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Mathematica Bohemica, Vol. 124 (1999), No. 2-3, 149–166

Persistent URL: <http://dml.cz/dmlcz/126255>

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POSITIVE SOLUTIONS OF CRITICAL QUASILINEAR
ELLIPTIC EQUATIONS IN \mathbb{R}^N

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(Received September 25, 1998)

Dedicated to Professor Alois Kufner on the occasion of his 65th birthday

Abstract. We consider the existence of positive solutions of

$$(1) \quad -\Delta_p u = \lambda g(x)|u|^{p-2}u + \alpha h(x)|u|^{q-2}u + f(x)|u|^{p^*-2}u$$

in \mathbb{R}^N , where $\lambda, \alpha \in \mathbb{R}$, $1 < p < N$, $p^* = Np/(N-p)$, the critical Sobolev exponent, and $1 < q < p^*$, $q \neq p$. Let $\lambda_1^+ > 0$ be the principal eigenvalue of

$$(2) \quad -\Delta_p u = \lambda g(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \quad \int_{\mathbb{R}^N} g(x)|u|^p > 0,$$

with $u_1^+ > 0$ the associated eigenfunction. We prove that, if $\int_{\mathbb{R}^N} f|u_1^+|^{p^*} < 0$, $\int_{\mathbb{R}^N} h|u_1^+|^q > 0$ if $1 < q < p$ and $\int_{\mathbb{R}^N} h|u_1^+|^q < 0$ if $p < q < p^*$, then there exist $\lambda^* > \lambda_1^+$ and $\alpha^* > 0$, such that for $\lambda \in [\lambda_1^+, \lambda^*)$ and $\alpha \in [0, \alpha^*)$, (1) has at least one positive solution.

Keywords: the p -Laplacian, positive solutions, critical exponent

MSC 1991: 35J70, 35P30

Research of the authors was supported by NSERC of Canada, Grant # 201/97/0395 of the Grant Agency of the Czech Republic, a University of Memphis Faculty Research Grant, and NATO Collaborative Research Grant OUTR.CRG 961190

1. INTRODUCTION

We study the existence of positive solutions to the following problem in \mathbb{R}^N

$$(1.1)_\lambda \quad -\Delta_p u = \lambda g(x)|u|^{p-2}u + \alpha h(x)|u|^{q-2}u + f(x)|u|^{p^*-2}u,$$

where $\lambda, \alpha \in \mathbb{R}$, $1 < p < N$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p -Laplacian, $p^* = Np/(N-p)$, $1 < q < p^*$, $q \neq p$, f, g and h satisfy $g^+ \neq 0$, $f^\pm \neq 0$, $h^+ \neq 0$, and other conditions. The problem is closely related to the following eigenvalue problem,

$$(1.2)_\lambda \quad -\Delta_p u = \lambda g(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \quad \int_{\mathbb{R}^N} g(x)|u|^p > 0.$$

It is known that $(1.2)_\lambda$ has an eigenvalue $\lambda_1^+ > 0$ associated with a positive eigenfunction u_1^+ (see [6]).

Equations involving critical Sobolev exponents have been studied extensively, and there exists a large body of literature. We refer to [5] and the references therein. Specifically, Swanson and Yu [12] studied $(1.1)_\lambda$ for the case $\lambda \in (0, \lambda_1^+)$ and $p < q < p^*$. It is shown in [12] that if $g \geq 0$, $g \in L^{N/p}(\mathbb{R}^N)$, $f \geq 0$, and $h \geq h_0 > 0$ in \mathbb{R}^N , then $(1.1)_\lambda$ has a positive solution if $\lambda \in (0, \lambda_1^+)$. Noussair, Swanson and Yang [11] investigated the problem

$$-\Delta_m u = p(x)u^\tau + q(x)u^\gamma$$

on an open connected smooth domain, where $2 \leq m < N$, $m-1 < \gamma < \tau$, and $\tau+1 = Nm/(N-m)$. The existence of at least one positive solution was obtained for both p and q nonnegative and satisfying other local conditions. More recently Noussair and Swanson [10] considered

$$(1.3) \quad -\Delta u = p|u|^{\tau-2}u + q|u|^{\gamma-2}u \quad \text{in } \mathbb{R}^N,$$

