Mathematica Bohemica

Hana Petzeltová

Local center manifold for parabolic equations with infinite delay

Mathematica Bohemica, Vol. 119 (1994), No. 3, 285–304

Persistent URL: http://dml.cz/dmlcz/126163

LOCAL CENTER MANIFOLD FOR PARABOLIC EQUATIONS WITH INFINITE DELAY

HANA PETZELTOVÁ, Praha

(Received June 24, 1993)

Summary. The existence and attractivity of a local center manifold for fully nonlinear parabolic equation with infinite delay is proved with help of a solution semigroup constructed on the space of initial conditions. The result is applied to the stability problem for a parabolic integrodifferential equation.

Keywords: parabolic functional equation, infinite delay, center manifold, solution semigroup

AMS classification: 45K05, 35R10, 35B35

1. INTRODUCTION

In this paper we prove the existence and attractivity of a local center manifold for the initial value problem for fully nonlinear equation with infinite delay:

$$\dot{u}(t) = Au(t) + Lu_t + g(u(t), u_t) \tag{E}$$

$$u(0) = x, \quad u(\tau) = \varphi(\tau) \text{ for } \tau < 0, \tag{I}$$

where u_t denotes the shift of the function u: $u_t(\tau) = u(t+\tau)$ for $\tau < 0$, A is a generator of an analytic semigroup in a Banach space X, L is a continuous linear operator from an appropriate function space Y into X and g is a nonlinear, sufficiently smooth function such that $g(0,0) = 0$, $Dg(0,0) = 0$. Thus the equation (E) can be considered a linearization of a fully nonlinear problem

$$\dot{u}(t) = F(u(t), u_t).$$

The following integrodifferential equation can serve as a typical example of the equation (E):

$$
\begin{aligned}
\text{(1)}\qquad \dot{u}(t,x) = \Delta u(t,x) + au(t,x) &+ \int_{-\infty}^{t} k(t-s)(\Delta + b)u(s,x)\,\mathrm{d}s \\
&+ f(u(t,x), Du(t,x), D^2u(t,x)) \\
&+ \int_{-\infty}^{t} l(t-s)h(u(s,x), Du(s,x), D^2u(s,x))\,\mathrm{d}s\,,
\end{aligned}
$$

where D denotes the differentiation with respect to space variables.

Such type of equations arises in the theory of heat conduction in materials with memory or in some models of population dynamics, see e.g. [5].

A center manifold theorem for semilinear equation (E) was proved in [8] and for quasilinear parabolic equations with the application to the reaction diffusion equations in [11]. The same problem for the fully nonlinear parabolic equation without delay was treated in [2], [7]. The existence of a stable and an unstable manifold for the equation (E) was proved in [9]. Here, we make use of results from [9] and show that the usual procedure used in the construction of center manifolds can be carried out also in the case of the equation (E).

To overcome the difficulties connected with the fully nonlinear character of the equation (E), we make use of interpolation spaces between X and $\mathcal{D}(A)$ and the corresponding spaces between Y and $\mathcal{D}(L)$ introduced in [12] and [9].

In the preceding paper [9], the resolvent operator for the linear equation (E) was constructed. It was proved that there exists an operator R satisfiyng the equation

$$\dot{R}(t) = AR(t) + LR_t, \quad R(0) = I, \quad R_0 = 0$$

and having most of the properties of an analytic semigroup. Due to the delay term, the operator R lacks the semigroup property. However, it was proved that the operators

$$S(t)\colon \begin{pmatrix} x \\ \varphi \end{pmatrix} \to \begin{pmatrix} u(t) \\ u_t \end{pmatrix},$$

where u is a solution of the equation

$$\text{(2)}\qquad \dot{u}(t) = Au(t) + Lu_t,\ u(0) = x,\ u_0 = \varphi,$$

form a C_0–semigroup on a certain subspace of $X \times Y$. The semigroup S and its generator B were proved to have the form

$$\text{(3)}\quad S(t)\begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} R(t)x + \int_0^t R(t-s)L\varphi_s\,\mathrm{d}s \\ R_t x + \int_0^t R_{t-s}L\varphi_s\,\mathrm{d}s + \varphi_t \end{pmatrix}, \quad B\begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} Ax + L\varphi \\ \dot{\varphi} \end{pmatrix}.$$

The equation (E) was then replaced by the equation

$$\dot{v}(t) = Bv(t) + f(v(t)), \quad v(0) = \begin{pmatrix} x \\ \varphi \end{pmatrix}, \tag{4}$$

$$\text{with } v(t) = \begin{pmatrix} u(t) \\ u_t \end{pmatrix}, \quad f(v(t)) = \begin{pmatrix} g(u(t), u_t) \\ 0 \end{pmatrix}.$$

The first component of this equation is the equation (E) while the second is the identity.

Due to the special form of the semigroup S, estimates for projections and convolutions of S, similar to those for analytic semigroup, were proved. The form of the nonlinear term in (4), namely the zero in its second component, plays an important role in the estimates, in spite of the fact that the projections of such elements to the eigenspaces do not keep this form. These estimates, which are not generally valid for C_0−semigroups, enable us to construct a center manifold for the fully nonlinear equation (E) in the usual way.

In Section 2, spaces and assumptions used in the paper are collected together with the results from [9], which we need in Section 3 for the proof of existence and attractivity of a center manifold.

The application of the results to the equation of the type (1) with Neumann boundary condition is given in Section 4. Contrary to the Neumann problem for the equation $\dot{u} = \Delta u + f(u, Du, D^2u)$, where the center manifold can be described by the identically zero function, here it is not easy to find an explicit expression even in the case of a unique simple isolated eigenvalue on the imaginary axis. Nevertheless, it is possible to state conditions under which the zero solution of (1) is locally stable.

2. Preliminaries

Let X be a Banach space and

(A) *let A be the generator of an analytic semigroup e^{At} in X.*

We introduce the interpolation spaces between $D(A)$ and X. Let $\|\cdot\|$ denote the norm in the space X, $\omega_0 = \sup\{\mathrm{Re}\,\lambda;\ \lambda \in \sigma(A)\}$. For $\omega_0 < 0$, $\alpha \in (0,1)$ we set

$$D_A(\alpha,\infty) = \{x \in X,\ |x|_\alpha = \sup_{\xi>0} \xi^{1-\alpha} \| A\mathrm{e}^{A\xi}x \| < \infty\},$$
$$D_A(\alpha+1,\infty) = \{x \in D(A);\ Ax \in D_A(\alpha,\infty)\}.$$

