T \|f(s, x(s))\|_0 \, \mathrm{d}s < \infty, \qquad \int_0^T \|\sigma(s, x(s))\|_Q^2 \, \mathrm{d}s < \infty; \tag{1.4}$$

(ii) for any $t \in [0,T]$ the equality

$$x(t) = U(t,0)\varphi + \int_0^t U(t,s) f(s, x(s)) \, \mathrm{d}s + \int_0^t U(t,s)\sigma(s, x(s)) \, \mathrm{d}w(s) \tag{1.5}$$

holds almost surely.

Note that the assumption (1.4) implies the existence of the integrals in (1.5).

First, let us adopt the following hypothesis.

(L_δ) *Let $\delta \in [0, \frac{1}{2}[$, let $f: [0,T] \times H_\delta \longrightarrow H$, $\sigma: [0,T] \times H_\delta \longrightarrow \mathcal{L}(Y, H)$ be measurable mappings such that*

$\exists K, K^* < \infty \quad \forall x, y \in H_\delta \quad \forall t \in [0,T]$

$$\|f(t,x) - f(t,y)\|_0 + \|\sigma(t,x) - \sigma(t,y)\|_Q \leqslant K\|x-y\|_\delta,$$

$$\|f(t,x)\|_0 + \|\sigma(t,x)\|_Q \leqslant K^*(1 + \|x\|_\delta).$$

Then the theorem on the existence of mild solutions reads as follows. (Recall that $\varrho > 0$ is the parameter introduced in the assumption (P3).)

Theorem 1.3. *Let the assumptions (A), (P), (I), (L_δ) be fulfilled. Then there exists a mild solution of the problem (1.2), (1.3). Moreover:*

(i) *The mild solution is unique — up to a modification — within the class of stochastic processes with sample paths in $L^2([0,T], H_\delta)$.*

(ii) *Let φ be H_α-valued almost surely for some $\alpha \in [0, \frac{1}{2}[$. Then the mild solution has a modification x such that $x(.,\omega) \in \mathcal{C}([0,T]; H_\beta) \cap \mathcal{C}^{0,\lambda}([a,T]; H_\varkappa)$ for almost every $\omega \in \Omega$ for any $a > 0$, $\varkappa \in [0, \frac{1}{2}[$, $\lambda \in [0, \frac{1}{2} - \varkappa[$, and $\beta \in [0, \alpha[$. If, moreover, $\alpha < \varrho$, then $x(.,\omega) \in \mathcal{C}([0,T]; H_\alpha)$ almost surely.*

(iii) *If $\varphi \in L^p(\Omega; H_\alpha)$, $p \geqslant 2$, $\alpha \in [0, \delta]$, then $x \in \mathcal{C}(]0,T]; L^p(\Omega; H_\delta))$ and*

$$\sup_{0 \leqslant t \leqslant T} t^{\delta - \alpha} \|x(t)\|_{p,\delta} \leqslant C^*(1 + \|\varphi\|_{p,\alpha}).$$

Furthermore, $x \in \mathcal{C}([0,T]; L^p(\Omega; H_\beta)) \cap \mathcal{C}^{0,\mu}([a,T]; L^p(\Omega; H_\delta))$ for any $a > 0$, $\mu \in [0, \frac{1}{2} - \delta[$, and $\beta \in [0, \alpha[$. If $\alpha < \varrho$, then $x \in \mathcal{C}([0,T]; L^p(\Omega; H_\alpha))$ as well.

(iv) *If, moreover, $p > 2$ and $\varphi \in L^p(\Omega; H_\delta)$, then for arbitrary $\gamma \in [0, \frac{1}{2} - \frac{1}{p}[$, $\gamma \leqslant \delta$, one has*

$$\left| \sup_{0 \leqslant t \leqslant T} \|x(t)\|_\gamma \right|_p \leqslant C^+(1 + \|\varphi\|_{p,\delta}).$$

For any $\gamma \in [0, (\frac{1}{2} - \frac{1}{p}) \wedge \delta[$, x belongs to the space $L^p(\Omega; \mathcal{C}([0,T]; H_\gamma))$; the same assertion holds for $\gamma = \delta$ provided $\delta < (\frac{1}{2} - \frac{1}{p}) \wedge \varrho$.

The constants C^* and C^+ depend only on K^*, T, p, δ, and U.

Remark. If the assumption (L_δ) is fulfilled, then (L_ϑ) holds for any $\vartheta \geqslant \delta$, thus we may replace δ by an arbitrary $\vartheta \in [\delta, \frac{1}{2}[$ in all statements of Theorem 1.3.

As in the case of Theorem 1.1, one can easily observe that no regularity of the evolution system is needed if $\delta = 0$. So we will virtually prove also the following theorem.

Theorem 1.4. *Let the assumptions* (A), (E), (I), (L_0) *be fulfilled. Then there exists a mild solution of the equation* (1.2), (1.3), *which is unique — up to a modification — within the class of stochastic processes with sample paths in* $L^2([0,T],H)$. *This mild solution has a modification* x *such that* $x(.,\omega) \in \mathcal{C}([0,T];H)$ *almost surely. If* $\varphi \in L^p(\Omega;H)$, $p \geqslant 2$, *then* $x \in \mathcal{C}([0,T];L^p(\Omega;H))$ *and*

$$\sup_{0 \leqslant t \leqslant T} \|x(t)\|_{p,0} \leqslant C^*(1 + \|\varphi\|_{p,0})$$

with a constant C^* *which depends only on* K^*, T, p, *and* $\|U\|_\infty$.

If, moreover, $p > 2$ *then* $x \in L^p(\Omega;\mathcal{C}([0,T];H))$ *and*

$$\left| \sup_{0 \leqslant t \leqslant T} \|x(t)\|_0 \right|_p \leqslant C^+(1 + \|\varphi\|_{p,0})$$

for a constant C^+ *which depends only on the same set of parameters as* C^*.

The next step consists in weakening the quite restrictive global Lipschitz continuity assumption (L_δ). We will assume that the nonlinear terms in (1.2) are lipschitzian only on bounded sets, that is, we adopt the following hypothesis:

(LL_δ) *Let* $\delta \in [0, \frac{1}{2}[$, *let* $f\colon [0,T] \times H_\delta \longrightarrow H$, $\sigma\colon [0,T] \times H_\delta \longrightarrow \mathcal{L}(Y,H)$ *be measurable functions and*

$$\forall N \in \mathbf{N} \quad \exists K_N < \infty \quad \forall t \in [0,T] \quad \forall x, y \in H_\delta,\ \|x\|_\delta, \|y\|_\delta \leqslant N$$

$$\|f(t,x) - f(t,y)\|_0 + \|\sigma(t,x) - \sigma(t,y)\|_Q \leqslant K_N \|x - y\|_\delta.$$

Hitherto, we could work with solutions which were defined on the whole (a priori fixed) interval $[0,T]$. It is well-known that even for ordinary differential equations the local lipschitzianity of the right-hand side of the equation is not sufficient for the existence of global solutions, so we have to generalize our concept of solution (cf. e.g. [26], Def. 34.4).

Definition 1.2. A pair (x,ε), where ε is a $]0,T]$-valued stopping time and $\{x(t), 0 \leqslant t < \varepsilon\}$ an $(\mathcal{F}_t)$-adapted H_δ-valued measurable stochastic process, is called a *local mild solution* to (1.2), (1.3) in H_δ with an *explosion time* ε provided

(a) $\limsup_{t \to \varepsilon(\omega)} \|x(t,\omega)\| = +\infty$ on the set $\{\omega; \varepsilon(\omega) < T\}$;

(b) there exists a sequence $\{\varepsilon^{(n)}\}_{n=1}^{\infty}$ of stopping times growing to ε such that

$$\int_0^{\varepsilon^{(n)}(\omega)} \|f(t,x(t,\omega))\|_0 \,\mathrm{d}t < \infty, \qquad \int_0^{\varepsilon^{(n)}(\omega)} \|\sigma(t,x(t,\omega))\|_Q^2 \,\mathrm{d}t < \infty$$

for any $n \in \mathbf{N}$ almost surely,

(c) for any $t \in [0,T]$ and every $n \in \mathbf{N}$ the equality

$$x(t \wedge \varepsilon^{(n)}) = U(t \wedge \varepsilon^{(n)},0)\varphi + \int_0^{t\wedge\varepsilon^{(n)}} U(t \wedge \varepsilon^{(n)},s)f(s,x(s))\,\mathrm{d}s + \int_0^{t\wedge\varepsilon^{(n)}} U(t \wedge \varepsilon^{(n)},s)\sigma(s,x(s))\,\mathrm{d}w(s)$$

holds almost surely.

The uniqueness of a local mild solution will be understood in the following sense: we say that there exists at most one local mild solution to the problem (1.2), (1.3) with paths continuous in H_δ, provided for any two local solutions $(x,\varepsilon),(y,\eta)$ satisfying $x(.,\omega) \in \mathcal{C}([0,\varepsilon(\omega)[;H_\delta)$, $y(.,\omega) \in \mathcal{C}([0,\eta(\omega)[;H_\delta)$ for almost every $\omega \in \Omega$ one has $\varepsilon = \eta$ almost surely and $x(t) = y(t)$ holds almost surely on the set $\{\omega; t < \varepsilon(\omega)\}$ for each $t \in [0,T]$.

If it is necessary to distinguish the two notions, we will call the solutions in the sense of Definition 1.1 global solutions. The analogy with a finite-dimensional case leads to the conjecture that global mild solutions exist if the assumption (LL_δ) is combined with the linear growth hypothesis

(LG_δ) $\exists K^* < \infty \quad \forall t \in [0,T] \quad \forall x \in H_\delta$

$$\|f(t,x)\|_0 + \|\sigma(t,x)\|_Q \leqslant K^*(1 + \|x\|_\delta).$$

We are not able to prove, however, that Theorem 1.3 remains valid with the assumptions (LL_δ) and (LG_δ) replacing (L_δ); we have in addition to require the initial data to be such that any solution is continuous in H_δ as $t \to 0^+$.

Theorem 1.5. *Let the assumptions* (A), (P), (I), *and* (LL_δ) *be fulfilled. Assume either that φ is H_δ-valued and $\delta < \varrho$, or that φ is H_ζ-valued for some $\zeta > \delta$. Suppose*

$$\sup_{0\leqslant t\leqslant T} \|f(t,0)\|_0 + \sup_{0\leqslant t\leqslant T} \|\sigma(t,0)\|_Q < \infty. \tag{1.6}$$

Then there exists a local mild solution of the problem (1.2), (1.3). *Moreover,*

(i) *This mild solution has a modification (x,ε) such that $x(.,\omega) \in \mathcal{C}([0,\varepsilon(\omega)[;H_\delta) \cap \mathcal{C}^{0,\lambda}(F;H_\varkappa)$ for almost all $\omega \in \Omega$, any compact set F in $]0,\varepsilon(\omega)[$, and for any $\varkappa \in [0,\frac{1}{2}[$, $\lambda \in [0,\frac{1}{2}-\varkappa[$.*

(ii) *A local mild solution is unique within the class of stochastic processes having a modification with trajectories continuous in H_δ.*

Furthermore, let the hypothesis (LG$_\delta$) *hold. Then*

(iii) *The solution x is global, $x(.,\omega) \in \mathcal{C}([0,T];H_\delta) \cap \mathcal{C}^{0,\lambda}([a,T];H_\varkappa)$ for almost all $\omega \in \Omega$ and any $a > 0$, $\varkappa \in [0,\frac{1}{2}[$, $\lambda \in [0,\frac{1}{2}-\varkappa[$.*

(iv) *If $\varphi \in L^p(\Omega;H_\delta)$, $p \geqslant 2$, then*

$$\sup_{0\leqslant t\leqslant T} \|x(t)\|_{p,\delta} \leqslant C^*(1+\|\varphi\|_{p,\delta}), \tag{1.7}$$

and if, moreover, $p > 2$ and $\delta \in [0,\frac{1}{2}-\frac{1}{p}[$, then

$$\left| \sup_{0\leqslant t\leqslant T} \|x(t)\|_\delta \right|_p \leqslant C^+(1+\|\varphi\|_{p,\delta}), \tag{1.8}$$

where C^*, C^+ are the constants introduced in Theorem 1.3.

Remark. a) An example of the ordinary differential equation $\dot{x} = \xi(t)$, ξ a measurable nonintegrable function, shows that it is inevitable to adopt some condition like (1.6). (Recall that Definition 1.2 requires the explosion time to be strictly positive almost surely.)

b) Like Theorems 1.1 and 1.3 the above result has its "non-parabolic" version which one gets by replacing the assumption (P) with (E) and setting $\delta = \varkappa = \lambda = 0$.

Remark. Only for the sake of a more lucid arrangement we have investigated the equation (1.2) assuming that the coefficients f and σ are nonrandom. Theorems 1.3, 1.4 and 1.5, however, remain valid (and their proofs unchanged), if $f\colon [0,T] \times H_\delta \times \Omega \longrightarrow H, \sigma\colon [0,T] \times H_\delta \times \Omega \longrightarrow \mathcal{L}(Y,H)$ are measurable mappings such that

(i) $f(.,x,.)$ and $\sigma(.,x,.)$ are $(\mathcal{F}_t)$-adapted measurable stochastic processes for any $x \in H_\delta$, and

(ii) $f(.,.,\omega)$, $\sigma(.,.,\omega)$ fulfil the estimates in (L$_\delta$) (or (LL$_\delta$), (LG$_\delta$)) for every $\omega \in \Omega$ with constants K, K^* (or K_N) independent of ω.