where $2 < \gamma < \tau = 2N/(N-2)$, and showed, under suitable assumptions, including nonnegativity of p and q , that (1.3) has two positive decaying solutions. The existence of two positive solutions of $(1.1)_\lambda$ was studied for the case $p < q < p^*$ and $f \equiv 0$ in [4], and for the case $h(x) \equiv 0$ in [5]. Various forms of the equation

$$(1.4) \quad -\Delta_p u + a(x)|u|^{p-2}u = \beta h(x)|u|^{q-2}u + k(x)|u|^{p^*-2}u \quad \text{in } \mathbb{R}^N$$

are treated by Alves, Gonçalves and Miyagaki in [1], [2] and [7], where a, h and k are nonnegative, $1 < q \neq p$, $q < p^*$, and $\beta \geq 0$. The existence of nonnegative solutions was obtained via Mountain Pass arguments. Specifically, [1] deals with the

case $1 < q < p$, $a \equiv 0$, $k \equiv 1$; [2] the case $1 < q < p$ and $\beta = 1$; and [7] the case $a \equiv 0$, $\beta = 1$, $k \equiv 1$, and $1 < q < p^*$, $q \neq p$, $p \geq 2$.

In this paper we are mainly concerned with the situation where $\lambda \geq \lambda_1^+$. We note that, for $\lambda \in (0, \lambda_1^+)$, the functional $\int_{\mathbb{R}^N} (|\nabla u|^p - \lambda g|u|^p)$ is always positive for $u \neq 0$, so one can use a Mountain Pass type argument to show that $(1.1)_\lambda$ has a positive solution. Assuming $h > 0$ in some open set in \mathbb{R}^N , one can even prove the existence of two positive solutions by first finding a local nonzero minimizer of the associated functional and then using the Mountain Pass Theorem to find a saddle point. This is the approach used in [1], [2] and [7]. For $\lambda \geq \lambda_1^+$, the situation is different. The problem is that in this case, the functional $\int_{\mathbb{R}^N} (|\nabla u|^p - \lambda g|u|^p)$ is no longer positive definite. Even a local minimizer is difficult to find. Specifically, for $\lambda > \lambda_1^+$, $\int_{\mathbb{R}^N} (|\nabla u|^p - \lambda g|u|^p)$ will always approach $-\infty$ as $\|u\| \rightarrow \infty$ in the direction of u_1^+ , while it can achieve positive values in other directions. For $\lambda = \lambda_1^+$, $\int_{\mathbb{R}^N} (|\nabla u|^p - \lambda g|u|^p)$ will always be zero in the direction of u_1^+ . This destroys the Mountain Pass structure. Here we use a procedure devised by Tarantello [13] and further utilized in [4] and [5]. The conditions

$$(1.5) \quad \int_{\mathbb{R}^N} f(u_1^+)^{p^*} < 0, \quad \int_{\mathbb{R}^N} h(u_1^+)^q > 0,$$

$$(1.6) \quad \int_{\mathbb{R}^N} f(u_1^+)^{p^*} < 0, \quad \int_{\mathbb{R}^N} h(u_1^+)^q < 0,$$

are essential in our presentation. Under further related local conditions on g , h and f , we can prove the existence of positive solutions of $(1.1)_\lambda$.

Main Result. *Assume (1.5) if $1 < q < p$ and (1.6) if $p < q < p^*$. Then there exist $\lambda^* > \lambda_1^+$ and $\alpha^* > 0$, such that for any $\lambda \in [\lambda_1^+, \lambda^*)$ and $\alpha \in [0, \alpha^*)$, $(1.1)_\lambda$ has a positive solution (see Theorems 3.8 and 4.7 for precise assumptions on f , g and h).*

In our setting, g and h are allowed more flexibility than in [1], [2] and [7], e.g., they may or may not change sign. But (1.5) forces f to change sign, and (1.6) forces both f and h to change sign. We note that here we need an additional condition that α is small enough. While this is the case for $1 < q < p$ in [1], [2] and [7], no such smallness restriction is postulated to h in [7] and [12], for the case $p < q < p^*$.

This paper is organized as follows: In Section 2 we study the geometric structure of certain solution manifolds of the associated functional. Section 3 provides the proof of the existence result for the case $1 < q < p$. The case $p < q < p^*$ is discussed in Section 4.