For $\omega_0 \geqslant 0$ we set $D_A(\alpha,\infty) = D_{A-2\omega_0}(\alpha,\infty)$, $D_A(\alpha+1,\infty) = D_{A-2\omega_0}(\alpha+1,\infty)$. The closure of $D(A)$ in $D_A(\alpha,\infty)$ in the norm

$$\|x\|_\alpha = \|x\| + |x|_\alpha$$

will be denoted by X^α. It was shown in [11] that $X^\alpha = \{x \in X;\ \lim_{\xi\to 0+} \xi^{1-\alpha} A e^{A\xi} x = 0\}$. Let $X^{\alpha+1}$ be the closed subspace of $D_A(\alpha+1,\infty)$: $X^{\alpha+1} = \{x \in D(A);\ Ax \in X^\alpha\}$,

$$\|x\|_{\alpha+1} = \|x\| + |Ax|_\alpha.$$

Let $\gamma > 0$. Denote by $Y^{\alpha+1}$ the space of all functions $\varphi\colon (-\infty,0) \to X^{\alpha+1}$ which are strongly measurable and

$$(5)\qquad \begin{aligned} |\varphi|_{Y^{\alpha+1}} &= \sup_{\xi>0} \xi^{1-\alpha} \int_{-\infty}^{0} \|e^{\gamma\tau} A^2 e^{A\xi} \varphi(\tau)\| \, d\tau < +\infty, \\ \lim_{\xi\to 0+} \xi^{1-\alpha} & \int_{-\infty}^{0} \|e^{\gamma\tau} A^2 e^{A\xi} \varphi(\tau)\| \, d\tau = 0 \end{aligned}$$

with the norm

$$\|\varphi\|_{Y^{\alpha+1}} = \int_{-\infty}^{0} e^{\gamma\tau} \|\varphi(\tau)\| \, d\tau + |\varphi|_{Y^{\alpha+1}}.$$

Let $\alpha \in (0,1)$ and

(L) *let L be a continuous linear operator from $Y^{\alpha+1}$ into X^α.*

In the sequel we shall need some information about the operator $L(\lambda)$ which is defined by

$$L(\lambda)\colon X^{\alpha+1} \to X^\alpha, \quad L(\lambda)x = L(\tau \to e^{\lambda\tau} x).$$

$L(\lambda)$ is a continuous linear operator from $X^{\alpha+1}$ into X^α for $\operatorname{Re}\lambda > -\gamma$, and

$$\|L(\lambda)x\|_\alpha \leqslant \frac{\|L\|}{\gamma + \operatorname{Re}\lambda} \|x\|_{\alpha+1}.$$

Moreover, we shall suppose that there exists $\beta > 0$ such that

$$(L_\lambda)\qquad \|L(\lambda)R(\lambda, A)\|_{\mathcal{L}(X^\alpha)} \leqslant \frac{C}{|\gamma+\lambda|^\beta} \quad \text{for } \operatorname{Re}\lambda > -\gamma$$

and

$$(\varrho)\qquad \begin{aligned} \varrho = \{\lambda \in \mathbb{C};\ & \lambda - A - L(\lambda) \text{ is invertible and } D(\lambda) = (\lambda - A - L(\lambda))^{-1} \in \\ & \in \mathcal{L}(X^\alpha, X^{\alpha+1})\} \supset \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda > \lambda^-\} \setminus \{\lambda_1, \dots, \lambda_n\}, \\ & \text{where } -\gamma \leqslant \lambda^- < 0,\ \operatorname{Re}\lambda_i \geqslant 0,\ i = 1,\dots,n,\ \lambda_i \text{ are poles of } D(\lambda). \end{aligned}$$

Further, let

$$
\begin{aligned}
&Z^\alpha = X^\alpha \times Y^{\alpha+1}, \quad \|\cdot\|_{Z^\alpha} = \|\cdot\|_{X^\alpha} + \|\cdot\|_{Y^{\alpha+1}}, \\
&Z = \Big\{ z = (x,\varphi) \in X^{\alpha+1} \times Y^{\alpha+1}; \\
&\qquad \int_{-\infty}^0 \mathrm{e}^{\gamma\tau} \|\varphi(\tau)\|_{\alpha+1}\,\mathrm{d}\tau < +\infty, \ \lim_{\tau\to 0^-} \varphi(\tau) = x \Big\}.
\end{aligned} \tag{6}
$$

Then Z is a Banach space with the norm

$$\|z\|_Z = \|x\|_{\alpha+1} + \int_{-\infty}^0 \mathrm{e}^{\gamma\tau} \|\varphi(\tau)\|_{\alpha+1}\,\mathrm{d}\tau$$

and we will suppose that

$$g \in C^1(Z, X^\alpha), \quad g(0) = 0, \quad Dg(0) = 0. \tag{g}$$

Finally, let $\mathbb{R}^+ = [0,+\infty)$, $\mathbb{R}^- = (-\infty, 0]$, let $\tilde{X}$ be a Banach space. For $\eta \in \mathbb{R}$ we denote by $C_\eta((a,b),\tilde{X})$ the set of all $f\colon (a,b) \to \tilde{X}$ such that $t \to \mathrm{e}^{\eta t} f(t)$ is continuous and bounded. These spaces are endowed with the norms

$$\|f\|_{C_\eta((a,b),\tilde{X})} = \sup_{t\in(a,b)} \|\mathrm{e}^{\eta t} f(t)\|_{\tilde{X}}.$$

It is proved in [9] that there exists an operator-valued function $R\colon \mathbb{R} \to \mathcal{L}(X)$ such that

$$R(t) = 0 \text{ for } t < 0, \ R(0) = I, \ \dot{R}(t) = AR(t) + LR_t \text{ for } t > 0. \tag{7}$$

The Laplace transform $\hat{R}(\lambda) = D(\lambda) = (\lambda - A - L(\lambda))^{-1}$. $D(\lambda)$ plays the same role in the investigation of the equation (E) and the construction of the resolvent operator R, as $(\lambda - A)^{-1}$ for a parabolic equation without delay and the analytic semigroup e^{At}. The operator R has most of the properties of an analytic semigroup e^{At}, but the operators $R(t)$, $t \geqslant 0$ do not form a semigroup. Each strong solution of the problem (2) satisfies the integral equation

$$u(t) = R(t)x + \int_0^t R(t-s) L\varphi_s\,\mathrm{d}s\,, \tag{8}$$

and, on the other hand, if $(x,\varphi) \in Z^\alpha$, then $u \in C(\mathbb{R}^+, X^\alpha) \cap C((0,+\infty), X^{\alpha+1}) \cap C^1((0,+\infty), X^\alpha)$, i.e. u is a strong solution of (2). Here φ is extended as 0 onto $\mathbb{R}^+$, so that φ_s is defined for all $s \geqslant 0$. Then

$$u_t = R_t x + \int_0^t R_{t-s} L\varphi_s\,\mathrm{d}s + \varphi_t \tag{9}$$

and

$$S(t)\begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} u(t) \\ u_t \end{pmatrix} \tag{10}$$

form a semigroup on the space of initial conditions Z^α. From the expressions (9), (10) it is clear that S cannot be an analytic semigroup, because it does not improve the regularity in the second component. Some properties of S and its generator are collected in the following two propositions, the proofs of which can be found in [9].

Proposition 1. *Let $S(t)$ be defined by (8), (9), (10) for $t \geqslant 0$. Then $\{S(t)\}$ is a C_0-semigroup of linear operators in the space Z^α. Its generator B is given by*

$$D(B) = \{(x,\varphi) \in Z^\alpha;\, x \in X^{\alpha+1},\ \dot\varphi \in Y^{\alpha+1}, \lim_{\tau\to 0} \varphi(\tau) = x\},$$
$$B\begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} Ax + L\varphi \\ \dot\varphi \end{pmatrix}.$$

$\lambda \in \mathbb{C}$ is in $\varrho(B)$, the resolvent set of B, iff $\operatorname{Re}\lambda > -\gamma$ *and $D(\lambda) \in \mathcal{L}(X^\alpha, X^{\alpha+1})$. Moreover, $S(t)Z \subset Z$ for all $t \geqslant 0$.*

Now, if we denote $v(t) = (u(t), u_t)$, then the problem (E) can be rewritten in the form (4). The local existence of a solution v can be proved with help of the variation of parameters formula

$$v(t) = S(t)\begin{pmatrix} x \\ \varphi \end{pmatrix} + \int_0^t S(t-s)\begin{pmatrix} g(u(s), u_s) \\ 0 \end{pmatrix} \mathrm{d}s\,.$$

The contraction principle together with the properties of the operator R yield the solution $v \in C([0,T), Z)$ with $u \in C^1([0,T), X^\alpha)$ provided that $v(0) \in Z$ is small enough.

