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JÓNSSON'S LEMMA FOR NORMALLY PRESENTED VARIETIES

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Varieties presented by normal identities were treated in [1]. Let us recall the basic concepts. Let τ be a similarity type and $\{x_1, x_2, \ldots\}$ a set of variables. For an n-ary term $p(x_1, \ldots, x_n)$ of type τ we denote by $\operatorname{var} p = \{x_1, \ldots, x_n\}$ the set of all variables occuring in p. For n-ary terms p, q of type τ the identity

$$p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n)$$

is said to be *normal* if it is either trivial, i.e. $x_1 = x_1$, or $p \notin \operatorname{var} p$ and $q \notin \operatorname{var} q$, i.e. neither p nor q is a single variable. A variety $\mathscr{V}$ of type τ is *normally presented* if $\operatorname{Id} \mathscr{V}$ contains only normal identities.

If $\mathscr{V}$ is a variety of type τ, denote by $N(\mathscr{V})$ the variety satisfying all normal identities of $\mathscr{V}$. Hence, $\mathscr{V}$ is a subvariety of $N(\mathscr{V})$ and if $\mathscr{V} \neq N(\mathscr{V})$ then $N(\mathscr{V})$ covers $\mathscr{V}$ in the lattice of all varieties of type τ, see [3].

Since every congruence identity is characterized by a Mal'tsev condition (see [4]) and because every Mal'tsev condition contains an identity which is not normal, we obtain the following

Observation. For every variety $\mathscr{V}$, the variety $N(\mathscr{V})$ satisfies no congruence identity.

In particular, $N(\mathscr{V})$ is never a congruence distributive variety. Despite of this fact, $N(\mathscr{V})$ satisfies the assertion of Jónsson's Lemma provided $\mathscr{V}$ is congruence distributive:

Theorem. *Let $\mathscr{V}$ be a congruence distributive variety of type τ and let $N(\mathscr{V})$ be generated by a class $\mathscr{K}$ of algebras of type τ. Then $Si(N(\mathscr{V})) = \mathbf{HSP_U}(\mathscr{K})$ and, therefore, $N(\mathscr{V}) = \mathbf{IP_S HSP_U}(\mathscr{K})$.*

Proof. Let $\mathscr{V}$ be a congruence distributive variety of type τ. Denote by $\mathscr{B} = (\{0,1\}, F)$ an algebra of type τ such that $f(x_1, \ldots, x_n) = 0$ for every $x_1, \ldots, x_n$ of

$\{0,1\}$. $\mathscr{B}$ is the so called *constant algebra* in the sense of [1]. As was pointed out in Theorem 3 of [1], $Si(N(\mathscr{V}) = Si(\mathscr{V}) \cup \mathscr{B}$. By Jónsson's Lemma, we have

$$Si(N(\mathscr{V})) = \mathbf{HSP_U}(\mathscr{K}) \cup \mathscr{B}.$$

If $\mathscr{B} \notin \mathbf{HSP_U}(\mathscr{K})$ then $\mathscr{B} \notin \mathbf{HSP}(\mathscr{K})$ and thus, by [1], $\mathbf{HSP}(\mathscr{K})$ is not normally presented, a contradiction with $N(\mathscr{V}) = \mathbf{HSP}(\mathscr{K})$. Hence $\mathscr{B} \in \mathbf{HSP_U}(\mathscr{K})$ and $Si(N(\mathscr{V})) = \mathbf{HSP_U}(\mathscr{K})$. □

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