2. GEOMETRY OF THE SOLUTION MANIFOLDS FOR $1 < q < p$

We collect our basic assumptions and recall some known results. We assume throughout this paper that $1 < p < N$, $p^* = Np/(N-p)$, $1 < q < p^*$ and $q \neq p$. We also assume

- (g0) $g(x) = g^+(x) - g^-(x)$, $g^+, g^- \geq 0$, and $g^+ \in L_{\text{loc}}^\infty(\mathbb{R}^N) \cap L^{N/p}(\mathbb{R}^N)$,
 $g^- \in L_{\text{loc}}^\infty(\mathbb{R}^N)$,
(h0) $h \in L_{\text{loc}}^\infty(\mathbb{R}^N) \cap L^Q(\mathbb{R}^N)$, where $Q = Np[Np - q(N-p)]^{-1}$.

Let

$$\omega(x) = \frac{1}{(1+|x|)^p}, \quad x \in \mathbb{R}^N,$$

$$w(x) = \max\{g^-(x), \omega(x)\} > 0, \quad x \in \mathbb{R}^N.$$

Let V be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|$ defined by

$$\|u\| = \left(\int |\nabla u|^p + \int w(x)|u|^p \right)^{1/p}.$$

Here and henceforth the integrals are taken on \mathbb{R}^N unless otherwise stated. Then V is a uniformly convex Banach space. In this paper $\|\cdot\|_p$ will denote the usual L^p norm, and $D^{1,p}(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_D = \left(\int |\nabla u|^p \right)^{1/p}.$$

Note that since $V \subset D^{1,p}(\mathbb{R}^N)$, a weakly convergent sequence in V is also weakly convergent in $D^{1,p}(\mathbb{R}^N)$. By Hardy's inequality, $D^{1,p}(\mathbb{R}^N)$ is embedded continuously in $L^p(\mathbb{R}^N, \omega(x))$, so a strongly convergent sequence in $D^{1,p}(\mathbb{R}^N)$ is also strongly convergent in $L^p(\mathbb{R}^N, \omega(x))$.

Throughout this paper the function f is always assumed to satisfy

- (f0) $f^\pm \not\equiv 0$ and $f(x) \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$.

We have (from Lemma 2.3 of [6]):

Proposition 2.1. *Assume the above conditions are satisfied. Then there exists a unique, simple isolated eigenvalue $\lambda_1^+ > 0$, such that the eigenvalue problem (1.2) $_\lambda$ has a positive eigenfunction $u_1^+ \in V$ associated with λ_1^+ .*

Next we introduce the following functional

$$(2.1) \quad I_\lambda(u) = \frac{1}{p} \int (|\nabla u|^p - \lambda g|u|^p) - \frac{\alpha}{q} \int h|u|^q - \frac{1}{p^*} \int f|u|^{p^*}.$$

It is clear that the functional J_λ is well defined on V . Obviously a critical point of J_λ in V is a (weak) solution of (1.1) $_\lambda$. We can always assume that critical points of J_λ are nonnegative functions since J_λ is an even functional. For simplicity, we will assume in the sequel that $\alpha > 0$, for the case $\alpha = 0$ has been covered in [5].

Define

$$\begin{aligned} J_\lambda(u) &= \int (|\nabla u|^p - \lambda g|u|^p), \\ \Lambda_\lambda &= \{u \in V : \Psi_\lambda(u) := \langle J'_\lambda(u), u \rangle = 0\} \\ &= \left\{ u \in V : J_\lambda(u) = \alpha \int h|u|^q + \int f|u|^{p^*} \right\}, \end{aligned}$$

and

$$(2.2) \quad \Lambda_\lambda^- = \{u \in \Lambda_\lambda : \langle \Psi'_\lambda(u), u \rangle < 0\}.$$

We list the following equivalent expressions of this set.

$$(2.3) \quad \begin{aligned} \Lambda_\lambda^- &= \left\{ u \in \Lambda_\lambda : (p-q)J_\lambda(u) < (p^*-q) \int f|u|^{p^*} \right\} \\ &= \left\{ u \in \Lambda_\lambda : (p^*-p)J_\lambda(u) > \alpha(p^*-q) \int h|u|^q \right\} \\ &= \left\{ u \in \Lambda_\lambda : \alpha(p-q) \int h|u|^q < (p^*-p) \int f|u|^{p^*} \right\}. \end{aligned}$$

We note that it is not entirely clear whether Λ_λ^- is nonempty for general g , h and f . To show that $\Lambda_\lambda^- \neq \emptyset$, we introduce other conditions on g , f and h .