Remark on previous research. We would like to mention here a few papers related to some of the topics treated in this paper.

a) The factorization method was introduced in the paper [5] and applied to semilinear problems in [7]. Theorems 1 and 2 from the paper [7] correspond to Theorems 1.2 and 1.4 of the present paper, respectively, with the only difference that C_0-semigroups are used in [7] instead of evolution systems.

b) A weak-type maximal inequality for stochastic convolution integrals with exponentially bounded evolution systems was proved in [18]. Maximal inequalities related to the one stated in Theorem 1.2 appeared in [19] for $p = 2$ and in [34] for $p \geqslant 2$, in both cases for contraction semigroups (or contraction type evolution systems). We must emphasize that the estimates obtained in [34] are more precise than ours. The results from [34] were further extended in [16], again for contraction type semigroups.

Maximal inequality in the spaces H_α was proved in [20] in the case $p = 2$. All the above mentioned results have immediate consequences for the continuity (or Hölder continuity) of paths of stochastic integrals.

c) The regularity of paths of solutions to linear stochastic parabolic problems was established by the factorization method in [5], for alternative proofs see [20] or [11] (cf. also [12]). The existence of solutions continuous in space and time to equations with fairly general drift terms, but with additive noise was discussed e.g. in [23], [24], [25] using random fields and in [6], [8] in the infinite dimensional setting.

Results closely related to our Theorem 1.3 for equations with scalar Wiener process and with A generating a contractive holomorphic semigroup can be found in [4]. (Cf. also the paper [35] where the non-autonomous case is mentioned.) Theorems on regularity of a different type were recently established in [10].

d) Local solutions for stochastic equations in Hilbert spaces with bounded coefficients were treated in [26] and [22], for stochastic evolution equations e.g. in [15] and [19] in the "non-parabolic case", and in [4] in the situation described above.

2. Four auxiliary propositions

In the course of proofs we will need repeatedly the following results.

Lemma 2.1. (i) *Let U be an evolution system satisfying the assumption* (E), *let* $\alpha \in]0,1]$, $p \in]1,\infty]$, $\alpha > \frac{1}{p}$. *Let* $0 \leqslant a \leqslant b \leqslant T$. *For any* $f \in L^p([a,b];H)$ *define*

$$(R_\alpha f)(t) = \int_a^t (t-s)^{\alpha-1} U(t,s) f(s)\,\mathrm{d}s, \quad a \leqslant t \leqslant b.$$

Then R_α is a bounded linear mapping from $L^p([a,b];H)$ into $C([a,b];H)$.

(ii) *If, moreover, the hypothesis* (P) *holds,* $\delta \in [0,1[$ *and* $\lambda^* \equiv \alpha - \frac{1}{p} - \delta > 0$, *then* $R_\alpha \in \mathcal{L}(L^p([a,b];H), C^{0,\lambda}([a,b];H_\delta))$ *for arbitrary* $\lambda \in [0,\lambda^*[$.

The first assertion of this lemma and the Hölder continuity of $R_\alpha f$ were proved by Da Prato, Kwapień and Zabczyk ([5], Lemma 1 and 3, respectively) under the additional hypothesis $U(t,s) = T(t-s)$, $T(t)$ being a strongly continuous (holomorphic, resp.) semigroup in H, but one can repeat their proof literally if one considers the fact that the assumption (P) implies the estimates (0.7), (0.8). The fact that under the assumptions of the assertion (ii), $R_\alpha \in \mathcal{L}(L^p([a,b];H), C([a,b];H_\delta))$ holds, can be established by the following simple argument: By (0.7) and the Hölder inequality we have

$$\begin{aligned}\|R_\alpha f(t)\|_\delta &\leqslant C_\delta \int_a^t (t-s)^{\alpha-\delta-1}\|f(s)\|_0\,\mathrm{d}s\\ &\leqslant C_\delta\Big(\int_a^t \|f(s)\|_0^p\,\mathrm{d}s\Big)^{1/p}\Big(\int_0^{t-a} v^{(\alpha-\delta-1)p/(p-1)}\,\mathrm{d}v\Big)^{(p-1)/p}\\ &\leqslant C_\delta((\alpha p-\delta p-1)/(p-1))^{1/p-1}(b-a)^{\alpha-\delta-1/p}\Big(\int_a^b \|f(s)\|_0^p\,\mathrm{d}s\Big)^{1/p}\end{aligned}$$

for any $t \in [a,b]$.

Further, we will need an estimate for stochastic integrals that is a very particular case of the Burkholder-Davis-Gundy inequality, the Hilbert space version of which follows from [17], Th. 3.1.

Lemma 2.2. *Let the assumption* (A) *be fulfilled. Then for any $p \in [1,\infty[$ there exists a constant $C_p > 0$ such that for any $(\mathcal{F}_t)$-adapted measurable stochastic process $G\colon [0,T]\times\Omega \longrightarrow \mathcal{L}(Y,H)$ satisfying $\int_0^T \|G(t)\|_Q^2\,\mathrm{d}t < \infty$ almost surely one has*

$$\mathsf{E}\sup_{0\leqslant t\leqslant T}\Big\|\int_0^t G(s)\,\mathrm{d}w(s)\Big\|^p \leqslant C_p\mathsf{E}\Big(\int_0^T \|G(s)\|_Q^2\,\mathrm{d}s\Big)^{p/2}.$$

The next result is a useful generalization of Gronwall's inequality; its proof is given in [3], Corollary 8.11 (cf. also Lemma 7.1.1 in [14]).

Lemma 2.3. *Let $g \in L^1([0,T])$, $h \in L^q([0,T])$ be nonnegative functions, $q \in [1,\infty]$. Let $f \in L^1([0,T])$ be such that*

$$f(t) \leqslant h(t) + \int_0^t g(t-s)f(s)\,\mathrm{d}s,\quad 0\leqslant t\leqslant T.$$

Then

$$f(t) \leqslant \sum_{n=0}^{\infty}(G^n h)(t),\quad 0\leqslant t\leqslant T,$$

where G is the Volterra operator given by $Gh(t) = \int_0^t g(t-s)h(s)ds$, $G^0 = I$, and the series on the right-hand side converges in $L^q([0,T])$. There exists a constant $L > 0$, dependent only on the function g, such that

$$\Big\|\sum_{n=0}^{\infty} G^n h\Big\|_{L^q([0,T])} \leqslant L\|h\|_{L^q([0,T])}.$$

In particular, if $h = 0$, then $f = 0$.

Finally, let us quote Theorem 3 from [7], which is a stochastic version of the Fubini theorem and enables us to interchange stochastic and Bochner integrals.

Theorem 2.4. *Let* (A) *be fulfilled, let* $(G, \mathcal{G}, \mu)$ *be a measure space, let* $h\colon [0,T] \times \Omega \times G \longrightarrow \mathcal{L}(Y, H)$ *be a* $\mathcal{B}([0,T]) \otimes \mathcal{F} \otimes \mathcal{G}$*-measurable mapping such that* $h(.,.,x)$ *is an* $(\mathcal{F}_t)$*-adapted stochastic process for each* $x \in G$ *and*

$$\int_G \left(\int_0^T \mathsf{E}\|h(t,x)\|_Q^2 \,\mathrm{d}t \right)^{1/2} \mathrm{d}\mu(x) < \infty.$$

Then

$$\int_G \left(\int_0^T h(t,x)\,\mathrm{d}w(t) \right) \mathrm{d}\mu(x) = \int_0^T \left(\int_G h(t,x)\,\mathrm{d}\mu(x) \right) \mathrm{d}w(t)$$

P*-almost surely.*

3. Proof of Theorem 1.1

Let us choose $\lambda \in [0, \frac{1}{2} - \frac{1}{p} - \delta[$, fix $\alpha \in]\frac{1}{p} + \delta + \lambda, \frac{1}{2}[$ and set

$$Y(s) = \int_0^s (s-r)^{-\alpha} U(s,r)\psi(r)\,\mathrm{d}w(r), \tag{3.1}$$

$0 \leqslant s \leqslant T$. Since

$$\int_0^s \mathsf{E}\|(s-r)^{-\alpha} U(s,r)\psi(r)\|_Q^2 \,\mathrm{d}r \leqslant \|U\|_\infty^2 \int_0^s (s-r)^{-2\alpha} \mathsf{E}\|\psi(r)\|_Q^2 \,\mathrm{d}r$$

and the right-hand side is finite for almost all $s \in [0,T]$ as a convolution of two L^1-functions, the stochastic integral (3.1) is defined for almost every s, which is sufficient in what follows. First we will check that

$$\mathsf{E}\|Y\|_{L^p([0,T];H)}^p = \mathsf{E}\int_0^T \|Y(t)\|^p \,\mathrm{d}t < \infty.$$

Indeed, by Lemma 2.2

$$\begin{aligned}\mathsf{E}\|Y(s)\|^p &\leqslant C_p\, \mathsf{E}\left(\int_0^s \|(s-r)^{-\alpha} U(s,r)\psi(r)\|_Q^2 \,\mathrm{d}r \right)^{p/2} \\ &\leqslant C_p \|U\|_\infty^p\, \mathsf{E}\left(\int_0^s (s-r)^{-2\alpha} \|\psi(r)\|_Q^2 \,\mathrm{d}r \right)^{p/2},\end{aligned}$$

so in virtue of the Young inequality for convolutions we obtain

$$\begin{aligned}\mathsf{E}\int_0^T \|Y(s)\|^p \,\mathrm{d}s &\leqslant C_p\|U\|_\infty^p\, \mathsf{E}\int_0^T \Big(\int_0^s (s-r)^{-2\alpha}\|\psi(r)\|_Q^2 \,\mathrm{d}r\Big)^{p/2} \mathrm{d}s\\ &\leqslant C_p\|U\|_\infty^p \Big(\int_0^T s^{-2\alpha}\,\mathrm{d}s\Big)^{p/2}\Big(\int_0^T \mathsf{E}\|\psi(r)\|_Q^p \,\mathrm{d}r\Big)\\ &= C_p\|U\|_\infty^p (1-2\alpha)^{-p/2} T^{p(1-2\alpha)/2}\Big(\int_0^T \mathsf{E}\|\psi(r)\|_Q^p\,\mathrm{d}r\Big).\end{aligned}$$

Hence, in particular, $Y \in L^p([0,T];H)$ almost surely.

We will complete the proof using the identity

$$\int_0^t U(t,r)\psi(r)\,\mathrm{d}w(r) = \frac{\sin\pi\alpha}{\pi}[R_\alpha Y](t) \quad \text{a.s.} \tag{3.2}$$

valid for any $t \in [0,T]$, where

$$(R_\alpha Y)(t) = \int_0^t (t-s)^{\alpha-1}U(t,s)Y(s)\,\mathrm{d}s,$$

i.e. R_α is the generalized Riemann-Liouville operator introduced in Lemma 2.1. The use of the representation (3.2) is the very core of the factorization method as treated in [5] and [7]. Note that the formula (3.2) follows easily from Therem 2.4 in which we set (for $t \in [0,T]$ arbitrary fixed)

$$h(r,s) = (t-s)^{\alpha-1}(s-r)^{-\alpha}\chi_{[0,s]}(r)U(t,r)\psi(r).$$

By Lemma 2.1 we know that $R_\alpha Y(.,\omega) \in \mathcal{C}^{0,\lambda}([0,T];H_\delta)$ for almost all $\omega \in \Omega$. Finally,

$$\begin{aligned}&\mathsf{E}\sup_{0\leqslant t\leqslant T}\Big\|\int_0^t U(t,s)\psi(s)\,\mathrm{d}w(s)\Big\|_\delta^p\\ &= \Big(\frac{\sin\pi\alpha}{\pi}\Big)^p \mathsf{E}\sup_{0\leqslant t\leqslant T}\|R_\alpha Y(t)\|_\delta^p\\ &= \Big(\frac{\sin\pi\alpha}{\pi}\Big)^p \mathsf{E}\|R_\alpha Y\|_{\mathcal{C}([0,T];H_\delta)}^p\\ &\leqslant \Big(\frac{\sin\pi\alpha}{\pi}\Big)^p |||R_\alpha|||^p\, \mathsf{E}\|Y\|_{L^p([0,T];H)}^p\\ &\leqslant \Big(\frac{\sin\pi\alpha}{\pi}\Big)^p C_p|||R_\alpha|||^p\|U\|_\infty^p(1-2\alpha)^{-p/2}T^{p(1-2\alpha)/2}\Big(\int_0^T \mathsf{E}\|\psi(r)\|_Q^p\,\mathrm{d}r\Big),\end{aligned}$$

where, obviously, $|||R_\alpha|||$ denotes the norm of R_α in the space $\mathcal{L}(L^p([0,T];H), \mathcal{C}([0,T];H_\delta))$. □

4. Proof of Theorem 1.3

We aim at proving Theorem 1.3; as the first step we will treat the case of integrable initial conditions. Then the proof can be based on the Banach fixed point principle and is quite standard.