A direct computation yields the expression for $(\lambda - B)^{-1}$, $\lambda \in \varrho(B)$:

$$(11)\ (\lambda - B)^{-1}\begin{pmatrix} x \\ \varphi \end{pmatrix}$$
$$= \begin{pmatrix} D(\lambda)(x + L(\theta \to \int_\theta^0 \mathrm{e}^{\lambda(\theta-\sigma)}\varphi(\sigma)\,\mathrm{d}\sigma)) \\ \tau \to \mathrm{e}^{\lambda\tau} D(\lambda)(x + L(\theta \to \int_\theta^0 \mathrm{e}^{\lambda(\theta-\sigma)}\varphi(\sigma)\,\mathrm{d}\sigma)) + \int_\tau^0 \mathrm{e}^{\lambda(\tau-\sigma)}\varphi(\sigma)\,\mathrm{d}\sigma \end{pmatrix}.$$

The last assumption deals with the spectrum of the operator B:

$$(\sigma)\quad \begin{aligned} &\sigma(B) = \sigma^-(B) \cup \{\lambda_1, \ldots, \lambda_n\},\ \operatorname{Re}\lambda_i \geqslant 0,\ \operatorname{Re}\sigma^-(B) < \lambda^- < 0, \\ &\lambda_1, \ldots, \lambda_n \text{ are eigenvalues of finite algebraic multiplicity.} \end{aligned}$$

We shall denote by P^+ the projection operator

$$P^+ = \frac{1}{2\pi i} \int_\Gamma (\lambda - B)^{-1}\, d\lambda\,,$$

where Γ is a suitable path around the set $\sigma^+(B) = \{\lambda_1, .., \lambda_n\}$. Further, let $P^- = I - P^+, Z^- = P^-(Z)$, $Z^+ = P^+(Z)$,

$$S^+(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t}(\lambda - B)^{-1}\, d\lambda\,, \quad t \in \mathbb{R}, \quad S^-(t) = S(t) - S^+(t), \quad t \geqslant 0. \tag{12}$$

Then $S^+(t)z = P^+S(t)z = S(t)P^+z$, $S^-(t)z = P^-S(t)z = S(t)P^-z$, $z \in Z$. Estimates of the norms of the operators $S^+(t)$ are quite straightforward. The proof of the corresponding estimates for the operators $S^-(t)$ is much more complicated. It requires a certain decomposition of the operator $D(\lambda)$, which allows to obtain an exact expression for S^-. In the following, we shall also need estimates of convolutions of S^- by functions of the form $\begin{pmatrix} h(s) \\ 0 \end{pmatrix}$. It is easily seen from (11), (12) that

$$S^+(t)\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} R^+(t)x \\ R_t^+ x \end{pmatrix}, \quad S^-(t)\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} R^-(t)x \\ R_t^- x \end{pmatrix}, \tag{13}$$

where

$$R^+(t)x = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} D(\lambda) x\, d\lambda\,, \quad R^-(t) = R(t) - R^+(t) \text{ for } t \in \mathbb{R}.$$

It means that $R^-(t) = -R^+(t)$ for $t < 0$, which, together with the estimates for convolutions of R, which are proved in [9], give the desired estimates. They are collected in

Proposition 2. *Let* (σ) *hold,* $\delta > \operatorname{Re}\lambda_i$, $\lambda_i \in \sigma^+(B)$, $\omega < 0$, $0 < a < \min(\gamma/2, -\lambda^-)$, $|\mu| < a$, $0 < T \leqslant +\infty$, $h \in C_\mu([0,T), X^\alpha)$, $k \in C_{-\mu}(\mathbb{R}^-, X^\alpha)$. *Then there are constants* K_i, $C(\mu)$, $C(-\mu)$, *such that the following estimates hold:*

$$\|S^+(t)\|_{\mathcal{L}(Z^\alpha, Z)} \leqslant \begin{cases} K_1 e^{\omega t} & \text{for } t \leqslant 0, \\ K_2 e^{\delta t} & \text{for } t \geqslant 0, \end{cases} \tag{14}$$

$$|S^-(t)\|_{\mathcal{L}(Z)} \leqslant K_3 e^{-at} \quad \text{for } t \geqslant 0, \tag{15}$$

$$\sup_{t\in[0,T)} \|e^{\mu t} \int_0^t S^-(t-s)\begin{pmatrix} h(s) \\ 0 \end{pmatrix} ds\,\|_Z \leqslant C(\mu) \sup_{t\in[0,T)} \|e^{\mu t} h(t)\|_\alpha, \tag{16}$$

$$\text{(17)} \qquad \sup_{t\leqslant 0}\left\| e^{-\mu t}\int_{-\infty}^{t} S^{-}(t-s)\begin{pmatrix} k(s)\\ 0\end{pmatrix} ds\right\|_Z \leqslant C(-\mu)\sup_{t\leqslant 0}\|e^{-\mu t}k(t)\|_{\alpha}.$$

3. The center manifold

In this section we shall prove the existence of a center unstable manifold for the equation (4) under the assumption (σ) on the spectrum of the operator B. As usual, we introduce a smooth truncation function ψ and a corresponding function g^r:

$$\psi\colon Z^+\to \mathbb{R},\ |\psi(z)|\leqslant 1,\ \psi(z)=1 \text{ iff } \|z\|_Z\leqslant 1,\ \psi(z)=0 \text{ for } \|z\|_Z\geqslant 2,$$

$$g^r(z)=g\Big(\psi\Big(\frac{P^+z}{r}\Big)P^+z+P^-z\Big).$$

The space $Z^+=P^+(Z)$ is finite-dimensional, so such a smooth function ψ exists and g^r satisfies (g). For the functions v such that $\|P^+v(t)\|_Z\leqslant r$ for $t\geqslant 0$, the equation (4) is equivalent to the equation

$$\text{(18)} \qquad \dot v(t)=Bv(t)+f^r(v(t)),\quad v(0)=v_0=\begin{pmatrix} x\\ \varphi\end{pmatrix}\in Z,$$

where $f^r(z)=\begin{pmatrix} g^r(z)\\ 0\end{pmatrix}$. For this equation we can prove the global existence of the solution provided that the projection of the initial data onto Z^- is sufficiently small.

Proposition 3. *Let* (A), (L), (L$_\lambda$), (g) *and* (σ) *be fulfilled. Then there is* $r>0$ *such that the solution of* (18) *exists in the space* $C([0,\infty),Z)$ *provided that* $\|P^-v_0\|_Z<r$.