- (f1) $f(0) = \|f\|_\infty$ and for some $r > 0$, $f(x) > 0$ for $x \in B(0, 2r)$,
- (h1) $h(x) \geq h_0 > 0$ in $B(0, 2r)$,
- (g1) $g(x) \geq g_0 > 0$ in $B(0, 2r)$.

Lemma 2.2. *Suppose (f0), (f1), (g0), (g1), (h0) and (h1) hold. Then for $\lambda > 0$ in any bounded interval, there exists $\alpha_1 > 0$ such that $\Lambda_\lambda^- \neq \emptyset$ provided $\alpha \in (0, \alpha_1)$.*

Proof. Define, for $\varepsilon > 0$,

$$u_\varepsilon(x) = \frac{\psi(x)}{(\varepsilon + |x|^{p/(p-1)}(N-p)/p)}, \quad v_\varepsilon(x) = \frac{u_\varepsilon(x)}{\|u_\varepsilon(x)\|_{p^*}},$$

where $\psi \in C_0^\infty(B(0, 2r))$ is such that $0 \leq \psi(x) \leq 1$ and $\psi(x) \equiv 1$ on $B(0, r)$.

Consider for $t > 0$,

$$\Psi_\lambda(tv_\varepsilon) = t^p J_\lambda(v_\varepsilon) - \alpha t^q \int h|v_\varepsilon|^q - t^{p^*} \int f|v_\varepsilon|^{p^*}.$$

Let $s_\alpha(t) = at^p - abt^q - ct^{p'}$, with $a = J_\lambda(v_\varepsilon)$, $b = \int h|v_\varepsilon|^q$ and $c = \int f|v_\varepsilon|^{p'}$. It is clear that $b > 0$, $c > 0$. By continuous dependence of the principal eigenvalue on the domain, $a > 0$ for $\varepsilon > 0$ small enough. Fix this ε so a , b , c are fixed and let α vary. One easily sees that $s_\alpha(t) \rightarrow -\infty$ as $t \rightarrow \infty$. Moreover, as $b \rightarrow 0$, $s_\alpha(t) \rightarrow s_0(t) := at^p - ct^{p'}$ in C^1 with respect to t . Let t_0 be such that $s_0(t_0) = 0$ and $s_0(t) > 0$ for $t < t_0$. Then $s'_0(t_0) < 0$. By C^1 convergence of s_α to s_0 , we easily conclude that there exist $\alpha_1 > 0$ and $\tau > 0$, such that if $0 < \alpha < \alpha_1$, $s_\alpha(t_\alpha) = 0$ and $s'_\alpha(t_\alpha) < 0$ for some $t_\alpha \in (t_0 - \tau, t_0 + \tau)$, that is, $t_\alpha v_\varepsilon \in \Lambda_\lambda^-$. This completes the proof. \square

Next we study the geometry of the set Λ_λ^- for $\lambda > 0$. We will seek a critical point of I_λ on Λ_λ^- . Observe that for any $u \in \Lambda_\lambda$,

$$(2.4) \quad \begin{aligned} I_\lambda(u) &= \frac{1}{N} \int f|u|^{p'} + \alpha \left(\frac{1}{p} - \frac{1}{q} \right) \int h|u|^q \\ &= \frac{1}{N} J_\lambda(u) - \alpha \left(\frac{1}{q} - \frac{1}{p'} \right) \int h|u|^q. \end{aligned}$$

We also assume that $\|u_1^+\| = 1$.

The next lemma requires the following conditions.

- (f2) $\int f(u_1^+)^{p'} < 0$.
(h2) $\int h(u_1^+)^q > 0$.

Lemma 2.3. *Assume $p > q$, (f0), (f1), (f2), (g0), (g1), (h0), (h1) and (h2) hold. Then there exist $\lambda^* > \lambda_1^+$ and $\alpha_2 > 0$ with $\alpha_2 \leq \alpha_1$, such that for any $\bar{\lambda} \in (0, \lambda_1^+)$, there exists $\sigma > 0$, such that for any $\lambda \in [\bar{\lambda}, \lambda^*)$ and $\alpha \in (0, \alpha_2)$, we have $J_\lambda(u) \geq \sigma \|u\|^p$ for any $u \in \Lambda_\lambda^-$.*