Proposition 4.1. *Let the assumptions* (A), (P), (L_δ) *and* (I) *be fulfilled.*

(i) *Let* $p \geqslant 2$, $\alpha \in [0,\delta]$, $\varphi \in L^p(\Omega; H_\alpha)$. *Then there exists a unique mild solution* x *of the equation* (1.2), (1.3) *in* $C(]0,T]; L^p(\Omega; H_\delta))$ *such that*

$$\sup_{0\leqslant t\leqslant T} t^{\delta-\alpha}\|x(t)\|_{p,\delta} \leqslant C^*(1+\|\varphi\|_{p,\alpha}), \tag{4.1}$$

for a constant C^* *dependent only on* p,δ,T,K^* *and* U. *Furthermore,* $x \in C([0,T]; L^p(\Omega; H_\beta)) \cap C^{0,\mu}([a,T]; L^p(\Omega; H_\delta))$ *for any* $a>0$, $\mu \in [0,\frac{1}{2}-\delta[$, $\beta \in [0,\alpha[$. *If* $\alpha < \varrho$, *then* $x \in C([0,T]; L^p(\Omega; H_\alpha))$ *as well.*

(ii) *Assume, in addition, that* $p>2$ *and* $\varphi \in L^p(\Omega; H_\delta)$; *let* $\gamma \in [0,\frac{1}{2}-\frac{1}{p}[$, $\gamma \leqslant \delta$, *be arbitrary. Then there exists a constant* C^+, *dependent only on* p,δ,T,K^*, *and* U, *such that*

$$\Big|\sup_{0\leqslant t\leqslant T}\|x(t)\|_\gamma\Big|_p \leqslant C^+(1+\|\varphi\|_{p,\delta}). \tag{4.2}$$

For $\gamma \in [0,(\frac{1}{2}-\frac{1}{p})\wedge\delta[$ *the mild solution* x *has continuous sample paths in* H_γ *and lies in* $L^p(\Omega; C([0,T]; H_\gamma))$; *the same assertion is true for* $\gamma=\delta$ *provided* $\delta < (\frac{1}{2}-\frac{1}{p})\wedge\varrho$.

Proof. As mentioned above, we will prove Proposition 4.1 using a standard fixed-point argument. In what follows, constants depending only on p, δ, T, K, K^* and U will be denoted by D_i. We set for brevity $\delta-\alpha=\nu$. Denote

$$\begin{aligned}\mathcal{E} \equiv \{h \in C(]0,T]; L^p(\Omega; H_\delta)),\ h(t)\ \ \mathcal{F}_t\text{-measurable}, 0\leqslant t\leqslant T,\\ \|h\|_{\mathcal{E}} \equiv \sup_{0\leqslant t\leqslant T} t^\nu\|h(t)\|_{p,\delta} < \infty\}.\end{aligned}$$

Obviously, the space $\mathcal{E}$ equipped with the norm $\|.\|_{\mathcal{E}}$ is a Banach space. Let us define

$$\mathfrak{K}h(t) = U(t,0)\varphi + \int_0^t U(t,s)f(s,h(s))\,\mathrm{d}s + \int_0^t U(t,s)\sigma(s,h(s))\,\mathrm{d}w(s),$$

$0\leqslant t\leqslant T$, $h\in\mathcal{E}$. We want to prove that $\mathfrak{K}h \in \mathcal{E}$. As a first step, we check that $\mathfrak{K}h(t) \in L^p(\Omega; H_\delta)$ for each $t>0$; in so doing we establish estimates ensuring the existence of integrals in the definition of $\mathfrak{K}$. We have

$$\begin{aligned}\|\mathfrak{K}h(t)\|_{p,\delta} \leqslant \|U(t,0)\varphi\|_{p,\delta} + \Big\|\int_0^t U(t,s)f(s,h(s))\,\mathrm{d}s\Big\|_{p,\delta}\\ + \Big\|\int_0^t U(t,s)\sigma(s,h(s))\,\mathrm{d}w(s)\Big\|_{p,\delta} \equiv I_1+I_2+I_3.\end{aligned}$$

Now

$$I_1 \leqslant \|U(t,0)\|_{H_\alpha \to H_\delta} \|\varphi\|_{p,\alpha} \leqslant C_\delta t^{-\nu} \|\varphi\|_{p,\alpha}$$

by (0.7), and further

$$\begin{aligned} I_2 &\leqslant \int_0^t \|U(t,s)f(s,h(s))\|_{p,\delta}\,\mathrm{d}s \leqslant C_\delta \int_0^t (t-s)^{-\delta} \|f(s,h(s))\|_{p,0}\,\mathrm{d}s \\ &\leqslant C_\delta K^* \int_0^t (t-s)^{-\delta}(1+\|h(s)\|_{p,\delta})\,\mathrm{d}s \\ &\leqslant C_\delta K^*(1+T^\nu) \int_0^t (t-s)^{-\delta} s^{-\nu}(1+s^\nu \|h(s)\|_{p,\delta})\,\mathrm{d}s \\ &\leqslant D_1(1+\|h\|_{\mathcal{E}}) \int_0^t (t-s)^{-\delta} s^{\alpha-\delta}\,\mathrm{d}s = D_1(1+\|h\|_{\mathcal{E}}) t^{1+\alpha-2\delta} B(1-\delta, 1-\nu) \\ &\leqslant D_2(1+\|h\|_{\mathcal{E}}), \end{aligned}$$

where B is the Euler Beta function. Using Lemma 2.2 we obtain

$$\begin{aligned} I_3 &\leqslant C_p \Big(\int_0^t C_\delta^2 (t-s)^{-2\delta} \|\sigma(s,h(s))\|_{p,Q}^2\,\mathrm{d}s \Big)^{1/2} \\ &\leqslant C_p C_\delta K^* \Big(\int_0^t (t-s)^{-2\delta}(1+\|h(s)\|_{p,\delta})^2\,\mathrm{d}s \Big)^{1/2} \\ &\leqslant D_3(1+\|h\|_{\mathcal{E}}) \Big(\int_0^t (t-s)^{-2\delta} s^{-2\nu}\,\mathrm{d}s \Big)^{1/2} \\ &= D_3(1+\|h\|_{\mathcal{E}}) \big(t^{1+2\alpha-4\delta} B(1-2\delta, 1-2\nu) \big)^{1/2} \\ &\leqslant D_3(1+\|h\|_{\mathcal{E}}) T^{1/2-\delta} \big(B(1-2\delta, 1-2\nu) \big)^{1/2} \, t^{\alpha-\delta}. \end{aligned}$$

Consequently,

$$\sup_{0 \leqslant t \leqslant T} t^\nu \|\mathfrak{K}h(t)\|_{p,\delta} < \infty.$$

To proceed further we have to prove

$$\lim_{z \to 0} \mathfrak{K}h(t+z) = \mathfrak{K}h(t) \text{ in } L^p(\Omega; H_\delta)$$

for any $0 < t \leqslant T$; in fact, we can prove much more. Towards this end, let us fix $a > 0$ and choose $u, v \in [a, T]$ arbitrary, for definiteness let us assume $u < v$. Then

$$\begin{aligned} \mathfrak{K}h(v) - \mathfrak{K}h(u) &= [U(v,0) - U(u,0)]\varphi + \int_0^u [U(v,s) - U(u,s)] f(s,h(s))\,\mathrm{d}s \\ &+ \int_0^u [U(v,s) - U(u,s)]\sigma(s,h(s))\,\mathrm{d}w(s) + \int_u^v U(v,s) f(s,h(s))\,\mathrm{d}s \\ &+ \int_u^v U(v,s)\sigma(s,h(s))\,\mathrm{d}w(s) \equiv Z_1 + \ldots + Z_5. \end{aligned}$$

Relying on the estimate (0.8), we obtain

$$\begin{aligned}\|Z_1\|_{p,\delta} &\leqslant \|[U(v,u)-I]U(u,0)\varphi\|_{p,\delta} \leqslant \widehat{C}|v-u|^{\mu}\|U(u,0)\varphi\|_{p,\delta+\mu} \\ &\leqslant C_{\delta+\mu}\widehat{C}u^{\alpha-\delta-\mu}\|\varphi\|_{p,\alpha}|v-u|^{\mu} \leqslant D_4 a^{-\nu-\mu}\|\varphi\|_{p,\alpha}|v-u|^{\mu}\end{aligned}$$

for $\delta+\mu<1$; further

$$\begin{aligned}\|Z_4\|_{p,\delta} &\leqslant \int_u^v \|U(v,s)f(s,h(s))\|_{p,\delta}\,\mathrm{d}s \leqslant C_\delta K^* \int_u^v (v-s)^{-\delta}(1+\|h(s)\|_{p,\delta})\,\mathrm{d}s \\ &\leqslant C_\delta K^*\Big(1+\sup_{a\leqslant t\leqslant T}\|h(s)\|_{p,\delta}\Big)\Big(\int_0^{v-u} r^{-\delta}\,\mathrm{d}r\Big) \leqslant C_\delta K^* a^{-\nu}(1+\|h\|_{\mathcal{E}})|v-u|^{1-\delta},\end{aligned}$$

and

$$\begin{aligned}\|Z_5\|_{p,\delta} &\leqslant C_p C_\delta K^*\Big(\int_u^v (v-s)^{-2\delta}(1+\|h(s)\|_{p,\delta})^2\,\mathrm{d}s\Big)^{1/2} \\ &\leqslant D_5 a^{-\nu}(1+\|h\|_{\mathcal{E}})|v-u|^{1/2-\delta}.\end{aligned}$$

Invoking again (0.8) we can estimate

$$\begin{aligned}\|Z_2\|_{p,\delta} &\leqslant \int_0^u \|[U(v,u)-I]U(u,s)f(s,h(s))\|_{p,\delta}\,\mathrm{d}s \\ &\leqslant \widehat{C}|v-u|^{\mu}\int_0^u \|U(u,s)f(s,h(s))\|_{p,\delta+\mu}\,\mathrm{d}s \\ &\leqslant D_6|v-u|^{\mu}\int_0^u (u-s)^{-(\delta+\mu)}(1+\|h(s)\|_{p,\delta})\,\mathrm{d}s \\ &\leqslant D_6(1+T^{\nu})(1+\|h\|_{\mathcal{E}})\Big(\int_0^u (u-s)^{-(\delta+\mu)}s^{-\nu}\,\mathrm{d}s\Big)|v-u|^{\mu} \\ &\leqslant D_6(1+T^{\nu})(1+\|h\|_{\mathcal{E}})B(1-\delta-\mu,1-\nu)Ta^{-(\delta+\mu+\nu)}|v-u|^{\mu}\end{aligned}$$

provided $\delta+\mu<1$. Analogously

$$\begin{aligned}\|Z_3\|_{p,\delta} &\leqslant D_7|v-u|^{\mu}(1+\|h\|_{\mathcal{E}})\Big(\int_0^u (u-s)^{-2(\delta+\mu)}s^{-2\nu}\,\mathrm{d}s\Big)^{1/2} \\ &\leqslant D_7(1+\|h\|_{\mathcal{E}})\big(B(1-2\delta-2\mu,1-2\nu)\big)^{1/2}T^{1/2}a^{-(\delta+\mu+\nu)}|v-u|^{\mu}\end{aligned}$$

provided $\delta+\mu<\frac{1}{2}$. So we have established the inequality

$$\|\mathfrak{K}h(v)-\mathfrak{K}h(u)\|_{p,\delta} \leqslant F(1+\|h\|_{\mathcal{E}})|v-u|^{\mu}$$

valid for any $0\leqslant\mu<\frac{1}{2}-\delta$, $a>0$, with a constant F dependent on μ and a, but independent of u, v and h.