Proof. We find a solution of (18) as a fixed point of the operator Ψ:

$$\Psi(v)(t)=S(t)v_0+\int_0^t S(t-s)f^r(v(s))\,ds.$$

First, we shall prove that there exist $T>0$, $r>0$ such that Ψ is a contraction on the ball

$$B^T(S(.)v_0,r)=\{v\in C([0,T),Z);\ |\!|\!|v-S(.)v_0|\!|\!|_T<r\},\ \text{where}$$

$$|\!|\!|w|\!|\!|_T=\sup_{t\in[0,T)}[e^{-\eta t}\|P^+w(t)\|_Z+\|P^-w(t)\|_Z],\quad \eta>\delta,\ \delta \text{ given in (14)}$$

and that Ψ maps $B^\infty(S(.)v_0, r)$ into $B^\infty(S(.)v_0, r/2)$ for any $v_0 \in Z$ satisfying $\|P^- v_0\|_Z < r$. This will enable us to continue the procedure and prove the existence of a global solution.

Let $D_r = \{z \in Z;\ \|P^- z\|_Z \leqslant r(K_3+1)\}, K_3$ given in (15),

$$
\begin{aligned}
&M(r) = \sup\{\|g^r(z)\|_\alpha,\ z \in D_r,\},\\
(19)\qquad &N(r) = \sup\left\{\frac{\|g^r(z_1) - g^r(z_2)\|_\alpha}{\|z_1 - z_2\|_Z};\ z_i \in D_r\right\}.
\end{aligned}
$$

If $v \in B^\infty(S(.)v_0, r)$, then $v(t) \in D_r$ for each $t \geqslant 0$ and, according to (g) and the definition of g^r, we get

$$
(20)\qquad \frac{M(r)}{r} \to 0, \quad N(r) \to 0 \ \text{ for } r \to 0.
$$

Now, we make use of (14), (16), (19) and the inequality $\|v(t)\| \leqslant \mathrm{e}^{\eta t} \|v\|_T$ for $t \leqslant T$ to estimate

$$
\|\Psi(v) - S(.)v_0\|_\infty, \quad \|\Psi(v_1) - \Psi(v_2)\|_T \ \text{ for } v,\ v_1, v_2 \in B^\infty(S(.)v_0, r):
$$

(21)

$$
\begin{aligned}
\|\Psi(v) - S(.)v_0\|_\infty &= \left\| \int_0^t S^+(t-s) f^r(v(s))\,\mathrm{d}s + \int_0^t S^-(t-s) f^r(v(s))\,\mathrm{d}s \right\|_\infty \\
&\leqslant \sup_{t\in R^+} \mathrm{e}^{-\eta t} \int_0^t K_2 \mathrm{e}^{\delta(t-s)} M(r)\,\mathrm{d}s + C(0) M(r) \\
&\leqslant M(r)[K_2/\delta + C(0)],
\end{aligned}
$$

$$
\begin{aligned}
&\|\Psi(v_1) - \Psi(v_2)\|_T \\
&\leqslant \sup_{t\in[0,T]} \left[\mathrm{e}^{-\eta t} \int_0^t K_2 \mathrm{e}^{\delta(t-s)} N(r) \|v_1(s) - v_2(s)\|_Z\,\mathrm{d}s + C(0) N(r) \|v_1(t) - v_2(t)\|_Z\right] \\
&\leqslant \sup_{t\in[0,T]} \left[\mathrm{e}^{-\eta t} \int_0^t K_2 \mathrm{e}^{\delta(t-s)} \mathrm{e}^{\eta s} N(r) \|v_1 - v_2\|_T\,\mathrm{d}s + C(0) N(r) \mathrm{e}^{\eta t} \|v_1 - v_2\|_T\right] \\
&\leqslant N(r)\Big[\frac{K_2}{\eta - \delta} + C(0)\mathrm{e}^{\eta T}\Big] \|v_1 - v_2\|_T.
\end{aligned}
$$

It follows from (20) that Ψ is a contraction of $B^T(S(.)v_0, r)$ into $B^T(S(.)v_0, r/2)$ if r is so small that

$$
(22)\qquad \Big(\frac{K_2}{\eta - \delta} + C(0)\mathrm{e}^{\eta T}\Big) N(r) < 1, \quad M(r)(K_2/\delta + C(0)) < \frac{r}{2}.
$$

Let v_1 be its fixed point. Then $v_1 \in B^T(S(.)v_0, r/2)$ and we can prove in the same way, that the mapping Ψ_1:

$$\Psi_1(v)(t) = S(t)v_1(T) + \int_0^t S(t-s)f^r(v(s))\,\mathrm{d}s\,,\ t \in [0,T],$$

is a contraction of the ball $B^T(S(.)v_1(T), r/2) \subset B^T(S(.)v_0, r)$ into itself. If we denote by v_2 its fixed point, we get the fixed point of Ψ in $B^{2T}(S(.)v_0, r)$:

$$V(t) = \begin{cases} v_1(t) & \text{for } 0 \leqslant t \leqslant T, \\ v_2(t-T) & \text{for } T \leqslant t \leqslant 2T. \end{cases}$$

Moreover, according to (21), (22), $V \in B^{2T}(S(.)v_0, r/2)$, so we can continue in the same way to get the assertion of the proposition. □

In the following, we shall state the existence of a finite dimensional manifold $\mathcal{C}$ for the truncated equation (18) under the assumption that r is small enough. $\mathcal{C}$ attracts the orbits which start close to it. Then we show that the null solution of (4) is stable or unstable iff it has this property with respect to the restriction of the flow to $\mathcal{C}$.

We will find a manifold $\mathcal{C}$ formed by the initial values of solutions of (18) which are defined for all $t \in \mathbb{R}$ and a map $\chi\colon Z^+ \to Z^-$ such that $\mathcal{C} = \mathit{graph}\ \chi$ and $\mathcal{C}$ is invariant with respect to (18).

We look for the solution v satisfying $P^-v(t) = \chi(P^+v(t))$. If we denote $P^+v(t) = \xi(t)$, $P^+v(0) = \zeta$, then

$$\xi(t) = S^+(t)\zeta + \int_0^t S^+(t-s)f^r(\xi(s) + \chi(\xi(s)))\,\mathrm{d}s\,, \tag{23}$$

$$\chi(\xi(t)) = S^-(t-t_0)\chi(\xi(t_0)) + \int_{t_0}^t S^-(t-s)f^r(\xi(s) + \chi(\xi(s)))\,\mathrm{d}s\,. \tag{24}$$

Letting $t_0 \to -\infty$ and then setting $t = 0$ gives

$$\chi(\zeta) = \int_{-\infty}^0 S^-(-s)f^r(\xi(s) + \chi(\xi(s)))\,\mathrm{d}s\,. \tag{25}$$

On the other hand, let χ be such that (25) holds whenever $\zeta \in Z^+$ and (23) holds. Then

$$\begin{aligned}
\chi(\xi(t)) &= \int_{-\infty}^{0} S^-(-s) f^r(\xi(t+s) + \chi(\xi(t+s)))\, \mathrm{d}s \\
&= \int_{-\infty}^{t} S^-(t-s) f^r(\xi(s) + \chi(\xi(s)))\, \mathrm{d}s \\
&= S^-(t-t_0) \int_{-\infty}^{t_0} S^-(t_0-s) f^r(\xi(s) + \chi(\xi(s)))\, \mathrm{d}s \\
&\quad + \int_{t_0}^{t} S^-(t-s) f^r(\xi(s) + \chi(\xi(s)))\, \mathrm{d}s \\
&= S^-(t-t_0)\chi(\xi(t_0)) + \int_{t_0}^{t} S^-(t-s) f^r(\xi(s) + \chi(\xi(s)))\, \mathrm{d}s
\end{aligned}$$

so that (24) is satisfied and $v(t) = \xi(t) + \chi(\xi(t))$ is the global solution of (18) with the initial condition $v(0) = \zeta + \chi(\zeta)$. These considerations imply also the invariance of the graph of χ with respect to (18), if χ is found as a fixed point of the operator Λ:

$$\Lambda(\chi)(\zeta) = \int_{-\infty}^{0} S^-(-s) f^r(\xi(s) + \chi(\xi(s)))\, \mathrm{d}s\,, \tag{26}$$

where $\xi = \xi(\zeta, \chi)$ is a solution of the finite dimensional equation (23).