Proof. We argue by contradiction. Suppose there exist λ_n , α_n and $u_n \in \Lambda_{\lambda_n}^-$ such that

$$(2.5) \quad \alpha_n \rightarrow 0, \quad \lambda_n \rightarrow \bar{\lambda} \in [\bar{\lambda}, \lambda_1^+], \quad J_{\lambda_n}(u_n) < \frac{1}{n} \|u_n\|^p.$$

We explicitly note that here $\Lambda_{\lambda_n}^-$ also depends on α_n . Let $v_n = u_n / \|u_n\|$. Without loss of generality we may assume $v_n \rightarrow v_0$ weakly in V . Then we have $\int g^+ |v_n|^p \rightarrow \int g^+ |v_0|^p$ by compactness. We then derive by weak lower semicontinuity of the norm that

$$(2.6) \quad \begin{aligned} 0 &\leq \int |\nabla v_0|^p - \bar{\lambda} \int g^+ |v_0|^p + \bar{\lambda} \int g^- |v_0|^p \\ &\leq \liminf_{n \rightarrow \infty} \left(\int |\nabla v_n|^p - \lambda_n \int g^+ |v_n|^p + \lambda_n \int g^- |v_n|^p \right) \\ &= \liminf_{n \rightarrow \infty} J_{\lambda_n}(v_n) \leq \lim_{n \rightarrow \infty} \frac{1}{n} = 0. \end{aligned}$$

There are two possibilities: (1) $v_0 = 0$, and (2) $v_0 = ku_1^+$ for some $k \neq 0$, and $\bar{\lambda} = \lambda_1^+$. If $v_0 = 0$, it follows from (2.6) that $\int |\nabla v_n|^p \rightarrow 0$ and $\int g^- |v_n|^p \rightarrow 0$. Thus $v_n \rightarrow 0$ in V , contradicting $\|v_n\| = 1$. If $v_0 = ku_1^+$ for some $k \neq 0$, and $\bar{\lambda} = \lambda_1^+$, then we have, by the weak convergence of v_n to ku_1^+ and (2.6),

$$\begin{aligned} \lambda_1^+ \int g^+ |ku_1^+|^p &= \int (|\nabla ku_1^+|^p + \lambda_1^+ g^- |ku_1^+|^p) \\ &\leq \liminf_{n \rightarrow \infty} \int |\nabla v_n|^p + \liminf_{n \rightarrow \infty} \int g^- |v_n|^p \\ &\leq \liminf_{n \rightarrow \infty} \int (|\nabla v_n|^p + \lambda_n g^- |v_n|^p) \\ &= \lim_{n \rightarrow \infty} \lambda_n \int g^+ |v_n|^p = \lambda_1^+ \int g^+ |ku_1^+|^p. \end{aligned}$$

It then follows that

$$\liminf_{n \rightarrow \infty} \int |\nabla v_n|^p = \int |\nabla ku_1^+|^p, \quad \liminf_{n \rightarrow \infty} \int g^- |v_n|^p = \int g^- |ku_1^+|^p.$$

We deduce that (passing to a subsequence if necessary) $v_n \rightarrow ku_1^+$ strongly in V . We then derive from (2.3) that

$$(2.7) \quad \|u_n\|^{p-p^*} J_{\lambda_n}(v_n) < \frac{p^*-q}{p-q} \int f |v_n|^{p^*} \rightarrow \frac{p^*-q}{p-q} \int f |ku_1^+|^{p^*} < 0.$$

This contradicts (2.6) if $\|u_n\| \not\rightarrow 0$ or $J_{\lambda_n}(u_n) \geq 0$. Suppose $\|u_n\| \rightarrow 0$ and $J_{\lambda_n}(u_n) < 0$. It follows from (2.3) that $\int h |u_n|^q < 0$. That is, $\int h |v_n|^q \leq 0$, which contradicts (h2). This proves the lemma. \square

Remark 2.4. For $\lambda \in (0, \lambda_1^+)$, conditions (f2) and (h2) are not needed because $J_\lambda(u) \geq 0$ for all u . Assumptions (f2) and (h2) are introduced to compensate for the possibility that $J_\lambda(u)$ is negative.