Hence we see that $\mathfrak{K}$ maps $\mathcal{E}$ into itself, since it is obvious that $\mathfrak{K}h$ is $(\mathcal{F}_t)$-adapted. In order to prove contractivity of $\mathfrak{K}$ we endow $\mathcal{E}$ with an equivalent norm

$$|||h||| = \sup_{0 \leqslant t \leqslant T} \mathrm{e}^{-qt} t^{\nu} \|h(t)\|_{p,\delta},$$

where $q > 0$ will be specified later. Let us choose $h, g \in \mathcal{E}$ and estimate $(t > 0)$

$$\begin{aligned} \|\mathfrak{K}h(t) - \mathfrak{K}g(t)\|_{p,\delta} &\leqslant \Big\| \int_0^t U(t,s)[f(s,h(s)) - f(s,g(s))]\,\mathrm{d}s \Big\|_{p,\delta} \\ &\quad + \Big\| \int_0^t U(t,s)[\sigma(s,h(s)) - \sigma(s,g(s))]\,\mathrm{d}w(s) \Big\|_{p,\delta} \equiv J_1 + J_2. \end{aligned}$$

The assumption (L_δ) yields

$$J_1 \leqslant C_\delta K \int_0^t (t-s)^{-\delta} \|h(s) - g(s)\|_{p,\delta}\,\mathrm{d}s$$

and

$$\begin{aligned} J_2 &\leqslant C_p \Big(\int_0^t C_\delta^2 (t-s)^{-2\delta} \|\sigma(s,h(s)) - \sigma(s,g(s))\|_{p,Q}^2\,\mathrm{d}s \Big)^{1/2} \\ &\leqslant D_8 \Big(\int_0^t (t-s)^{-2\delta} \|h(s) - g(s)\|_{p,\delta}^2\,\mathrm{d}s \Big)^{1/2}, \end{aligned}$$

that is

$$\|\mathfrak{K}h(t) - \mathfrak{K}g(t)\|_{p,\delta}^2 \leqslant D_9 \int_0^t (t-s)^{-2\delta} \|h(s) - g(s)\|_{p,\delta}^2\,\mathrm{d}s.$$

This implies

$$\begin{aligned} &\mathrm{e}^{-2qt} t^{2\nu} \|\mathfrak{K}h(t) - \mathfrak{K}g(t)\|_{p,\delta}^2 \\ &\quad \leqslant D_9 t^{2\nu} \int_0^t \mathrm{e}^{-2q(t-s)} (t-s)^{-2\delta} s^{-2\nu} (\mathrm{e}^{-2qs} s^{2\nu} \|h(s) - g(s)\|_{p,\delta}^2)\,\mathrm{d}s \\ &\quad \leqslant D_9 |||h - g|||^2 t^{1-2\delta} \int_0^1 \mathrm{e}^{-2qt(1-v)} (1-v)^{-2\delta} v^{-2\nu}\,\mathrm{d}v, \end{aligned}$$

so

$$|||\mathfrak{K}h - \mathfrak{K}g||| \leqslant D_{10} \Big(\sup_{0 \leqslant t \leqslant T} t^{1-2\delta} \int_0^1 \mathrm{e}^{-2qtv} (1-v)^{-2\nu} v^{-2\delta}\,\mathrm{d}v \Big)^{1/2} |||h - g|||.$$

Since

$$\lim_{q \to \infty} \sup_{0 \leqslant t \leqslant T} \Big(t^{1-2\delta} \int_0^1 \mathrm{e}^{-2qtv} (1-v)^{-2\nu} v^{-2\delta}\,\mathrm{d}v \Big)^{1/2} = 0,$$

we can ensure the contractivity of $\mathfrak{K}$ by choosing $q > 0$ sufficiently large.

Since mild solutions to (1.2), (1.3) in $\mathcal{E}$ are precisely the fixed points of $\mathfrak{K}$, we see that there exists a unique $x \in \mathcal{E}$ being the mild solution to (1.2), (1.3). Above we have actually derived an estimate

$$\|x(t)\|_{p,\delta}^2 \leqslant D_{11} t^{-2\nu} \|\varphi\|_{p,\alpha}^2 + D_{11} \int_0^t (t-s)^{-2\delta} (1 + \|x(s)\|_{p,\delta}^2) \, \mathrm{d}s.$$

Straightforward calculations based on Lemma 2.3 now yield

$$\|x(t)\|_{p,\delta} \leqslant D_{12} t^{\alpha-\delta} (1 + \|\varphi\|_{p,\alpha}), \quad 0 < t \leqslant T,$$

and (4.1) follows. Moreover, take $\beta \in [0, \alpha]$ arbitrary, then

$$\|x(z) - \varphi\|_{p,\beta} \leqslant \|U(z,0)\varphi - \varphi\|_{p,\beta} + \Big\| \int_0^z U(z,r) f(r, x(r)) \, \mathrm{d}r \Big\|_{p,\beta} + \Big\| \int_0^z U(z,r)\sigma(r, x(r)) \, \mathrm{d}w(r) \Big\|_{p,\beta} \equiv J_3 + J_4 + J_5.$$

We can easily estimate

$$\begin{aligned} J_5 &\leqslant C_p \Big(\int_0^z C_\beta^2 (z-r)^{-2\beta} \|\sigma(r, x(r))\|_{p,Q}^2 \, \mathrm{d}r \Big)^{1/2} \\ &\leqslant D_{13} \Big(\int_0^z (z-r)^{-2\beta} r^{2\alpha - 2\delta} (1 + r^\nu \|x(r)\|_{p,\delta})^2 \, \mathrm{d}r \Big)^{1/2} \\ &\leqslant D_{13} (1 + \|x\|_{\mathcal{E}}) (B(1-2\beta, 1-2\nu))^{1/2} T^{\alpha-\beta} z^{1/2-\delta} \to 0, \quad z \to 0^+; \end{aligned}$$

analogously $J_4 \to 0$, $z \to 0^+$. Further, if $\alpha < \varrho$, then

$$\lim_{z \to 0+} \|U(z,0)\varphi(\omega) - \varphi(\omega)\|_\alpha = 0$$

for almost all $\omega \in \Omega$ by (0.10), as φ is almost surely H_α-valued. On the other hand, if $\varrho \leqslant \alpha$, then for any $\beta < \alpha$ we have

$$\|U(z,0)\varphi(\omega) - \varphi(\omega)\|_\beta \leqslant C z^{\alpha-\beta} \|\varphi(\omega)\|_\alpha$$

by (0.8). In both cases the dominated convergence theorem applies and we conclude that $x \in C([0,T]; L^p(\Omega; H_\beta))$ for $0 \leqslant \beta \leqslant \alpha$ (or for $0 \leqslant \beta < \alpha$, respectively).

Let us turn to the proof of the statement (ii). In order to prove it let us assume that $\varphi \in L^p(\Omega; H_\delta)$, $p > 2$. Note that the estimate (4.1) now implies

$$\sup_{0 \leqslant t \leqslant T} \|x(t)\|_{p,\delta} \leqslant C^*(1 + \|\varphi\|_{p,\delta}),$$

thus

$$\int_0^T \mathsf{E}\|\sigma(r,x(r))\|_Q^p\,\mathrm{d}r \leqslant 2^{p-1}(K^*)^p\int_0^T(1+\|x(r)\|_{p,\delta}^p)\,\mathrm{d}r$$
$$\leqslant 2^{p-1}(K^*)^p(1+C^*)^p(1+\|\varphi\|_{p,\delta})^pT<\infty,$$

so

$$\int_0^{\cdot} U(.,s)\sigma(s,x(s))\,\mathrm{d}w(s)\in \mathcal{C}^{0,\lambda}([0,T];H_\gamma)$$

almost surely for any $\gamma\in[0,\frac{1}{2}-\frac{1}{p}[$, $\lambda\in[0,\frac{1}{2}-\frac{1}{p}-\gamma[$ according to Theorem 1.1. Analogously,

$$\int_0^{\cdot} U(.,s)f(s,x(s))\,\mathrm{d}s\in \mathcal{C}^{0,\lambda}([0,T];H_\gamma)$$

almost surely (with the same γ, λ). Furthermore, fix $\gamma\in[0,\frac{1}{2}-\frac{1}{p}[$, $\gamma\leqslant\delta$ arbitrarily, then

$$\Big|\sup_{0\leqslant t\leqslant T}\|x(t)\|_\gamma\Big|_p \leqslant \Big|\sup_{0\leqslant t\leqslant T}\|U(t,0)\varphi\|_\gamma\Big|_p+\Big|\sup_{0\leqslant t\leqslant T}\Big\|\int_0^t U(t,s)f(s,x(s))\,\mathrm{d}s\Big\|_\gamma\Big|_p$$
$$+\Big|\sup_{0\leqslant t\leqslant T}\Big\|\int_0^t U(t,s)\sigma(s,x(s))\,\mathrm{d}w(s)\Big\|_\gamma\Big|_p$$
$$\equiv M_1+M_2+M_3.$$

Obviously we have

$$M_1\leqslant C'_\gamma\|\varphi\|_{p,\gamma}\leqslant D_{14}\|\varphi\|_{p,\delta}$$

by (0.9), and further

$$M_2\leqslant C_\gamma K^*\Big|\sup_{0\leqslant t\leqslant T}\int_0^t(t-s)^{-\gamma}(1+\|x(s)\|_\delta)\,\mathrm{d}s\Big|_p$$
$$\leqslant C_\gamma K^*\left|\sup_{0\leqslant t\leqslant T}\Big(\int_0^t(t-s)^{-\gamma p/(p-1)}\,\mathrm{d}s\Big)^{(p-1)/p}\Big(\int_0^t(1+\|x(s)\|_\delta)^p\,\mathrm{d}s\Big)^{1/p}\right|_p$$
$$\leqslant D_{15}\Big(\int_0^T(1+\|x(s)\|_{p,\delta}^p)\,\mathrm{d}s\Big)^{1/p}.$$

In view of Theorem 1.1 we have

$$M_3\leqslant C\Big(\int_0^T\|\sigma(s,x(s))\|_{p,Q}^p\,\mathrm{d}s\Big)^{1/p}\leqslant D_{16}\Big(\int_0^T(1+\|x(s)\|_{p,\delta}^p)\,\mathrm{d}s\Big)^{1/p}.$$

Hence

$$\mathsf{E}\sup_{0\leqslant t\leqslant T}\|x(t)\|_\gamma^p\leqslant D_{17}\|\varphi\|_{p,\delta}^p+D_{17}\int_0^T(1+\mathsf{E}\|x(s)\|_\delta^p)\,\mathrm{d}s. \tag{4.3}$$

Using (4.1) to estimate the right-hand side of (4.3) we obtain the desired inequality (4.2) at once.

We have already noted that $U(.,0)\varphi(\omega) \in \mathcal{C}([0,T];H_\gamma)$ almost surely for any $\gamma \in [0,\delta]$ provided $\delta < \varrho$, and for any $\gamma \in [0,\delta[$ in the opposite case. Hence under the restrictions on γ adopted in Proposition the sample paths of the mild solution x are continuous in H_γ almost surely, $x(.,\omega) \in \mathcal{C}([0,T];H\gamma)$. Thus, by the finiteness of the right-hand side of (4.3), $x \in L^p(\Omega;\mathcal{C}([0,T];H_\gamma))$. The proof of Proposition 4.1 is completed. □

To extend the existence proof to more general initial conditions we will use the same procedure as applied in [7], which is based on the following local uniqueness lemma. Its proof is contained in the proof of Theorem 2 in [7]; we repeat the simple proof here for completeness. (Recall that χ_B stands for an indicator of the set B.)

Lemma 4.2. *Let the assumptions* (A), (P) *and* (L_δ) *be fulfilled. Let $\varphi, \psi \in L^2(\Omega;H)$ be $\mathcal{F}_0$-measurable. Let $x, y \in \mathcal{E}$ be mild solutions of the equation* (1.2) *with initial conditions $x(0) = \varphi, y(0) = \psi$, respectively. Set $\Sigma = \{\omega \in \Omega; \varphi(\omega) = \psi(\omega)\}$. Then $\chi_\Sigma x(t) = \chi_\Sigma y(t)$ almost surely for any $t \in [0,T]$ fixed.*

Proof. By the definition of a mild solution,

$$x(t) - y(t) = U(t,0)[\varphi - \psi] + \int_0^t U(t,r)[f(r,x(r)) - f(r,y(r))]\,\mathrm{d}r$$
$$+ \int_0^t U(t,r)[\sigma(r,x(r)) - \sigma(r,y(r))]\,\mathrm{d}w(r).$$

Let us realize that $\chi_\Sigma[\varphi - \psi] = 0$ and

$$\chi_\Sigma \int_0^t U(t,r)[\sigma(r,x(r)) - \sigma(r,y(r))]\,\mathrm{d}w(r)$$
$$= \int_0^t U(t,r)\chi_\Sigma[\sigma(r,x(r)) - \sigma(r,y(r))]\,\mathrm{d}w(r)$$

as χ_Σ is an $\mathcal{F}_0$-measurable random variable. Consequently

$$\|\chi_\Sigma(x(t) - y(t))\|_{2,\delta} \leqslant C_\delta K \int_0^t (t-r)^{-\delta}\|\chi_\Sigma(x(r) - y(r))\|_{2,\delta}\,\mathrm{d}r$$
$$+ C_\delta K\Big(\int_0^t (t-r)^{-2\delta}\|\chi_\Sigma(x(r) - y(r))\|_{2,\delta}^2\,\mathrm{d}r\Big)^{1/2},$$

and Lemma 2.3 implies

$$\sup_{0\leqslant t\leqslant T}\|\chi_\Sigma(x(t) - y(t))\|_{2,\delta} = 0.$$

□

Proposition 4.3. *Let the assumptions* (A), (P), (I) *and* (L_δ) *be fulfilled. Let φ be H_α-valued almost surely for some $\alpha \in [0, \frac{1}{2}[$. Then there exists a mild solution x of the problem* (1.2), (1.3) *whose sample paths lie in $\mathcal{C}([0,T]; H_\beta) \cap \mathcal{C}^{0,\lambda}([a,T]; H_\varkappa)$ almost surely for any $a > 0$, $\varkappa \in [0, \frac{1}{2}[$, $\lambda \in [0, \frac{1}{2} - \varkappa[$ and $\beta \in [0, \alpha[$. If, moreover, $\alpha < \varrho$, then $x(\cdot, \omega) \in \mathcal{C}([0,T]; H_\alpha)$ for almost all $\omega \in \Omega$ as well.*

Remark. In Proposition 4.1 we have established the existence of a unique mild solution to (1.2), (1.3) if $\varphi \in L^p(\Omega; H)$. Proposition 4.3 states, in particular, that this solution has a modification with continuous sample paths.