Theorem 1. *Let* (A), (L), (L_λ), (g) *and* (σ) *be satisfied. Then there is $\varrho > 0$ such that for $r < \varrho$ there exists a Lipschitz continuous function $\chi\colon Z^+ \to Z^-$ such that*

(i) *its graph is invariant with respect to* (18),

(ii) *$\chi \in C^{k-1}$ whenever $g \in C^k$, $\chi^{(k-1)}$ is Lipschitz continuous and $\chi(\zeta) \in \mathcal{D}(B)$ if $k \geqslant 2$, $\zeta \in Z^+$. In that case*

$$\chi'(\zeta)(B\zeta + P^+ f^r(\zeta + \chi(\zeta)) = B\chi(\zeta) + P^- f^r(\zeta + \chi(\zeta)). \tag{27}$$

(iii) *$C^r = graph \chi / \{\zeta \in Z^+, \|\zeta\|_Z < r\}$ is the locally invariant manifold for the equation* (4) *with respect to the set $Z(r) = \{z \in Z,\ \|P^+ z\|_Z < r,\ \|P^- z\|_Z < r\}$.*

(iv) *If $v\colon \mathbb{R} \to Z$ is a global solution of* (4) *such that $v(t) \in Z(r)$ for all $t \in \mathbb{R}$, then $P^- v(t) = \chi(P^+ v(t))$, $t \in \mathbb{R}$ and $P^+ v(.)$ is a solution of* (23).

P r o o f. To prove the existence of χ, we show that Λ is a contraction on the set

$$\begin{aligned}
\mathcal{H} = \{\chi\colon Z^+ \to Z^-,\ \chi(0) = 0,\ \|\chi\|_{\mathcal{H}} = \sup_{\zeta \in Z^+} \|\chi(\zeta)\|_Z \leqslant r, \\
\|\chi(\zeta_1) - \chi(\zeta_2)\|_Z \leqslant b\|\zeta_1 - \zeta_2\|_Z\}.
\end{aligned}$$

For $\chi \in \mathcal{H}$ we have, according to (26), (17),

$$\|\Lambda(\chi(\zeta)\|_Z \leqslant C(0) \sup_{s\leqslant 0} \|f^r(\xi(s,\zeta,\chi) + \chi(\xi(s,\zeta,\chi)))\|_\alpha \leqslant C(0)M(r).$$

To estimate $\|\Lambda(\chi)(\zeta_1) - \Lambda(\chi)(\zeta_2)\|_Z$ and $\|\Lambda(\chi_1)(\zeta) - \Lambda(\chi_2)(\zeta)\|_Z$, we use Gronwall's lemma for the first equation. We find that there is a constant $C = C(\omega, b)$ such that for $t \leqslant 0$, $\omega < 0$

$$\|\xi(t,\zeta_1,\chi_1) - \xi(t,\zeta_2,\chi_2)\|_Z \leqslant C(\omega,b)N(r)\mathrm{e}^{-\nu t}(\|\zeta_1 - \zeta_2\|_Z + \|\chi_1 - \chi_2\|_{\mathcal{H}})$$

with

$$\nu = \nu(\omega, r) = K_1 N(r)(1+b) - \omega, \quad K_1 \text{ given in (14).} \tag{28}$$

Then

$$f^r(\xi(.,\zeta_1,\chi_1) + \chi_1(\xi(.,\zeta_1,\chi_1))) - f^r(\xi(.,\zeta_2,\chi_2) + \chi_2(\xi(.,\zeta_2,\chi_2))) \in C_{-\mu}(\mathbb{R}^-, X^\alpha)$$

for $\mu < -\nu$. If we take $\omega > -a$ (a given in Proposition 2) and then r so small that $-a < -\nu$, we can take $\mu \in (-a, -\nu)$ and, according to (17), we get

$$\|\Lambda(\chi)(\zeta_1) - \Lambda(\chi)(\zeta_2)\|_Z \leqslant C(-\mu)N(r)(1+b)C(\omega,b) \sup_{s\leqslant 0} \mathrm{e}^{(-\mu-\nu)s}\|\zeta_1 - \zeta_2\|_Z, \tag{29}$$

$$\begin{aligned} &\|\Lambda(\chi_1)(\zeta) - \Lambda\chi_2(\zeta)\|_Z \\ &\quad\leqslant C(-\mu)N(r)\Big[(1+b)C(\omega,b)N(r)\sup_{s\leqslant 0}\mathrm{e}^{-\nu s} + 1\Big]\|\chi_1 - \chi_2\|_{\mathcal{H}}. \end{aligned} \tag{30}$$

Now, it is clear that, taking r even smaller if necessary, we obtain the coefficients in (29), (30) less than b or 1, respectively, and then Λ is a contraction on $\mathcal{H}$.

The differentiability of χ we obtain by replacing $\mathcal{H}$ by $\mathcal{H}_k = \{\chi \in \mathcal{H}, \|\chi^{(i)}(\zeta)\| \leqslant b,\ i = 1,\dots,k-1,\ \|\chi^{(k-1)}(\zeta_1) - \chi^{(k-1)}(\zeta_2)\|_Z \leqslant b\|\zeta_1 - \zeta_2\|_Z\}$ and proving that Λ maps $\mathcal{H}_k$ into itself. This can be done by using (17) together with Gronwall's lemma repeatedly.

To prove that $\chi(\zeta) \in \mathcal{D}(B)$ it is sufficient to show that its second component is differentiable on $\mathbb{R}^-$ and its derivative belongs to $Y^{\alpha+1}$. Let $\chi(\zeta) = (x, \varphi)$. Then, due to (25), (13),

$$\varphi(\tau) = \int_{-\infty}^{0} R^-(\tau - s)h(s)\,\mathrm{d}s\,, \text{ where } h(s) = g^r(\xi(s,\zeta,\chi) + \chi(\xi(s,\zeta,\chi))),$$

$$\begin{aligned} \varphi'(\tau) &= \int_{\tau}^{\infty} R^-(t)h'(\tau - t)dt - R^-(\tau)h(0) \\ &= \int_{-\infty}^{\tau} R^-(\tau - s)h'(s)\,\mathrm{d}s - \int_{\tau}^{0} R^+(\tau - s)h'(s)\,\mathrm{d}s + R^+(\tau)h(0). \end{aligned}$$

