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GRAPHS IN WHICH EACH PAIR OF VERTICES HAS EXACTLY TWO COMMON NEIGHBOURS

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Summary. The paper studies graphs in which each pair of vertices has exactly two common neighbours. It disproves a conjecture by P. Hliněný concerning these graphs.

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At the Czechoslovak conference on graph theory [1] at Zemplínska Šírava in June 1991, P. Hliněný proposed the problem to describe all connected graphs G with the property that for any two distinct vertices of G there exist exactly two vertices which are adjacent to both of them in G. He conjectured that such graphs are only two, namely K_4 and $K_4 \times K_4$. In this paper we disprove this conjecture by showing another graph with this property, and present some other results concerning this topic.

We consider finite undirected graphs without loops and multiple edges. We use the standard terminology of the graph theory, as it is for example in [2]. The symbol $V(G)$ denotes the vertex set of a graph G. By the symbol K_n we denote the complete graph with n vertices, i.e. a graph with n vertices such that each pair of distinct vertices is joined by an edge in it. The direct product $G_1 \times G_2$ of two graphs G_1, G_2 is the graph whose vertex set is the Cartesian product $V(G_1) \times V(G_2)$ and in which two vertices $[u_1, u_2]$, $[v_1, v_2]$ are adjacent if and only if either u_1, v_1 are adjacent in G_1 and $u_2 = v_2$, or $u_1 = v_1$ and u_2, v_2 are adjacent in G_2. If $u \in V(G)$, then by $N(u)$ we denote the set of vertices of G adjacent to u and $\overline{N}(u) = V(G) - \big(N(u) \cup \{u\}\big)$. The wheel W_n is the graph obtained from the circuit C_n of length n by adding a new vertex (centre of the wheel) and joining it by edges with all vertices of C_n.

Now we shall disprove the conjecture of P. Hliněný.

Theorem 1. *There exists a regular connected graph G of degree 6 with 16 vertices such that any two distinct vertices of G have exactly two common neighbours in G and G is not isomorphic to $K_4 \times K_4$.*

Proof. We shall describe this graph G. The vertex set of G is $\{u_1, \dots, u_{12}, v_0, v_1, v_2, v_3\}$. The edges are $u_i u_{i+1}$ for each $i \in \{1, \dots, 12\}$, $u_i u_{i+2}$ for all even $i \in \{1, \dots, 12\}$, $u_i u_{i+4}$ for all odd $i \in \{1, \dots, 12\}$, the subscripts being always taken modulo 12, and further $u_i u_0$ for all even $i \in \{1, \dots, 12\}$, $u_i v_1$ for all $i \in \{1, 2, 3, 7, 8, 9\}$, $u_i v_2$ for all $i \in \{3, 4, 5, 9, 10, 11\}$, $u_i v_3$ for all $i \in \{1, 5, 6, 7, 11, 12\}$. Evidently the graph thus described is not $K_4 \times K_4$. By help of a computer it was verified that it has the mentioned property. □

Now we shall show some properties of the graphs mentioned. If $M \subseteq V(G)$, then by $\langle M \rangle$ we denote the subgraph of G induced by the set M.

Proposition 1. *Let G be a graph in which any two distinct vertices have exactly two common neighbours. The for each $u \in V(G)$ the graph $\langle N(u) \rangle$ is regular of degree 2.*

Proof is straightforward. □

Proposition 2. *Let G be a graph in which any two distinct vertices have exactly two common neighbours. Then no graph $\langle N(u) \rangle$ for $u \in V(G)$ contains a circuit of length 4.*

Proof. Suppose that such a vertex u exists. Then the subgraph $\langle N(u) \cup \{u\} \rangle$ contains the wheel W_4 as its connected component; this wheel contains two pairs of vertices having three common neighbours, which is a contradiction. □

Theorem 2. *Let G be a connected graph in which any two distinct vertices have exactly two common neighbours. Let G contain a vertex u of degree $r \geqslant 5$. Then G is regular of degree r and its number of vertices is $n = \frac{1}{2}(r^2 - r + 2)$.*

Proof. If $x \in N(u)$, then the vertices u, x have exactly two common neighbours in $\langle N(u) \cup \{u\} \rangle$. Two vertices of $N(u)$ have exactly two common neighbours in $\langle N(u) \cup \{u\} \rangle$ if and only if they are not connected by a path of length 2 in $\langle N(u) \rangle$. Otherwise they have exactly one common neighbour u. Therefore for any two distinct vertices x, y of $N(u)$ which are not connected by a path of length 2 in $\langle N(u) \rangle$ there exists exists exactly one vertex $p(x, y) \in \overline{N}(u)$ adjacent to both x and y. On the other hand, let $v \in \overline{N}(u)$. As u and v have exactly two common neighbours in G, the vertex v is adjacent to exactly two vertices of $N(u)$. These two vertices are not connected by a path of length 2 in $\langle N(u) \rangle$, because otherwise they would have three common neighbours in G, namely u, v and the inner vertex of the mentioned path in $\langle N(u)\rangle$. We see that p is a bijection of the set of all two-element subsets of $N(u)$ with the property that their elements are not connected by a path of length 2 in $\langle N(u)\rangle$, onto $\overline{N}(u)$. The graph $\langle N(u)\rangle$ is regular of degree 2 and none of its connected components is isomorphic to C_4. Therefore there are exactly r paths of length 2 in $\langle N(u)\rangle$ and also r pairs of vertices which are connected by such a path. The number of pairs which are not connected by such a path is $\frac{1}{2}r(r-1)-r=\frac{1}{2}r(r-3)$ and therefore $|\overline{N}(u)|=\frac{1}{2}r(r-3)$. As $V(G)$ is the union of pairwise disjoint sets $\{u\}$, $N(u)$, $\overline{N}(u)$ and $|N(u)|=r$, we have $|V(G)|=1+r+\frac{1}{2}r(r-3)=\frac{1}{2}(r^2-r+2)$. Each vertex of $N(u)$ is adjacent to u, to two vertices of $N(u)$ and to $r-3$ vertices of $\overline{N}(u)$ (as the number of vertices not connected to it by a path of length 2 in $\langle N(u)\rangle$ is $r-3$); therefore its degree is r, i.e. the same as that of u. As G is connected, we may proceed by induction and prove that G is regular of degree r. □

Theorem 3. *Let* $r\equiv 1 \pmod 4$. *Then there exists no connected graph* G *in which any two distinct vertices would have exactly two common neighbours and which would contain a vertex of degree* r.

Proof. Suppose that such a graph exists. Then, according to Theorem 2, it is a regular graph of degree r with $\frac{1}{2}(r^2-r+2)$ vertices. But if $r\equiv 1 \pmod 4$, then both r and $\frac{1}{2}(r^2-r+2)$ are odd. Thus the graph G is a regular graph of an odd degree with an odd number of vertices, which is impossible. □

At the end we express a problem.

Problem. Find all graphs in which each pair of vertices has exactly two common neighbours.

References

[1] Graphs '91, The Czechoslovak conference on graph theory held at Zemplínska Šírava in June 1991.

[2] *Ore O.*: Theory of Graphs, AMS Colloq. Publ. Providence, Rhode Island, 1962.

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