Lemma 2.5. *Assume $p > q$, (f0), (f1), (f2), (g0), (g1), (h0), (h1) and (h2) hold. For any $\bar{\lambda} \in (0, \lambda_1^+)$, there exist $\varrho > 0$ and $\alpha^* > 0$ with $\alpha^* \leq \alpha_2$, such that for any $\lambda \in [\bar{\lambda}, \lambda^*)$, $\alpha \in (0, \alpha^*)$ and $u \in \Lambda_\lambda^-$, we have $-(\Psi'_\lambda(u), u) \geq \varrho$.*

Proof. We first claim that there exists $\zeta > 0$, independent of λ , such that $\|u\| > \zeta$ for all $u \in \Lambda_\lambda^-$. If this were not true, then for some $u_n \in \Lambda_{\lambda_n}^-$, $\lambda_n \in [\bar{\lambda}, \lambda^*)$, $u_n \rightarrow 0$. Dividing (2.3) by $\|u_n\|^p$ we obtain, using Lemma 2.3,

$$(2.8) \quad 0 < \sigma \leq J_{\lambda_n}(v_n) < \frac{p^*-q}{p-q} \int f |v_n|^{p^*} \cdot \|u_n\|^{p-p^*} \rightarrow 0,$$

a contradiction, where $v_n = u_n/\|u_n\|$.

Now, by Young's inequality and Lemma 2.3, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} -\langle \Psi'_\lambda(u), u \rangle &= (p^* - p)J_\lambda(u) - \alpha(p^* - q) \int h|u|^q \\ &\geq (p^* - p)\sigma \|u\|^p - \alpha(p^* - q) \|h\|_Q \cdot \|u\|^q \\ &\geq ((p^* - p)\sigma - \varepsilon)\zeta^p - \alpha C_\varepsilon \|h\|_Q^{p/(p-q)}. \end{aligned}$$

The proof is complete. \square

Corollary 2.6. *Under the conditions of Lemma 2.5, for any $\bar{\lambda} \in (0, \lambda_1^+)$, there exists $\alpha^* > 0$ such that $\Lambda_{\bar{\lambda}}^-$ is a closed set for $\lambda \in [\bar{\lambda}, \lambda^*)$ provided $\alpha \in (0, \alpha^*)$.*

3. PROOF OF EXISTENCE OF SOLUTIONS FOR $1 < q < p$

Lemma 3.1. *Assume (f0), (f1), (f2), (g0), (g1), (h0), (h1) and (h2) hold. Then I_λ is bounded below on $\Lambda_{\bar{\lambda}}^-$ for $\lambda \in (0, \lambda^*)$ and $\alpha \in (0, \alpha^*)$, where α^* and λ^* are given in Lemma 2.5.*

Proof. Suppose for some $u_n \in \Lambda_{\bar{\lambda}}^-$, $I_\lambda(u_n) \rightarrow -\infty$. Then $\|u_n\| \rightarrow \infty$. Since $u_n \in \Lambda_{\bar{\lambda}}^-$, $\int f|u_n|^{p^*} > 0$ by (2.3) and Lemma 2.3. Dividing $I_\lambda(u_n)$ by $\|u_n\|^p$ we obtain from (2.4) that

$$\frac{I_\lambda(u_n)}{\|u_n\|^p} = \frac{1}{N} \int f|v_n|^{p^*} \cdot \|u_n\|^{p^*-p} - \alpha \left(\frac{1}{q} - \frac{1}{p} \right) \int h|v_n|^q \cdot \|u_n\|^{q-p} \rightarrow \ell \leq 0,$$

with $v_n = u_n/\|u_n\|$. It then follows that $\int f|v_n|^{p^*} \cdot \|u_n\|^{p^*-p} \rightarrow N\ell \leq 0$. On the other hand, dividing

$$J_\lambda(u_n) = \int f|u_n|^{p^*} + \alpha \int h|u_n|^q$$

by $\|u_n\|^p$ we obtain, using Lemma 2.3,

$$0 < \sigma \leq J_\lambda(v_n) = \int f|v_n|^{p^*} \cdot \|u_n\|^{p^*-p} + \alpha \int h|v_n|^q \cdot \|u_n\|^{q-p} \rightarrow N\ell \leq 0,$$

a contradiction. So I_λ is bounded below on $\Lambda_{\bar{\lambda}}^-$. \square

Thus we can define $c_0 = \inf_{\Lambda_{\bar{\lambda}}^-} I_\lambda(u)$.