As $\|h(s)\|_\alpha \leqslant M(r)$, $\|h'(s)\|_\alpha \leqslant N(r)(1+b)\|\xi'(s)\|_Z \leqslant Ce^{\omega s}$, we get, using (13), (14), (17), $\varphi' \in C_{\omega'}(\mathbb{R}^-, X^{\alpha+1})$ with $-\omega < \omega' < \gamma$, which implies that $\varphi' \in Y^{\alpha+1}$. The assertion (27) we obtain by projecting the equation (18) onto Z^+, Z^- and realizing that

$$\xi'(t) = P^+B\xi(t) + P^+f^r(\xi(t) + \chi(\xi(t))),$$

$$\chi'(\xi(t))\xi'(t) = \mathrm{d}/\mathrm{d}t\,\chi(\xi(t)) = P^-B\chi(\xi(t)) + P^-f^r(\xi(t) + \chi(\xi(t))).$$

The considerations above the statement of the theorem together with the equivalence of (4) and (18) for small solutions give the remaining parts of the assertions. The procedure is the same as in the case of ODE's, see e.g. [13], so we omit the details. □

R e m a r k. It is possible to prove even better smoothness of χ, namely $\chi \in C^k$ for $g \in C^k$, but it requires a lengthy calculation. The proof follows the treatment given in Sect. 1.3 of [13] for the finite dimensional case and the proof of Theorem 2 in [14], so it will be omitted here.

Theorem 2. *Let the assumptions of the preceding theorem be satisfied, $\eta \in (0, a)$, $r < \varrho$, $v(0) \in D_r$. Then there are constants C_1, C_2 such that*

(i) *the solution v of the equation* (18) *exists in $C(\mathbb{R}^+, Z)$ and the following estimate holds:*

$$\|P^-v(t) - \chi(P^+v(t))\|_Z \leqslant Ce^{-\eta t}\|P^-v_0 - \chi(P^+v_0)\|_Z, \quad t \geqslant 0. \tag{31}$$

(ii) *There exists $\bar\zeta \in Z^+$ and a solution $\bar\xi$ of* (23) *with $\bar\xi(0) = \bar\zeta$ such that*

$$\|P^+v(t) - \bar\xi(t)\|_Z + \|P^-v(t) - \chi(\bar\xi(t))\|_Z \leqslant C_1e^{-\eta t}\|P^-v_0 - \chi(P^+v_0)\|_Z. \tag{32}$$

P r o o f. The proof follows the ideas of [2] and [4] for parabolic equations, so we only sketch it.

Let $v(t)$ be a solution of (18) with $v(0) = v_0$ that ensures its existence in the large. Then we estimate the norm of the difference

$$\Delta(t) = P^-v(t) - \chi(P^+v(t)).$$

In the case that χ is differentiable, we have using (27)

$$\begin{aligned}\Delta'(t) &= P^-B\Delta(t) + P^-[f^r(v(t)) - f^r(P^+v(t) + \chi(P^+v(t)))]\\ &\quad - \chi'(P^+(v(t))P^+[f^r(v(t)) - f^r(P^+v(t) + \chi(P^+v(t)))]\\ &= P^-B\Delta(t) + k(v,\chi)(t).\end{aligned}$$

The function k is estimated by $\|k(t)\|_\alpha \leqslant CN(r)\|\Delta(t)\|_Z$, so we can take r so small that the solution of the above equation satisfies

$$\|\Delta(t)\|_Z \leqslant C\mathrm{e}^{-\eta t}\|\Delta(0)\|_Z,\ \eta < a,\ t \geqslant 0.$$

If χ is not differentiable, then we get the same result with help of the expression

$$\begin{aligned}\Delta(t) = S^-(t)\Delta(0) &+ \int_0^t S^-(t-s)[f^r(v(s)) - f^r(\xi(s,t) + \chi(\xi(s,t)))]\,\mathrm{d}s \\ &+ \int_{-\infty}^0 S^-(t-s)[f^r(\xi(s,0) + \chi(\xi(s,0))) - f^r(\xi(s,t) + \chi(\xi(s,t)))]\,\mathrm{d}s,\end{aligned}$$

where $\xi(s,t)$ is a solution of the problem

$$(33)\quad \begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\xi(s,t) &= B^+\xi(s,t) + P^+f^r(\xi(s,t) + \chi(\xi(s,t))),\ s \in \mathbb{R},\ B^+ = B/Z^+, \\ \xi(t,t) &= \xi(t) = P^+v(t).\end{aligned}$$

With help of Gronwall's inequality we estimate

$$(34)\quad \begin{aligned}\|\xi(s,t) - \xi(s)\|_Z &\leqslant K_1N(r)\int_s^t \mathrm{e}^{\nu(\sigma-s)}\|\Delta(\sigma)\|_Z\,\mathrm{d}\sigma \quad \text{for } 0 \leqslant s \leqslant t, \\ \|\xi(s,t) - \xi(s,0)\|_Z &\leqslant K_1N(r)\int_0^t \mathrm{e}^{\nu(\sigma-s)}\|\Delta(\sigma)\|_Z\,\mathrm{d}\sigma \quad \text{for } s \leqslant 0 \leqslant t,\end{aligned}$$

where K_1 is given in (14), $N(r)$ in (19) and ν in (28). Then, using (16) and (17), we obtain the same estimate as above.

To prove (ii), we make use of the solution of (33) and the estimate (34). Let $\xi(t,t_n)$be a solution of (33) with $\xi(t_n,t_n) = P^+v(t_n)$. Let $\bar{\zeta} = \lim_{t_n\to+\infty} \xi(0,t_n)$, $\bar{\xi}(.)$ be the solution of (23) with $\bar{\xi}(0) = \bar{\zeta}$. Then $\bar{\xi}(t) = \lim_{t_n\to+\infty} \xi(t,t_n)$ for every $t > 0$ and, according to (34), (31), we get

$$\|P^+v(t) - \bar{\xi}(t)\| \leqslant C\mathrm{e}^{-\eta t}\Delta(0),$$

which together with (28) and the Lipschitz property of χ gives (32). □

It follows from this theorem that C is uniformly asymptotically stable with asymptotic phase. The asymptotic behaviour of small solutions of (4) and, consequently, of (E) depends on the behaviour of the solutions of the finite dimensional equation (23).

Corollary. *The zero solution of* (4) *is stable (asymptotically stable, unstable) in* Z, *iff the zero solution of the finite dimensional problem* (23) *is stable (asymptotically stable, unstable). For the equation* (E) *it means that, once* $\|x\|_{X^{\alpha+1}}, \|\varphi\|_{Y^{\alpha+1}}$ *are sufficiently small, then* $\|u(t)\|_{X^{\alpha+1}}$, $t \in \mathbb{R}^+$ *remains small or tends to zero if the same property have the* Z*-norms of the solutions of* (23) *with small initial data.*