Proof. For any $N \in \mathbf{N}$ set $\Omega_N = \{\omega \in \Omega; \|\varphi(\omega)\|_\alpha \leqslant N\}$ and define

$$\varphi_N(\omega) = \begin{cases} \varphi(\omega), & \omega \in \Omega_N \\ 0, & \text{otherwise,} \end{cases}$$

thus $\varphi_N \in L^\infty(\Omega; H_\alpha)$. Let x_N be the mild solution of the problem

$$\mathrm{d}x_N(t) = [A(t)x_N(t) + f(t, x_N(t))]\,\mathrm{d}t + \sigma(t, x_N(t))\,\mathrm{d}w(t),$$
$$x_N(0) = \varphi_N.$$

First, choose $a > 0$, $\varkappa \in [0, \frac{1}{2}[$, $\lambda \in [0, \frac{1}{2} - \varkappa[$ arbitrarily. We can find $r \in]2, \infty[$ such that $\varkappa < \frac{1}{2} - \frac{1}{r}$, $\lambda < \frac{1}{2} - \frac{1}{r} - \varkappa$. By the definition of a mild solution we have

$$\begin{aligned} x_N(t) &= U(t,a)x_N(a) + \int_a^t U(t,s) f(s, x_N(s))\,\mathrm{d}s \\ &\quad + \int_a^t U(t,s)\sigma(s, x_N(s))\,\mathrm{d}w(s) \end{aligned} \tag{4.4}$$

for $t \in [a, T]$. We will proceed as in the proof of Theorem 1.1 using the representation

$$\int_a^t U(t,s)\sigma(s, x_N(s))\,\mathrm{d}w(s) = \frac{\sin \pi\mu}{\pi}(R_\mu Y)(t), \quad a \leqslant t \leqslant T,$$

$$Y(s) = \int_a^s (s-v)^{-\mu} U(s,v)\sigma(v, x_N(v))\,\mathrm{d}w(v), \quad a \leqslant s \leqslant T,$$

with $\mu \in]\frac{1}{r} + \varkappa, \frac{1}{2}[$. As $\varphi_N \in L^r(\Omega; H)$, we have $x_N \in \mathcal{C}([a,T]; L^r(\Omega; H_\delta))$ by Proposition 4.1(i), and this implies

$$\begin{aligned} \mathsf{E}\int_a^T \|Y(s)\|^r\,\mathrm{d}s &\leqslant C_r \|U\|_\infty^r \int_a^T \mathsf{E}\Big(\int_a^s (s-v)^{-2\mu} \|\sigma(v, x_N(v))\|_Q^2\,\mathrm{d}v\Big)^{r/2}\mathrm{d}s \\ &\leqslant D_{18}\Big(\int_0^T v^{-2\mu}\,\mathrm{d}v\Big)^{r/2} \Big(\int_a^T (1 + \|x_N(v)\|_{r,\delta}^r)\,\mathrm{d}v\Big) \\ &\leqslant D_{19}(1 + \sup_{a \leqslant v \leqslant T} \|x_N(v)\|_{r,\delta})^r < \infty \end{aligned}$$

by Lemma 2.2 and the Young inequality. Thus almost every path of Y lies in $L^r([a,T];H)$ and Lemma 2.1 can be used. The Bochner integral in (4.4) can be treated in a similar way, hence

$$\int_a^{\cdot} U(.,s)f(s,x_N(s))\,\mathrm{d}s + \int_a^{\cdot} U(.,s)\sigma(s,x_N(s))\,\mathrm{d}w(s) \in C^{0,\lambda}([a,T];H_\varkappa)$$

almost surely. Now, let us take $h \in H$, $2a \leqslant s \leqslant t \leqslant T$ arbitrary. Using (0.8) we obtain

$$\begin{aligned}\|U(t,a)h - U(s,a)h\|_\varkappa = \|[U(t,s) - I]U(s,a)h\|_\varkappa \leqslant \widehat{C}(t-s)^\lambda \|U(s,a)h\|_{\lambda+\varkappa} \leqslant \\ \leqslant D_{20}(t-s)^\lambda (s-a)^{-(\lambda+\varkappa)}\|h\|_0 \leqslant D_{20}a^{-(\lambda+\varkappa)}\|h\|_0|t-s|^\lambda.\end{aligned}$$

So $x_N \in C^{0,\lambda}([2a,T];H_\varkappa)$ almost surely for any $N \in \mathbb{N}$ and $a > 0$.

Further, taking into account that φ (and hence also φ_N) is H_α-valued we establish the desired continuity of paths of x_N on the whole interval $[0,T]$, modifying slightly the argument used above. First, let us assume that $\alpha < \delta$ and take $q \in]2,\infty[$ such that $\delta < \alpha + \frac{1}{q} < \frac{1}{2}$. Again we will use the factorization

$$\int_0^{\cdot} U(.,r)\sigma(r,x_N(r))\,\mathrm{d}w(r) = \frac{\sin \pi\mu}{\pi} R_\mu Y, \tag{4.5}$$

where

$$Y(r) = \int_0^r (r-v)^{-\mu} U(r,v)\sigma(v,x_N(v))\,\mathrm{d}w(v),$$

$\mu \in]\alpha + \frac{1}{q}, \frac{1}{2}[$. By Lemma 2.1, the stochastic integral on the left-hand side of (4.5) has sample paths in $C([0,T];H_\alpha)$ provided $Y(.,\omega) \in L^q([0,T];H)$ almost surely. But using (4.1) and the fact that $q(\delta - \alpha) < 1$ we obtain in a similar manner as above

$$\begin{aligned}\mathsf{E}\int_0^T \|Y(t)\|^q\,\mathrm{d}t &\leqslant D_{21}\Big(\int_0^T (1 + \|x_N(v)\|_{q,\delta}^q)\,\mathrm{d}v\Big) \\ &\leqslant D_{22}\Big(1 + \Big(\sup_{0\leqslant t\leqslant T} t^{\delta-\alpha}\|x_N(t)\|_{q,\delta}\Big)^q\Big)\Big(\int_0^T t^{-q(\delta-\alpha)}\,\mathrm{d}t\Big) < \infty.\end{aligned}$$

So we have proved that

$$\int_0^{\cdot} U(.,r)\sigma(r,x_N(r))\,\mathrm{d}w(r) \in C([0,T];H_\alpha) \quad \text{a.s.}$$

in the case $\alpha < \delta$; the same result for $\alpha \geqslant \delta$ has been already established in the proof of Proposition 4.1(ii). One can obtain easily that

$$\int_0^{\cdot} U(.,r)f(r,x_N(r,\omega))\,\mathrm{d}r \in C([0,T];H_\alpha)$$

almost surely. Moreover, we know that $U(.,0)\varphi_N(\omega) \in \mathcal{C}([0,T]; H_\beta)$ almost surely for $\beta < \alpha$ (or for $\beta \leqslant \alpha$, if $\alpha < \varrho$).

So we see that the paths of the mild solution x_N have the required regularity properties. Now we will use the processes x_N to construct a solution of the problem (1.2), (1.3).

Let $N, M \in \mathbb{N}$, $M \geqslant N$. By Lemma 4.2, $x_N(t) = x_M(t)$ almost everywhere on Ω_N for any t; the continuity of trajectories implies

$$\sup_{M \geqslant N} \sup_{0 \leqslant t \leqslant T} \|x_M(t,\omega) - x_N(t,\omega)\| = 0 \quad \text{a.s. on } \Omega_N.$$

Hence the limit

$$x(.\,,\omega) = \lim_{M\to\infty} x_M(.\,,\omega)$$

is well defined in $\mathcal{C}([0,T]; H)$ for almost all $\omega \in \Omega_N$; in fact, $x_N = x$ almost surely on Ω_N. As $\mathsf{P}(\Omega_N) \to 1$, $N \to \infty$, we obtain x as a stochastic process whose sample paths share all the continuity properties of the paths of x_N. It can be seen easily that x is a mild solution to (1.2). (Recall that the equality $\sigma(r, x(r)) = \sigma(r, x_N(r))$ a.s. on Ω_N implies

$$\int_0^t U(t,r)\sigma(r,x(r))\,\mathrm{d}w(r) = \int_0^t U(t,r)\sigma(r,x_N(r))\,\mathrm{d}w(r)$$

almost surely on Ω_N.) $\square$

To complete the proof of Theorem 1.3 it remains to establish the uniqueness.

Lemma 4.4. *Let the assumptions* (A), (P), (I), (L_δ) *be fulfilled. Then there exists — up to a modification — at most one mild solution of the problem* (1.2), (1.3) *within the class of processes with* $L^2([0,T]; H_\delta)$*-trajectories.*

Proof. Let x, y be two mild solutions of (1.2), (1.3) such that

$$\text{(4.6)} \qquad \int_0^T \|x(t)\|_\delta^2\,\mathrm{d}t < \infty, \qquad \int_0^T \|y(t)\|_\delta^2\,\mathrm{d}t < \infty \quad \text{a.s.}.$$

(Note that (4.6) implies (1.4) from Definition 1.1.) For $R \in \mathbb{N}$ arbitrary set

$$\tau_R = \inf\{t \in [0,T];\ \int_0^t \|x(s)\|_\delta^2\,\mathrm{d}s + \int_0^t \|y(s)\|_\delta^2\,\mathrm{d}s \geqslant R\}$$

(with the convention $\inf\emptyset = T$). Let us have R fixed for the meantime and define $\xi(t,\omega) = \chi_{[0,\tau_R(\omega)]}(t)$. We have

$$\xi(t)(x(t) - y(t)) = \xi(t)\int_0^t U(t,s)[f(s,x(s)) - f(s,y(s))]\,\mathrm{d}s$$
$$+\,\xi(t)\int_0^t U(t,s)[\sigma(s,x(s)) - \sigma(s,y(s))]\,\mathrm{d}w(s).$$

As $\xi(t) \leqslant \xi(s)$ for $t \geqslant s$ by definition, we obtain

$$\begin{aligned}\Big\|\xi(t)\int_0^t U(t,s)[f(s,x(s)) - f(s,y(s))]\,\mathrm{d}s\Big\|_\delta \\ &\leqslant \int_0^t \xi(s)\|U(t,s)[f(s,x(s)) - f(s,y(s))]\|_\delta\,\mathrm{d}s \\ &\leqslant C_\delta K\int_0^t (t-s)^{-\delta}\xi(s)\|x(s)-y(s)\|_\delta\,\mathrm{d}s.\end{aligned}$$

Further, set $\psi(s) = U(t,s)[\sigma(s,x(s)) - \sigma(s,y(s))]$ for brevity. We want to prove that

$$\Big\|\xi(t)\int_0^t \psi(s)\,\mathrm{d}w(s)\Big\|_\delta \leqslant \Big\|\int_0^t \xi(s)\psi(s)\,\mathrm{d}w(s)\Big\|_\delta \tag{4.7}$$

holds almost surely. Indeed, put $B = \{\omega \in \Omega;\ \xi(t,\omega) = 1\}$. On $\Omega \setminus B$, the left-hand side of (4.7) is zero. On the other hand, $\xi(s)\psi(s) = \psi(s)$, $0 \leqslant s \leqslant t$, on B, hence

$$\xi(t)\int_0^t \psi(s)\,\mathrm{d}w(s) = \int_0^t \xi(s)\psi(s)\,\mathrm{d}w(s) \quad \text{a.s. on } B.$$

It follows that

$$\begin{aligned}\xi(t)\|x(t)-y(t)\|_\delta &\leqslant C_\delta K\int_0^t (t-s)^{-\delta}\xi(s)\|x(s)-y(s)\|_\delta\,\mathrm{d}s \\ &\quad + \Big\|\int_0^t U(t,s)\xi(s)[\sigma(s,x(s)) - \sigma(s,y(s))]\,\mathrm{d}w(s)\Big\|_\delta,\end{aligned}$$

therefore

$$|||\xi(t)[x(t)-y(t)]|||_{2,\delta}^2 \leqslant D_{23}\int_0^t (t-s)^{-2\delta}|||\xi(s)[x(s)-y(s)]|||_{2,\delta}^2\,\mathrm{d}s.$$

By the definition of ξ,

$$\mathsf{E}\int_0^T \|\xi(s)(x(s)-y(s))\|_\delta^2\,\mathrm{d}s \leqslant 2R < \infty,$$

so $\mathsf{E}\|\xi(\cdot)(x(\cdot)-y(\cdot))\|_\delta^2$ is a function in $L^1([0,T])$ and the generalized Gronwall lemma implies

$$|||\xi(t)[x(t)-y(t)]|||_{2,\delta} = 0, \quad 0 \leqslant t \leqslant T.$$

This means that $x(t) = y(t)$ almost surely on the set $\{\omega; t \leqslant \tau_R(\omega)\}$ and it suffices to realize that, by (4.6), $\tau_R \to T$ almost surely as $R \to \infty$. □

Combining the assertions of all propositions proved in this section we obtain Theorem 1.3.