Lemma 3.2. *Assume (f0), (f1), (f2), (g0), (g1), (h0), (h1) and (h2) hold. Then for any $\lambda \in (0, \lambda^*)$, $\alpha \in (0, \alpha^*)$, there exists a minimizing sequence $\{u_n\} \subset \Lambda_{\bar{\lambda}}^-$ of I_λ on $\Lambda_{\bar{\lambda}}^-$ which converges weakly to a solution u of (1.1) $_\lambda$.*

Proof. We first show that any minimizing sequence of I_λ on Λ_λ^- is bounded. Suppose $\{u_n\}$ is an unbounded minimizing sequence of I_λ on Λ_λ^- . Dividing $J_\lambda(u_n)$ by $\|u_n\|^q$, we conclude that, since $I_\lambda(u_n)$ is bounded, $\int f|u_n|^{p^*} \cdot \|u_n\|^{-q}$ is bounded by (2.4). Thus $J_\lambda(u_n) \cdot \|u_n\|^{-q}$ is also bounded by (2.3). Let $v_n = u_n/\|u_n\|$. Then $\sigma \leq J_\lambda(v_n) \rightarrow 0$, a contradiction. Thus any minimizing sequence $\{u_n\}$ in Λ_λ^- is bounded.

Since Λ_λ^- is a closed set by Corollary 2.6, it follows from Theorem 4.1 and Remark 4.1 of [9] that we can replace $\{u_n\}$ by another minimizing sequence $\{z_n\} \subset \Lambda_\lambda^-$ such that $\|u_n - z_n\| < 1/n$, and for any $y \in \Lambda_\lambda^-$,

$$(3.1) \quad I_\lambda(y) > I_\lambda(z_n) - \frac{1}{n}\|y - z_n\|.$$

We want to show that $I'_\lambda(z_n) \rightarrow 0$. Choose w_n of unit norm so that

$$\langle I'_\lambda(z_n), w_n \rangle \geq \|I'_\lambda(z_n)\| - o(1)$$

as $n \rightarrow \infty$. It will suffice to show that

$$(3.2) \quad \langle I'_\lambda(z_n), w_n \rangle \rightarrow 0.$$

For each n , let $g_n(t, s) = \Psi_\lambda(tz_n - sw_n)$. Then $g_n(1, 0) = 0$ and

$$\frac{\partial g_n}{\partial t} = \langle \Psi'_\lambda(z_n), z_n \rangle \neq 0 \text{ at } t = 1, s = 0.$$

It follows from the C^1 Implicit Function Theorem that for each n , for small enough s , there exists $t_n \in C^1$ so that $\Psi_\lambda(t_n(s)z_n - sw_n) = 0$, i.e. $t_n(s)z_n - sw_n \in \Lambda_\lambda$ and

$$(3.3) \quad \langle \Psi'_\lambda(z_n), z_n \rangle t'_n(0) - \langle \Psi'_\lambda(z_n), w_n \rangle = 0.$$

Since z_n is a bounded sequence, so is $\|\Psi'_\lambda(z_n)\|$, and we then conclude from (3.3) and Lemma 2.5 that

$$(3.4) \quad t'_n(0) \text{ is uniformly bounded in } n.$$

We fix n , and consider $v_n(s) = t_n(s)z_n - sw_n - z_n$. Since $\|w_n\| = 1$, we have

$$(3.5) \quad \|v_n(s)\| \leq |s|(1 + (|t'_n(0) + o(1)|)\|z_n\|)$$

as $s \rightarrow 0$. Moreover $z_n \in \Lambda_\lambda$ gives $\langle I'_\lambda(z_n), z_n \rangle = 0$, so

$$(3.6) \quad I_\lambda(z_n) - I_\lambda(t_n(s)z_n - sw_n) = \langle I'_\lambda(z_n), -v_n(s) \rangle + o(\|v_n(s)\|) = \langle I'_\lambda(z_n), sw_n \rangle + o(s)$$

follows from (3.5). By continuity of $\langle \Psi'_\lambda(u), u \rangle$, we have

$$\langle \Psi'_\lambda(t_n(s)z_n - sw_n), t_n(s)z_n - sw_n \rangle - \langle \Psi'_\lambda(z_n), z_n \rangle \rightarrow 0$$

as $s \rightarrow 0$. We then conclude from this and Lemma 3.4 that

$$\langle \Psi'_\lambda(t_n(s)z_n - sw_n), t_n(s)z_n - sw_n \rangle < 0$$

for s small enough, so $t_n(s)z_n - sw_n \in \Lambda_\lambda^-$.