4. Example

Consider the problem

$$
\begin{aligned}
\dot{u}(t,x) &= \Delta u(t,x) + bu(t,x) + \int_0^\infty k_1(s)(\Delta u(t-s,x) + cu(t-s,x))\,\mathrm{d}s \\
&\quad + h_1(u(t,x), Du(t,x), D^2u(t,x)) \\
&\quad + \int_0^\infty k_2(s)h_2(u(t-s,x), Du(t-s), D^2u(t-s,x))\,\mathrm{d}s\,, \\
&\frac{\partial u(t,x)}{\partial n} = 0 \quad \text{for } x \in \partial\Omega,\ t \in \mathbb{R}, \\
&u(\tau,x) = \varphi(\tau,x) \quad \text{for } \tau \leqslant 0,\ x \in \Omega,
\end{aligned}
\tag{35}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set with a smooth boundary, $Du = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$ and $\partial/\partial n$ denotes the normal derivative. We suppose that

$$
\begin{aligned}
&h_i \text{ are smooth functions vanishing at zero} \\
&\text{together with their first derivatives,} \\
&|h_2(p,q,r)| \leqslant C(|p| + |q| + |r|) \text{ for } p \in \mathbb{R}, \\
&q \in \mathbb{R}^n,\ r \in \mathbb{R}^{n^2},\ |d| = \max|d_i|,
\end{aligned}
\tag{h}
$$

$$
|k_i(s)| \leqslant C_i \mathrm{e}^{-\gamma s} \text{ for } i = 1,2, \quad |\hat{k}_1(\lambda)| \leqslant \frac{C}{\lambda^\beta},\ \beta > 0.
\tag{k}
$$

We rewrite the equation (35) in the form (E), setting

$$
\begin{aligned}
A &= \Delta + bI, \\
L\psi(x) &= \int_0^\infty k_1(s)(\Delta\psi(-s,x) + c\psi(-s,x))\,\mathrm{d}s\,, \\
g(v,\psi)(x) &= h_1(v(x), Dv(x), D^2v(x)) \\
&\quad + \int_0^\infty k_2(s)h_2(\psi(-s,x), D\psi(-s,x), D^2\psi(-s,x)\,\mathrm{d}s\,.
\end{aligned}
$$

It can be shown (see [1], [6]) that, taking

$$
X = C(\bar{\Omega}),\ D(A) = \{w \in C^2(\bar{\Omega});\ \partial w(x)/\partial n = 0 \text{ for } x \in \partial\Omega\},
$$

we get $X^\alpha = h^{2\alpha}(\bar\Omega)$, $X^{\alpha+1} = h_0^{2+2\alpha}\{w \in h^{2+2\alpha}(\bar\Omega);\ \partial w(x)/\partial n = 0 \text{ for } x \in \partial\Omega\}$, where $h^\theta(\bar\Omega)$ is the space of the so called small Hölder continuous functions, i.e. functions $w\colon \bar\Omega \to \mathbb{R}$ such that

$$\lim_{\delta\to 0}\ \sup_{|x-y|\leqslant\delta} \frac{|w(x)-w(y)|}{|x-y|^\theta} = 0, h^{2+\theta} = \{w \in C^2(\bar\Omega);\ \Delta w \in h^\theta\}.$$

Then $\varphi = (\,\varphi(0)\ \tau \to \varphi(\tau)\,) \in Z$ iff $\varphi \in L^1_\gamma(\mathbb{R}^-, h_0^{2+2\alpha})$, where

$$L^1_\gamma(\mathbb{R}^-, h_0^{2+2\alpha}) = \Big\{\varphi\colon \mathbb{R}^- \to h_0^{2+2\alpha};\ \int_{-\infty}^0 \mathrm{e}^{\gamma\tau}\|\varphi(\tau)\|_{h^{2+2\alpha}}\,\mathrm{d}\tau < +\infty,$$
$$\lim_{\tau\to 0^-}\varphi(\tau) = \varphi(0)\Big\},$$

$$\|\varphi\|_Z = \|\varphi(0)\|_{h^{2+2\alpha}} + \int_{-\infty}^0 \mathrm{e}^{\gamma\tau}\|\varphi(\tau)\|_{h^{2+2\alpha}}\,\mathrm{d}\tau\,.$$

Then, owing to the assumptions (h), (k), it is easy to verify that g maps the space Z into X^α, L is a continuous linear operator from $Y^{\alpha+1}$ into X^α satisfying (L_λ) and A is a generator of an analytic semigroup in X.

The relation between the eigenvalues of the Laplace operator and the Laplace transform of the kernel k_1 yields the values of the spectrum of the equation. In fact, for $w \in X^{\alpha+1}$ we have

$$L(\lambda)w = \int_0^\infty k_1(s)\mathrm{e}^{-\lambda s}(\Delta + c)w\,\mathrm{d}s = \hat{k}_1(\lambda)(\Delta + c)w.$$

Let $0 = \mu_1 > \mu_2 > \mu_3 > \ldots$ be the eigenvalues of the operator Δ. Then $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > -\gamma$ is in the spectrum of the operator B (see Proposition 1) iff

$$(\hat{k}_1(\lambda) + 1)\mu_n = \lambda - c\hat{k}_1(\lambda) - b \quad \text{for some } n \in \mathbb{N}$$

In the case of $k_1(s) = \mathrm{e}^{-\gamma s}$, we have $\hat{k}_1(\lambda) = \frac{1}{\gamma+\lambda}$ and it follows that for $b < \gamma$ and $c < -\gamma b$, the spectrum of B lies in the halfplane with negative real parts and 0 is an asymptotically stable solution of (35).

On the other hand, the zero solution is unstable whenever $b > \gamma$ or $c + \gamma b > 0$. If, moreover, $c + \mu_n + \gamma(\mu_n + b) \neq 0$ for $n = 2, 3, \ldots$, then we get the saddle point property of the zero solution (see [9]).

In the critical case of stability, namely, when

$$(k_1) \qquad k_1(s) = \mathrm{e}^{-\gamma s}, \quad \gamma > b,\ c = -\gamma b,$$

the spectrum of B consists of the simple eigenvalue 0 and of the part which is in the halfplane $\operatorname{Re}\lambda \leqslant \frac{1}{2}(\mu_2 + \max[0, \mu_2^2 + 4\mu_2(\gamma+1)]^{1/2}) < 0$. Z^+ is now the eigenspace corresponding to the eigenvalue 0, which is the space of constant functions on $(-\infty, 0] \times \Omega$.

In the case of the parabolic equation with Neumann boundary condition without delay, i.e.

$$\dot{u}(t,x) = \Delta u(t,x) + h_1(u(t,x), Du(t,x), D^2u(t,x)), \tag{36}$$

$$\frac{\partial u(t,x)}{\partial n} = 0 \quad \text{for } x \in \partial\Omega, \; t \in \mathbb{R},$$

$$u(0,x) = u_0(x) \quad \text{for} \quad x \in \Omega,$$

we get the eigenspace corresponding to the simple eigenvalue 0 consisting of constant functions on Ω, and the corresponding projector is given by

$$\Pi x(y) = \frac{1}{\mu(\Omega)} \int_\Omega x(\eta)\, \mathrm{d}\eta\,, \quad y \in \Omega. \tag{37}$$

The nonlinearity f then maps the space $\Pi(X^\alpha)$ into itself, which allows to express the center manifold explicitly, namely $\chi \equiv 0$ is the solution of the equation

$$\chi(\zeta) = \int_{-\infty}^{0} \mathrm{e}^{-\Delta s}(I - \Pi) f\big(\xi(s) + \chi(\xi(s))\big),$$

which corresponds to (25) with ξ satisfying the equation corresponding to (23), which is in this case equivalent to the scalar equation

$$\dot{\xi}(t) = h_1(\xi(t), 0, 0) \tag{38}$$

(see [7]).