5. Proof of Theorem 1.5

Our procedure follows the finite-dimensional pattern of passing from lipschitzian to locally lipschitzian coefficients (see [9], §5.2) and so it is based on the following local uniqueness result.

Proposition 5.1. *Let the assumptions* (A) *and* (P) *be satisfied, let* $p > 2, \delta \in [0, \frac{1}{2} - \frac{1}{p}[$. *Let*

$$f_i : [0,T] \times H_\delta \longrightarrow H, \quad \sigma_i : [0,T] \times H_\delta \longrightarrow \mathcal{L}(Y,H), \quad i = 1,2,$$

be measurable functions such that both the pairs (f_1, σ_1) *and* (f_2, σ_2) *fulfil* (L_δ). *Let* $D \subseteq H_\delta$ *be a domain such that*

$$f_1 = f_2, \ \sigma_1 = \sigma_2 \text{ on } [0,T] \times D.$$

Let $\varphi_i \in L^p(\Omega; H_\delta)$, $i = 1,2$, *be* $\mathcal{F}_0$*-measurable. Set* $\Xi = \{\omega \in \Omega; \varphi_1(\omega) \in D$ *or* $\varphi_2(\omega) \in D\}$ *and assume*

$$\chi_\Xi \varphi_1 = \chi_\Xi \varphi_2 \text{ almost surely.} \tag{5.1}$$

Let us denote by x_i *the mild solutions to the problems*

$$\mathrm{d}x_i(t) = [A(t)x_i(t) + f_i(t, x_i(t))]\,\mathrm{d}t + \sigma_i(t, x_i(t))\,\mathrm{d}w(t),$$
$$x_i(0) = \varphi_i,$$

$i = 1,2$. *Set* $\tau_i(\omega) = \inf\{t \in [0,T]; x_i(t,\omega) \notin D\}$ *(with the convention* $\inf\emptyset = T$*). Then*

$$\mathsf{P}\{\omega; \tau_1(\omega) = \tau_2(\omega)\} = 1,$$

and

$$\mathsf{P}\{\omega; \sup_{0 \leqslant t \leqslant \tau_1(\omega)} \|x_1(t,\omega) - x_2(t,\omega)\|_\delta = 0\} = 1.$$

Proof. Set $\xi(t,\omega) = \inf\{\chi_D(x_1(s,\omega)), 0 \leqslant s \leqslant t\}$. By the definition of a mild solution

$$
\begin{aligned}
\xi(t)[x_1(t) - x_2(t)] &= \xi(t)U(t,0)[\varphi_1 - \varphi_2] \\
&\quad + \xi(t)\int_0^t U(t,s)[f_1(s,x_1(s)) - f_2(s,x_1(s))]\,\mathrm{d}s \\
&\quad + \xi(t)\int_0^t U(t,s)[f_2(s,x_1(s)) - f_2(s,x_2(s))]\,\mathrm{d}s \\
&\quad + \xi(t)\int_0^t U(t,s)[\sigma_1(s,x_1(s)) - \sigma_2(s,x_1(s))]\,\mathrm{d}w(s) \\
&\quad + \xi(t)\int_0^t U(t,s)[\sigma_2(s,x_1(s)) - \sigma_2(s,x_2(s))]\,\mathrm{d}w(s) \\
&\equiv \xi(t)U(t,0)[\varphi_1 - \varphi_2] + M_1 + \ldots + M_4.
\end{aligned}
$$

We have $\xi(t)[\varphi_1 - \varphi_2] = 0$ almost surely by (5.1). Further, if $\xi(t,\omega) = 1$, then $f_1(s,x_1(s,\omega)) = f_2(s,x_1(s,\omega))$, $0 \leqslant s \leqslant t$, hence $M_1 = 0$. If $t < \tau_1(\omega)$, then $\sigma_1(s,x_1(s,\omega)) = \sigma_2(s,x_1(s,\omega))$, $0 \leqslant s \leqslant t$, hence

$$
\int_0^t U(t,s)\sigma_1(s,x_1(s))\,\mathrm{d}w(s) = \int_0^t U(t,s)\sigma_2(s,x_1(s))\,\mathrm{d}w(s)
$$

almost surely on the set $\{\omega; t < \tau_1(\omega)\}$. But $\xi(t) = 0$ on $\{\omega; t \geqslant \tau_1(\omega)\}$, hence $M_3 = 0$ a.s.. The obvious fact $\xi(t) \leqslant \xi(s)$, $t \geqslant s$, implies

$$
\begin{aligned}
\|M_2\|_\delta &\leqslant C_\delta K \int_0^t (t-s)^{-\delta}\xi(t)\|x_1(s) - x_2(s)\|_\delta\,\mathrm{d}s \\
&\leqslant C_\delta K \int_0^t (t-s)^{-\delta}\xi(s)\|x_1(s) - x_2(s)\|_\delta\,\mathrm{d}s \\
&\leqslant D_{24}\Big(\int_0^t \xi(s)\|x_1(s) - x_2(s)\|_\delta^p\,\mathrm{d}s\Big)^{1/p}
\end{aligned}
$$

by the Hölder inequality.

Further, proceeding as in the proof of formula (4.7) we obtain

$$
\begin{aligned}
&\xi(t)\Big\|\int_0^t U(t,s)[\sigma_2(s,x_1(s)) - \sigma_2(s,x_2(s))]\,\mathrm{d}w(s)\Big\|_\delta \\
&\qquad \leqslant \Big\|\int_0^t U(t,s)\xi(s)[\sigma_2(s,x_1(s)) - \sigma_2(s,x_2(s))]\,\mathrm{d}w(s)\Big\|_\delta.
\end{aligned}
$$

By Theorem 1.1 we have

$$\begin{aligned}
&\Big| \sup_{0\leqslant t\leqslant r} \Big\| \int_0^t U(t,s)\xi(s)[\sigma_2(s,x_1(s)) - \sigma_2(s,x_2(s))]\,\mathrm{d}w(s)\Big\|_\delta \Big|_p \\
&\quad \leqslant C^{1/p}\Big(\int_0^r \mathsf{E}\|\xi(s)[\sigma_2(s,x_1(s)) - \sigma_2(s,x_2(s))]\|_Q^p\,\mathrm{d}s\Big)^{1/p} \\
&\quad \leqslant C^{1/p}K\Big(\int_0^r \mathsf{E}\|\xi(s)[x_1(s) - x_2(s)]\|_\delta^p\,\mathrm{d}s\Big)^{1/p}.
\end{aligned}$$

This implies

$$\mathsf{E}\sup_{0\leqslant t\leqslant r} \xi(t)\|x_1(t) - x_2(t)\|_\delta^p \leqslant D_{25}\int_0^r \mathsf{E}\xi(s)\|x_1(s) - x_2(s)\|_\delta^p\,\mathrm{d}s.$$

The terms on boths sides of the above inequality are finite by Theorem 1.3, hence the Gronwall lemma yields

$$\sup_{0\leqslant t\leqslant T} \xi(t)\|x_1(t) - x_2(t)\|_\delta^p = 0 \quad \text{almost surely,}$$

hence

$$\sup_{0\leqslant t<\tau_1} \|x_1(t) - x_2(t)\|_\delta = 0 \quad \text{almost surely}$$

and using the path continuity of x_1, x_2 we complete the proof. □

Proposition 5.2. *Let the assumptions* (A), (P), (I), *and* (LL$_\delta$) *be fulfilled. Let* (1.6) *hold and let* $\varphi \in L^p(\Omega; H_\delta)$, $p > 2$, $\delta \in [0, \frac{1}{2} - \frac{1}{p}[$, *and let either* $\delta < \varrho$, *or let* φ *be* H_ζ*-valued,* $\zeta > \delta$. *Then there exists a local mild solution* (x, ε) *of the problem* (1.2), (1.3) *such that* $x(.,\omega) \in C([0,\varepsilon(\omega)[; H_\delta) \cap C^{0,\lambda}(F; H_\varkappa)$ *for almost all* $\omega \in \Omega$, *any compact set* F *in* $]0,\varepsilon(\omega)[$, *and any* $\varkappa \in [0, \frac{1}{2}[$, $\lambda \in [0, \frac{1}{2} - \varkappa[$. *If, moreover,* (LG$_\delta$) *is fulfilled, then the solution* x *is global and* $x(.,\omega) \in C([0,T]; H_\delta) \cap C^{0,\lambda}([a,T]; H_\varkappa)$ *for almost all* $\omega \in \Omega$ *and any* $a > 0$, $\varkappa \in [0, \frac{1}{2}[$, $\lambda \in [0, \frac{1}{2} - \varkappa[$.

Proof. For $N \in \mathbf{N}$ arbitrary let us set

$$f_N(t,x) = \begin{cases} f(t,x), & t \geqslant 0,\ \|x\|_\delta \leqslant N \\ f(t,x)(2 - \|x\|_\delta/N), & t \geqslant 0,\ N < \|x\|_\delta \leqslant 2N \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$\sigma_N(t,x) = \begin{cases} \sigma(t,x), & t \geqslant 0,\ \|x\|_\delta \leqslant N \\ \sigma(t,x)(2 - \|x\|_\delta/N), & t \geqslant 0,\ N < \|x\|_\delta \leqslant 2N \\ 0, & \text{elsewhere.} \end{cases}$$

Then the functions f_N, σ_N fulfil the assumption (L_δ) for each $N \in \mathbf{N}$, the linear growth being implied by (1.6).

According to Theorem 1.3 there exists a unique mild solution x_N of the problem

$$\mathrm{d}x_N(t) = [A(t)x_N(t) + f_N(t, x_N(t))]\,\mathrm{d}t + \sigma_N(t, x_N(t))\,\mathrm{d}w(t), \qquad x_N(0) = \varphi; \tag{5.2}$$

this solution is such that $x_N \in \mathcal{C}([0,T]; H_\delta) \cap \mathcal{C}^{0,\lambda}([a,T]; H_\varkappa)$ almost surely, $a > 0$, $\lambda < \frac{1}{2} - \varkappa$. Define

$$\tau_N(\omega) = \inf\{t \in [0,T];\ \|x_N(t,\omega)\|_\delta \geqslant N\} \quad (\inf \emptyset \equiv T).$$

By Proposition 5.1 we have

$$\mathrm{P}\{\omega \in \Omega;\ \sup_{M \geqslant N} \sup_{0 \leqslant t \leqslant \tau_N(\omega)} \|x_M(t,\omega) - x_N(t,\omega)\|_\delta = 0\} = 1.$$

Note that $\{\tau_R\}_{R=1}^\infty$ is an almost surely nondecreasing and eventually strictly positive sequence by the continuity of sample paths of x_R in H_δ. Set $\varepsilon(\omega) = \lim_{R \to \infty} \tau_R(\omega)$. Setting

$$x(t,\omega) = \lim_{N \to \infty} x_N(t,\omega), \quad 0 \leqslant t < \varepsilon(\omega),$$

we obtain a well-defined stochastic process with sample paths of the required regularity. Let $\omega \in \Omega$ be such that $x(.\,,\omega)$ is continuous and $\varepsilon(\omega) < T$. As, by definition, $\|x(\tau_N(\omega),\omega)\|_\delta = \|x_N(\tau_N(\omega),\omega)\|_\delta = N$ and $\tau_N(\omega) \to \varepsilon(\omega)$, we have

$$\limsup_{t \to \varepsilon} \|x(t)\|_\delta = +\infty \quad \text{almost surely on} \quad \{\omega; \varepsilon(\omega) < T\},$$

and ε is the explosion time for x. We have to prove that x is a local mild solution. By the definition of a (global) mild solution and the continuity of sample paths we can obtain a null-set $\Gamma \in \mathcal{F}$ such that

$$x_N(t) = U(t,0)\varphi + \int_0^t U(t,s) f_N(s, x_N(s))\,\mathrm{d}s + \int_0^t U(t,s)\sigma_N(s, x_N(s))\,\mathrm{d}w(s)$$

holds for any $t \in [0,T]$ and $\omega \notin \Gamma$; thus also

$$x_N(t \wedge \tau_N) = U(t \wedge \tau_N, 0)\varphi + \int_0^{t \wedge \tau_N} U(t \wedge \tau_N, s) f_N(s, x_N(s))\,\mathrm{d}s + \int_0^{t \wedge \tau_N} U(t \wedge \tau_N, s)\sigma_N(s, x_N(s))\,\mathrm{d}w(s)$$

for all t and $\omega \notin \Gamma$. Our construction yields that $x_N(s,\omega) = x(s,\omega)$, $f_N(s, x_N(s,\omega)) = f(s, x(s,\omega))$, $\sigma_N(s, x_N(s,\omega)) = \sigma(s, x(s,\omega))$ for $0 \leqslant s \leqslant \tau_N(\omega)$, hence

$$x(t \wedge \tau_N) = U(t \wedge \tau_N, 0)\varphi + \int_0^{t\wedge\tau_N} U(t \wedge \tau_N, s) f(s, x(s))\,\mathrm{d}s + \int_0^{t\wedge\tau_N} U(t \wedge \tau_N, s)\sigma(s, x(s))\,\mathrm{d}w(s)$$

for any $N \in \mathbf{N}$ and $t \in [0,T]$ almost surely.