In our case, the projector P^+ maps elements of the type $(x, 0)$ into the constant elements $(C, \tau \to C) \in Z$, so that (23) cannot have the identically zero solution. The projector P^+ can be expressed explicitly with help of Π (see (37)): We have

$$D(\lambda) = \frac{\lambda + \gamma}{\lambda + \gamma + 1}(a(\lambda) - \Delta)^{-1}, \text{ where } a(\lambda) = \lambda \frac{\lambda + \gamma - b}{\lambda + \gamma + 1},$$

which, together with the power series representation of

$$(\mu - \Delta)^{-1}x = \frac{\Pi x}{\mu} + \sum_{n=0}^{\infty} \mu^n R_n x,$$

gives

$$Qx = \frac{1}{2\pi \mathrm{i}} \int_\Gamma D(\lambda)x\, \mathrm{d}\lambda = \frac{\gamma}{\gamma - b}\, \Pi x. \tag{39}$$

Here Γ is a smooth curve surrounding 0 such that $\operatorname{Re}\lambda > \operatorname{Re}\sigma^-(B)$ for $\lambda \in \Gamma$. Further

$$L\Big(\theta \to \int_\theta^0 e^{\lambda(\theta-\sigma)}\psi(\sigma)\,d\sigma\Big) = \frac{1}{\lambda+\gamma}(\Delta + c)\hat{\psi}_1(\gamma),$$

where $\hat{\psi}_1$ is the Laplace transform of $\psi_1 : \psi_1(s) = \psi(-s)$. (11) and (39) now give formulas for P^+:

$$(40)\quad P^+\begin{pmatrix} x \\ \psi \end{pmatrix} = \frac{1}{2\pi i}\int_\Gamma (\lambda - B)^{-1}\begin{pmatrix} x \\ \psi \end{pmatrix} d\lambda = \begin{pmatrix} \frac{1}{\gamma-b}\Pi(\gamma x + (\Delta + c)\hat{\psi}_1(\gamma)) \\ \tau \to \frac{1}{\gamma-b}\Pi(\gamma x + (\Delta + c)\hat{\psi}_1(\gamma)) \end{pmatrix},$$

$$(41)\qquad P^+\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} Qx \\ \tau \to Qx \end{pmatrix}, \quad P^-\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} (I-Q)x \\ \tau \to -Qx \end{pmatrix}.$$

The stability of the zero solution of (35) depends on the stability of the zero solution of (23), which, due to (40), is equivalent to the scalar equation

$$(42)\qquad \dot{\xi}(t) = P^+ f^r(\xi(t) + \chi(\xi(t))) = F(\xi(t)).$$

The stability or instability depends on the behaviour of F and its derivatives at 0.

Let e, e_1 denote the functions $e \equiv 1$ on $\mathbb{R}^- \times \Omega$, $e_1 \equiv 1$ on Ω, and

$$(43)\qquad d_n = \frac{\partial^n h_1(0,0,0)}{\partial u^n} + \frac{\partial^n h_2(0,0,0)}{\partial u^n}\int_0^\infty k_2(s)\,ds.$$

Then $f^{(n)}(0)(e,\ldots,e) = \begin{pmatrix} d_n e_1 \\ 0 \end{pmatrix}$ and according to (25) we have, since $\chi(0) = 0$ and $\xi \equiv 0$ is the solution of (23) with $\zeta = 0$,

$$\chi'(0)e = 0, \quad \chi''(0)(e,e) = \int_{-\infty}^0 S^-(-s)P^-(f^r)''(0)(e,e).$$

Further, due to (g), (41), (43) and the fact that f^r coincides with f on a neighbourhood of 0, we get

$$F'(0) = 0, \quad F''(0)(e,e) = P^+ f''(0)(e,e) = P^+\begin{pmatrix} d_2 e_1 \\ 0 \end{pmatrix} = \frac{\gamma d_2}{\gamma - b}e,$$

$$F'''(0)(e,e,e) = P^+ f'''(0)(e,e,e) + 3P^+ f''(0)(e, \chi''(0)(e,e)).$$

It follows that, if we suppose $d_2 = 0$, then the stability of the zero solution of (35) depends on the sign of d_3.

Theorem 3. *Let* (h), (k), (k_1) *be fulfilled,* $d_2 = 0$. *Then 0 is an asymptotically stable solution of* (35) *in* $L^1_\gamma(\mathbb{R}^-, h_0^{2\alpha+2})$ *if* $d_3 < 0$ *and it is unstable for* $d_3 > 0$. *If* $d_2 \neq 0$, *then the zero solution is unstable.*

Remark. The relations (25) and (41) show that the center manifold for the problem (35) belongs to the subspace of Z formed by functions which do not depend on the space variable. This space is invariant for the corresponding equation (4) and the flow restricted to this subspace is given by the ordinary differential equation with delay, namely

$$\dot{u}(t) = bu(t) + \int_0^\infty ck_1(s)u(t-s)\,\mathrm{d}s + h_1(u(t),0,0) + \int_0^\infty k_2(s)h_2(u(t-s),0,0)\,\mathrm{d}s.$$

Stability of the zero equilibrium of this equation is equivalent to stability of the zero equilibrium of the equation (35), analogously to the equations (36), (38).

References

[1] *P. Acquistapace, B. Tereni*: Hölder classes with boundary conditions as interpolation spaces. Math Z. *195* (1987), 451–471.

[2] *G. Da Prato, A. Lunardi*: Stability, instability and center manifold theorem for fully nonlinear autonomous parabolic equations in Banach space. Arch. Rat. Mech. Anal. *101* (1988), 115–141.

[3] *J. K. Hale, J. Kato*: Phase space for retarded equations with infinite delay. Funkcial. Ekvac. *21* (1978), 11–41.

[4] *D. Henry*: Geometric theory of semilinear parabolic equations. Lecture Notes in Math. 840, Springer Verlag, 1981.

[5] *S.O. Londen, J.A. Nohel*: Nonlinear Volterra integrodifferential equation occuring in heat flow. J. Int. Equations *6* (1984), 11–50.

[6] *A. Lunardi*: Interpolation spaces between domains of elliptic operators and spaces of continuous functions with applications to nonlinear parabolic equations. Math. Nachr. *121* (1985), 295–318.

[7] *A. Lunardi*: Stability and local invariant manifolds in fully nonlinear parabolic equations. Preprint.

[8] *J. Milota*: Asymptotic behaviour of parabolic equations with infinite delay. Volterra Integrodiff. Eqs. and Appl., Pitman Research Notes in Math. *190* (1989), 295–305.

[9] *H. Petzeltová*: Solution semigroup and invariant manifolds for functional equations with infinite delay. Mathematica Bohemica *118* (1993), no. 2, 175–193.

[10] *H. Petzeltová, J. Milota*: Resolvent operator for abstract functional differential equations with infinite delay. Numer. Funct. Anal. and Optimiz. *9* (1987), 779–807.

[11] *G. Simonett*: Zentrumsmannigfaltigkeiten für quasilineare parabolische Gleichungen. Thesis.

[12] *E. Sinestrari*: On the abstract Cauchy problem of parabolic type in the spaces of continuous functions. J. Math. And Appl. *107* (1985), 16–66.

[13] *A. Vanderbauwhede*: Center manifolds, normal forms and elementary bifurcations. Dynamics reported *2* (1989), 89–169.

[14] *A. Vanderbauwhede, G. Iooss*: Center manifold theory in infinite dimensions. Dynamics reported *1* (1992), 125–163.

Author's address: *Hana Petzeltová*, Matematický ústav AV ČR, Žitná 25, 115 67 Praha 1.