Finally, let us assume that (LG_δ) is fulfilled. Then

$$\sup_{N\in\mathbf{N}} \sup_{t\in[0,T]} [\|f_N(t,x)\|_0 + \|\sigma_N(t,x)\|_Q] \leqslant K^*(1 + \|x\|_\delta), \tag{5.3}$$

where K^* is the constant introduced in (LG_δ). Set $\Omega_N = \{\omega; \tau_N(\omega) = T\}$. Using the estimate stated in Theorem 1.3(iv) we obtain

$$\begin{aligned} \mathsf{P}(\Omega \setminus \Omega_N) &= \mathsf{P}\{\omega; \sup_{0\leqslant t\leqslant T} \|x_N(t,\omega)\|_\delta \geqslant N\} \leqslant N^{-p}\mathsf{E} \sup_{0\leqslant t\leqslant T} \|x_N(t,\omega)\|_\delta^p \\ &\leqslant N^{-p}(C^+)^p(1 + \mathsf{E}\|\varphi\|_\delta^p) \longrightarrow 0, \quad N \to \infty, \end{aligned}$$

as the constant C^+ is independent of N by (5.3). Hence it is easy to see that x solves (1.2) in the sense of Definition 1.1 and its sample paths have the required regularity. □

The following uniqueness result will play now the same role as Lemma 4.2 plays in the proof of Theorem 1.3.

Lemma 5.3. *Let the assumptions* (A), (P) *and* (LL_δ) *be fulfilled. Let* $\varphi, \psi\colon \Omega \longrightarrow H_\delta$ *be* $\mathcal{F}_0$*-measurable. Let* (x, ε_x), (y, ε_y) *be local mild solutions of the equation* (1.2) *with initial data* $x(0) = \varphi$, $y(0) = \psi$. *Assume that* $x(.,\omega) \in C([0, \varepsilon_x(\omega)[; H_\delta)$, $y(.,\omega) \in C([0, \varepsilon_y(\omega)[; H_\delta)$ *for almost every* $\omega \in \Omega$. *Set* $\Sigma = \{\omega \in \Omega; \varphi(\omega) = \psi(\omega)\}$. *Then*

$$\chi_\Sigma \varepsilon_x = \chi_\Sigma \varepsilon_y \quad \textit{almost surely}$$

and

$$\chi_\Sigma \chi_{[t<\varepsilon_x]} x(t) = \chi_\Sigma \chi_{[t<\varepsilon_y]} y(t) \quad \textit{almost surely}$$

for any $t \in [0,T]$. *In particular, if* x, y *are global mild solutions to* (1.2) *with sample paths in* $C([0,T]; H_\delta)$, *then*

$$\chi_\Sigma x(t) = \chi_\Sigma y(t) \quad \textit{almost surely}$$

for any $t \in [0,T]$.

R e m a r k. In particular, under (A), (P) and (LL_δ) there exists—up to a modification—at most one mild solution to (1.2) with trajectories continuous in H_δ for any $\mathcal{F}_0$-measurable initial condition $\varphi\colon \Omega \longrightarrow H_\delta$.

P r o o f. For $N \in \mathbf{N}$ let us define stopping times

$$\begin{aligned}
\tau_N^x(\omega) &= \inf\{t \in [0,T], \|x(t,\omega)\|_\delta \geqslant N\} \wedge \varepsilon_x^{(N)}(\omega),\\
\tau_N^y(\omega) &= \inf\{t \in [0,T], \|y(t,\omega)\|_\delta \geqslant N\} \wedge \varepsilon_y^{(N)}(\omega),\\
\tau_N &= \tau_N^x \wedge \tau_N^y,
\end{aligned}$$

where $\{\varepsilon_x^{(n)}\}$, $\{\varepsilon_y^{(n)}\}$ are the sequences of stopping times the existence of which is postulated in Definition 1.2. (As usual, $\inf \emptyset = T$.) Fix N for a while and set $\xi(t,\omega) = \chi_{[0,\tau_N(\omega)]}(t)$. Using Definition 1.2(c) and the $\mathcal{F}_0$-measurability of χ_Σ we obtain the equality

$$\begin{aligned}
\chi_\Sigma \xi(t)(x(t) - y(t)) = \xi(t) &\int_0^t U(t,s)\chi_\Sigma[f(s,x(s)) - f(s,y(s))]\,\mathrm{d}s\\
&+ \xi(t)\int_0^t U(t,s)\chi_\Sigma[\sigma(s,x(s)) - \sigma(s,y(s))]\,\mathrm{d}w(s).
\end{aligned}$$

Using again the same procedure as in the derivation of (4.7) we get

$$\begin{aligned}
\|\chi_\Sigma\xi(t)[x(t) - y(t)]\|_\delta \leqslant &\int_0^t \|U(t,s)\chi_\Sigma\xi(s)[f(s,x(s)) - f(s,y(s))]\|_\delta\,\mathrm{d}s\\
&+ \Big\|\int_0^t U(t,s)\chi_\Sigma\xi(s)[\sigma(s,x(s)) - \sigma(s,y(s))]\,\mathrm{d}w(s)\Big\|_\delta
\end{aligned}$$

almost surely, hence

$$\begin{aligned}
&\|\chi_\Sigma\xi(t)[x(t) - y(t)]\|_{2,\delta}\\
&\qquad\leqslant C_\delta\int_0^t (t-s)^{-\delta}\big|\chi_\Sigma\xi(s)\|f(s,x(s)) - f(s,y(s))\|_0\big|_2\,\mathrm{d}s\\
&\qquad\quad + C_\delta\Big(\int_0^t (t-s)^{-2\delta}\big|\chi_\Sigma\xi(s)\|\sigma(s,x(s)) - \sigma(s,y(s))\|_Q\big|_2^2\,\mathrm{d}s\Big)^{1/2}.
\end{aligned}$$

By the definition of τ_N, if $\omega \in \Omega$ is such that $\xi(t,\omega) = 1$, then

$$\begin{aligned}
\|f(s,x(s,\omega)) - f(s,y(s,\omega))\|_0 + \|\sigma(s,x(s,\omega)) - \sigma(s,y(s,\omega))\|_Q&\\
\leqslant K_N\|x(s,\omega) - y(s,\omega)\|_\delta,&
\end{aligned}$$

thus

$$\|\chi_\Sigma\xi(t)[x(t) - y(t)]\|_{2,\delta}^2 \leqslant D_{26}K_N\int_0^t (t-s)^{-2\delta}\|\chi_\Sigma\xi(s)[x(s) - y(s)]\|_{2,\delta}^2\,\mathrm{d}s.$$

As $\|\xi(s)[x(s)-y(s)]\|_{2,\delta}^2 \leqslant 4N^2$, $0 \leqslant s \leqslant T$, Lemma 2.3 may be applied and we have

$$\|\chi_\Sigma \xi(t)[x(t)-y(t)]\|_{2,\delta} = 0$$

for each $t \in [0,T]$. Thus $\chi_\Sigma x(t) = \chi_\Sigma y(t)$ almost surely on the set $\{\omega; t \leqslant \tau_N\}$ for arbitrary N, so by passing $N \to \infty$ we obtain

$$\chi_\Sigma x(t) = \chi_\Sigma y(t) \quad \text{on} \quad \{\omega, t < \varepsilon_x \wedge \varepsilon_y\} \tag{5.4}$$

for any $t \in [0,T]$ almost surely.

It remains to prove $\varepsilon_x = \varepsilon_y$ a.s. on Σ. To this end, let us realize that there exists a null set $\Gamma \in \mathcal{F}$, $\mathsf{P}(\Gamma) = 0$, such that the equality (5.4) holds for any $\omega \in \Omega \setminus \Gamma$ and $0 \leqslant t < \varepsilon_x(\omega) \wedge \varepsilon_y(\omega)$. Let us assume that $\tilde{\omega} \notin \Gamma$ is such that $x(.,\tilde{\omega})$ and $y(.,\tilde{\omega})$ are continuous and $\varepsilon_x(\tilde{\omega}) < \varepsilon_y(\tilde{\omega})$. We can find $R \in \mathbf{N}$ such that $\varepsilon_x(\tilde{\omega}) < \tau_R^y(\tilde{\omega})$. By (5.4), $x(s,\tilde{\omega}) = y(s,\tilde{\omega})$ for $0 \leqslant s \leqslant \tau_{R+1}^x(\tilde{\omega}) = \tau_{R+1}(\tilde{\omega}) < \varepsilon_x(\tilde{\omega})$. Thus $R+1 = \|x(\tau_{R+1}^x(\tilde{\omega}),\tilde{\omega})\|_\delta = \|y(\tau_{R+1}^x(\tilde{\omega}),\tilde{\omega})\|_\delta \leqslant R$; this contradiction proves the desired equality.

If x, y are global mild solutions, then (5.4) yields $\chi_\Sigma x(t) = \chi_\Sigma y(t)$ for any $t \in [0,T[$ and $\omega \notin \Gamma$. Now the continuity of sample paths implies $\chi_\Sigma x(T) = \chi_\Sigma y(T)$ a.s. as well. □

Further we can state a proposition on the existence of solutions to (1.2), (1.3) not assuming the integrability of initial data.

Proposition 5.4. *Let the assumptions* (A), (P), (I), (LL_δ) *and* (1.6) *be fulfilled. Assume either that φ is H_δ-valued and $\delta < \varrho$, or that φ is H_ζ-valued and $\zeta > \delta$. Then there exists a local mild solution (x,ε) of the problem* (1.2), (1.3) *such that $x(.,\omega) \in \mathcal{C}([0,\varepsilon(\omega)[;H_\delta) \cap \mathcal{C}^{0,\lambda}(F;H_\varkappa)$ almost surely for any compact set F in $]0,\varepsilon(\omega)[$, $\varkappa \in [0,\frac{1}{2}[$ and $\lambda \in [0,\frac{1}{2}-\varkappa[$.*

If, moreover, the hypothesis (LG_δ) *holds, then x is a global mild solution and $x(.,0) \in \mathcal{C}([0,T];H_\delta) \cap \mathcal{C}^{0,\lambda}([a,T];H_\varkappa)$ almost surely for any $a > 0$, $\varkappa \in [0,\frac{1}{2}[$ and $\lambda \in [0,\frac{1}{2}-\varkappa[$.*

Proof. Lemma 5.3 being available, we can proceed similarly as in the proof of Proposition 4.3. Details are left to the reader. □

The proof of Theorem 1.5 is almost completed, it remains only to establish the estimates (1.7), (1.8). The proof of (1.8) is straightforward. Let x_N be the solution of the problem (5.2), then

$$\Big| \sup_{0\leqslant t\leqslant T} \|x_N(t)\|_\delta \Big|_p \leqslant C^+(1+\|\varphi\|_{p,\delta})$$

by Theorem 1.3(iv); furthermore, we know that C^+ does not depend on $N \in \mathbb{N}$. In the proof of Proposition 5.2 we established

$$\sup_{0\leqslant t\leqslant T} \|x_N(t)\|_\delta^p \longrightarrow \sup_{0\leqslant t\leqslant T} \|x(t)\|_\delta^p, \quad N \to \infty,$$

almost surely. So the application of the Fatou lemma yields

$$\begin{aligned} \mathsf{E} \sup_{0\leqslant t\leqslant T} \|x(t)\|_\delta^p = \mathsf{E} \lim_{N\to\infty} \sup_{0\leqslant t\leqslant T} \|x_N(t)\|_\delta^p \leqslant \liminf_{N\to\infty} \mathsf{E} \sup_{0\leqslant t\leqslant T} \|x_N(t)\|_\delta^p \\ \leqslant [C^+(1+\|\varphi\|_{p,\delta})]^p. \end{aligned}$$

The proof of (1.7) uses only a slightly more complicated approximation procedure. Let us denote by $x_N^{(M)}$ the mild solutions to the problems

$$\begin{aligned} \mathrm{d}x_N^{(M)}(t) &= [A(t)x_N^{(M)}(t) + f_N(t, x_N^{(M)}(t))]\,\mathrm{d}t + \sigma_N(t, x_N^{(M)}(t))\,\mathrm{d}w(t), \\ x_N^{(M)}(0) &= \varphi_M, \end{aligned}$$

$N, M \in \mathbb{N}$, where the coefficients f_N, σ_N are the same as in the equation (5.2) and

$$\varphi_M(\omega) = \begin{cases} \varphi(\omega), & \|\varphi(\omega)\|_\delta \leqslant M \\ 0, & \text{otherwise.} \end{cases}$$

As $\varphi_M \in L^\infty(\Omega; H_\delta)$ we can use Proposition 5.1 in the same way as in the proof of Proposition 5.2 to obtain $x_N^{(M)}(t) \longrightarrow x^{(M)}(t), N \to \infty$, in H_δ for each $t \in [0,T]$, where $x^{(M)}$ solves the equation (1.2) with the initial condition $x^{(M)}(0) = \varphi_M$. Theorem 1.3(iii) implies

$$\|x_N^{(M)}(t)\|_{p,\delta} \leqslant C^*(1 + \|\varphi_M\|_{p,\delta}) \leqslant C^*(1 + \|\varphi\|_{p,\delta}),$$

thus by passing $N \to \infty$ we obtain

$$\|x^{(M)}(t)\|_{p,\delta} \leqslant C^*(1 + \|\varphi\|_{p,\delta}), \quad 0 \leqslant t \leqslant T,$$

by the Fatou lemma. Finally, $x^{(M)}(t) \longrightarrow x(t)$ almost surely in H_δ by Lemma 5.3, so we can again apply the Fatou lemma to obtain (1.7). □

Example 6.1. Let us discuss briefly a particular example in which our assumptions are fulfilled. Let $G \subseteq \mathbf{R}^N$ be a bounded domain with a boundary of class $\mathcal{C}^{2m}$. Set

$$\mathcal{A}(t,x,D) = \sum_{|\alpha| \leqslant 2m} a_\alpha(t,x) \left(\frac{\partial}{\partial x}\right)^\alpha$$

assuming

(i) $\mathcal{A}$ is strongly elliptic uniformly in $t \in [0,T]$, that is
$\exists c > 0 \ \forall t \in [0,T] \ \forall x \in \overline{G} \ \forall \xi \in \mathbf{R}^N$

$$(-1)^m \sum_{|\alpha|=2m} a_\alpha(t,x)\xi^\alpha \geqslant c\|\xi\|^{2m},$$

(ii) $a_\alpha(t,.) \in \mathcal{C}(\overline{G})$ for $|\alpha| = 2m$ and $0 \leqslant t \leqslant T, a_\alpha(t,.)$ are bounded measurable functions for $|\alpha| < 2m$, $t \in [0,T]$,

(iii) there exist $\varrho \in]0,1]$ and $L > 0$ such that

$$\max_{|\alpha| \leqslant 2m} \sup_{x \in G} |a_\alpha(t,x) - a_\alpha(s,x)| \leqslant L|t-s|^\varrho$$

for all $t, s \in [0,T]$.

Let $\{B_1, \ldots, B_m\}$ be a normal system of boundary operators on ∂G (see [32], Def. 3.7.1)-independent of t. Assume for simplicity that $\{B_j\}$ is of Dirichlet type, i.e. B_j are of orders ω_j, $\omega_1 < \ldots < \omega_m < m$. Set $Dom(A(t)) = W_B^{2m,2}$, $A(t)u = -\mathcal{A}(t,.,D)u(.)$, where

$$W_B^{2m,2} \equiv \{y \in W^{2m,2}(G), B_j y = 0 \text{ on } \partial G, j = 1, \ldots, m\},$$

and $W^{2m,2}(G)$ stands for the usual Sobolev space of L^2-functions with distributive derivatives up to order $2m$ belonging to $L^2(G)$. Then the operators $\{A(t) - kI, 0 \leqslant t \leqslant T\}$ satisfy (P1)–(P3) for some $k \geqslant 0$ (see [32], §5.2, cf. also [27], §7.6). Without a loss of generality we will assume $k = 0$, i.e. $a_0(t,x) \geqslant \widehat{a}$, where $\widehat{a}$ is a sufficiently large constant. As we mentioned in Section 1, the assumption (P4) is fulfilled e.g. if $A(t)$ are self-adjoint on $L^2(G)$. More generally, by [29], Th. 1, (0.6) (and hence (P4)) holds if the above assumptions are fulfilled and ∂G is of class $\mathcal{C}^\infty, a_\alpha(t,.) \in \mathcal{C}^\infty(\overline{G})$, and B_j have $\mathcal{C}^\infty$-coefficients (cf. also the example in [1], §7). Further, according to [30], Th. 4.1, one has

$$[L^2(G), W_B^{2m,2}]_\varkappa = W_B^{2m\varkappa,2}, \quad \varkappa \in]0,1[,$$

$W_B^{2m\varkappa,2}$ being a subspace in $W^{2m\varkappa,2}(G)$ determined by some boundary conditions (dependent on $\varkappa$); in particular,

$$W_B^{2m\varkappa,2} = \{y \in W^{2m\varkappa,2}(G), B_j y = 0 \text{ on } \partial G, j = 1, \dots, m\},$$

if $\varkappa \in](m - \frac{1}{2})/2m, 1[$. By $W^{s,2}(G)$, $s = r + \lambda$, $r \in \mathbf{N}$, $0 < \lambda < 1$, we denote the Sobolev-Slobodeckiĭ space (see [21], §8.3):

$$W^{s,2}(G) = \left\{ y \in W^{r,2}(G), \forall \alpha, |\alpha| = r, \int_G \int_G \frac{|D^\alpha y(x) - D^\alpha y(z)|^2}{|x - z|^{N+2\lambda}} \, dx \, dz < \infty \right\}.$$

By [33], Th. 4.6.1, the space $W^{2m\varkappa,2}(G)$ is continuously imbedded into the space $C^{k,\alpha}(\overline{G})$ of functions with α-Hölder continuous k-th derivative on $\overline{G}$ provided $2m\varkappa > k + \alpha + N/2$.

So, let us investigate the stochastic parabolic problem, formally written as

$$(6.1) \qquad \begin{aligned} \frac{\partial u}{\partial t}(t, x) + \mathcal{A}(t, x, D)u(t, x) &= F(u(t, x)) + \Sigma(u(t, x))\dot{w}(t, x), \\ B_j u(t, x) &= 0 \quad \text{on } \partial G, \; j = 1, \dots, m, \\ u(0, x) &= \varphi(x). \end{aligned}$$

Here $\mathcal{A}$ is the parabolic operator introduced above, $F\colon \mathbf{R} \longrightarrow \mathbf{R}$ and $\Sigma\colon \mathbf{R} \longrightarrow \mathbf{R}$ are Lipschitz continuous functions, Σ bounded, and $\dot{w}$ stands symbolically for a correlated space-time noise. We transform the equation (6.1) into the form (1.2) in the following straightforward way. Set $H = Y = L^2(G)$, let $w(t)$ be an $L^2(G)$- valued Wiener process on a probability space Ω, with a nuclear covariance operator. Let $\varphi\colon \Omega \longrightarrow L^2(G)$ be a random variable independent of the process w. The definition of the operators $A(t)$ is given above; suppose that the hypothesis (P4) holds. Finally, let us assume that $2m > N$ so that we can take $\delta \in [0, 1/2[$ such that $2m\delta > N/2$. The space H_δ is then continuously imbedded into $C^{0,\alpha}(\overline{G})$ (for some $\alpha \in]0, 2m\delta - N/2[$) and hence the mappings

$$\begin{aligned} f\colon H_\delta &\longrightarrow L^2(G), \;\; f(y) = F(y(.)), \quad y \in H_\delta, \\ \sigma\colon H_\delta &\longrightarrow \mathcal{L}(L^2(G)), \quad \sigma(y)h = \Sigma(y(.))h(.), \quad y \in H_\delta, \; h \in L^2(G), \end{aligned}$$

satisfy the assumption (L_δ). Indeed, for f this is obvious. Since Σ is a bounded function, $\Sigma(y(.)) \in L^\infty(G)$ for any $y \in H$ and multiplication by an L^∞-function is a continuous linear operator on $L^2(G)$. Furthermore,

$$\begin{aligned} \|\sigma(y)h - \sigma(z)h\|_H^2 &= \int_G |[\Sigma(y(\xi)) - \Sigma(z(\xi))]h(\xi)|^2 d\xi \\ &\leqslant \operatorname*{ess\,sup}_{\xi \in G} |\Sigma(y(\xi)) - \Sigma(z(\xi))|^2 \int_G |h(\xi)|^2 d\xi \\ &\leqslant \mathrm{Lip}(\Sigma)^2 \|y - z\|_{L^\infty(G)}^2 \|h\|_{L^2(G)}^2 \leqslant \text{const.}\; \|h\|_H^2 \|y - z\|_\delta^2 \end{aligned}$$

for any $y, z \in H_\delta$, $h \in L^2(G)$, as H_δ is imbedded in $\mathcal{C}^{0,\alpha}(\overline{G})$, hence a fortiori in $L^\infty(G)$. Moreover, we can choose $\delta > 1/2 - 1/4m$, so that any function $y \in H_\delta$ may fulfil all the boundary conditions $B_j y = 0$, $j = 1, \ldots, m$, on ∂G.

Applying Theorem 1.3 we obtain a unique mild solution x to (6.1) whose sample paths are continuous on $]0, T]$ as $W^{m-\varepsilon,2}(G)$ -valued functions for $\varepsilon > 0$ arbitrarily small but positive and $B_j x(t) = 0$ on ∂G, $j = 1, \ldots, m$, $t > 0$. In particular, let us investigate the problem (6.1) in one space dimension ($N = 1$), taking for $\mathcal{A}$ a differential operator of the second order with Dirichlet boundary data. The mild solution x then satisfies $x \in \mathcal{C}^{0,\lambda}([\tau, T]; \mathcal{C}^{0,\frac{1}{2}-\varepsilon}(\overline{G}))$ almost surely for any $\tau > 0$ and $0 < \lambda < \varepsilon < \frac{1}{2}$ and can be viewed as a random field with Hölder continuous sample paths.

Example 6.2. To give some idea of the scope of applicability of Theorem 1.4 we will consider a stochastic symmetric hyperbolic system

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{n} a_j(t,x)\frac{\partial u}{\partial x_j} + b(t,x)u + h(t,x,u) + g(t,x,u)\dot{w}(t), \tag{6.2}$$
$$x \in \mathbf{R}^n,\ 0 \leqslant t \leqslant T,\ u(0,x) = u_0(x).$$

Here $u = (u_1, \ldots, u_N)^T$ is the unknown N-tuple of stochastic processes (the superscript ^{T} denoting transposition), $w(t)$ is a standard d-dimensional Brownian motion, $a_j(t,x)$, $b(t,x)$ are $N \times N$- matrices, a_j Hermitian for any $(t,x) \in [0,T] \times \mathbf{R}^n$. Let the functions a_j, b fulfil

$$a_j \in \mathcal{C}([0,T]; \mathcal{C}_b^1(\mathbf{R}^n; \mathbf{M}_{N\times N})),\ b \in \mathcal{C}([0,T]; \mathcal{C}_b^0(\mathbf{R}^n; \mathbf{M}_{N\times N})),$$
$$\frac{\partial b}{\partial x_j} \in \mathcal{C}_b^0([0,T] \times \mathbf{R}^n; \mathbf{M}_{N\times N}),\ j = 1, \ldots, n,$$

where we denote by $\mathbf{M}_{k\times l}$ the set of all $k \times l$-matrices and by $\mathcal{C}_b^m$ the Banach space of all functions whose derivatives up to the degree m are continuous and bounded. Moreover, let

$$h\colon [0,T] \times \mathbf{R}^n \times \mathbf{R}^N \longrightarrow \mathbf{R}^N, \qquad g\colon [0,T] \times \mathbf{R}^n \times \mathbf{R}^N \longrightarrow \mathbf{M}_{N\times d}$$

be measurable functions such that for a function $k \in L^\infty(\mathbf{R}^n)$ and every $t \in [0,T]$, $x \in \mathbf{R}^n$, $r, v \in \mathbf{R}^N$ one has

$$\|h(t,x,r) - h(t,x,v)\| + \|g(t,x,r) - g(t,x,v)\| \leqslant k(x)\|r - v\|,$$

$$\|h(t,x,0)\| + \|g(t,x,0)\| \leqslant k(x).$$

We can easily transform (6.2) into the form (1.2). Set $H = [L^2(\mathbf{R}^n)]^N$, and for any $t \in [0,T]$ define

$$\mathcal{A}(t)u = \sum_{j=1}^{n} a_j(t,x)\frac{\partial u}{\partial x_j} + b(t,x)u \tag{6.3}$$

(by our assumptions, the right-hand side of (6.3) makes sense as a distribution for each $u \in H$), and

$$Dom(A(t)) = \{u \in H, \mathcal{A}(t)u \in H\}, \; A(t)u = \mathcal{A}(t)u, \; u \in Dom(A(t)).$$

Then there exists an evolution operator U for $\{A(t), 0 \leqslant t \leqslant T\}$ satisfying (E) (see [32], §4.6). Further, the mappings

$$f\colon [0,T] \times H \longrightarrow H, \qquad (t,\varphi) \longmapsto h(t,.,\varphi(.));$$
$$\sigma\colon [0,T] \times H \longrightarrow \mathcal{L}(\mathbf{R}^d, H), \qquad (t,\varphi) \longmapsto g(t,.,\varphi(.))$$

fulfil the assumption (L_0). Finally, suppose $u_0 \in H$ almost surely and let u_0, $w(t)$ be independent. Then Theorem 1.4 gives us a unique mild solution to (6.2), the paths of which are continuous as $[L^2(\mathbf{R}^n)]^N$-valued functions.

It should be remarked, however, that in the present case of a finite-dimensional noise much more regular solutions can be obtained by different methods, cf. e.g. [28], Prop. 3.2.

Remark added in proof. This paper having been submitted I learned that a result closely related to our Theorem 1.3 was obtained independently by D. Gątarek (A note on nonlinear stochastic equations in Hilbert spaces, to appear).

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