Dividing (3.6) by s and using (3.1) with $y = t_n(s)z_n - sw_n$ and (3.5), we obtain

$$|I'_\lambda(z_n, w_n)| \leq n^{-1}(1 + (I'_n(0))\|z_n\|) + o(1).$$

Letting $n \rightarrow \infty$ we conclude that $(I'_\lambda(z_n), w_n)$ tends to zero by boundedness of z_n and (3.4). This establishes (3.2).

Assume now that $z_n \rightarrow u$ weakly in V . We have, then, as in the proof of Lemma 3.1 of [5], since $I'_\lambda(z_n) \rightarrow 0$, that u is a weak solution of (1.1) $_\lambda$, i.e.,

$$-\Delta_p u = \lambda g|u|^{p-2}u + \alpha h|u|^{q-2}u + f|u|^{p^*-2}u$$

in V . This proves the lemma. \square

Thus we have obtained a weak solution of (1.1) $_\lambda$. To show that this solution is nontrivial, we need some preparation. Let S be the best Sobolev constant, i.e.,

$$S = \inf \left\{ \frac{\|\nabla u\|_2^p}{\|u\|_{p^*}^p} : u \in W_0^{1,p}(\mathbb{R}^N) \setminus \{0\} \right\},$$

and $S_0 = S^{N/p} \|f\|_{\infty}^{(p-N)/p} / N$. Recall the concentration-compactness principle of P.L. Lions ([8]).

Proposition 3.3. *Let $\{u_n\}$ converge weakly to u in $D^{1,p}(\mathbb{R}^N)$ such that $|u_n|^{p^*}$ and $|\nabla u_n|^p$ converge weakly to nonnegative measures ν and μ on \mathbb{R}^N respectively. Then, for some at most countable set J , we have*

- (i) $\nu = |u|^{p^*} + \sum_{j \in J} \nu_j \delta_{x_j}$;
- (ii) $\mu \geq |\nabla u|^p + \sum_{j \in J} \mu_j \delta_{x_j}$;
- (iii) $S \nu_j^{p/p^*} \leq \mu_j$,

where $x_j \in \mathbb{R}^N$, δ_{x_j} is the Dirac measure at x_j , and ν_j and μ_j are nonnegative constants.

Lemma 3.4. Assume (f0), (f1), (f2), (g0), (g1), (h0), (h1) and (h2) hold. For $\lambda \in [0, \lambda^*)$ and $\alpha \in (0, \alpha^*)$, any minimizing sequence $\{u_n\}$ of I_λ on Λ_λ^- satisfying $I_\lambda(u_n) < S_0$ either converges strongly to a solution $u \in \Lambda_\lambda^-$, hence $u \neq 0$, or converges weakly to a nontrivial solution $u \in \Lambda_\lambda$.

P r o o f. Let $\{u_n\}$ be such a minimizing sequence. We can assume without loss of generality that $\{u_n\}$ is bounded (cf. Lemma 3.2).

Assume that $u_n \rightarrow u$ weakly in V . We conclude as in the proof of Proposition 2.3 of [5] that

$$-\Delta_p u = \lambda g|u|^{p-2}u + f|u|^{p^*-2}u + \alpha h|u|^{q-2}u$$

in V , that is, $I'_\lambda(u) = 0$ and hence $u \in \Lambda_\lambda$.

Suppose that $u_n \not\rightarrow u$ strongly in V and $u = 0$. Then for some j, ν_j given by Proposition 3.3 is not zero. We obtain, using the fact that $\int h|u_n|^q \rightarrow 0$ (cf. Proposition 2.3 of [5]),

$$\begin{aligned} S_0 > I_\lambda(u_n) &= \frac{1}{N} \int f|u_n|^{p^*} + \alpha \left(\frac{1}{p} - \frac{1}{q} \right) \int h|u_n|^q \\ &\geq \frac{1}{N} \sum_{j \in J} f(x_j) \nu_j \geq \frac{1}{N} \sum_{j \in J} \frac{S^{N/p}}{f(x_j)^{(N-p)/p}} \geq S_0, \end{aligned}$$

a contradiction. Here we used the facts that $f(x_j) \nu_j = \mu_j$ and $\nu_j \geq (S/f(x_j))^{N/p}$, which follow from the proof of Proposition 2.4 of [5]. This proves the lemma. \square

We need more conditions on f . Assume

(f3) for $x \in B(0, 2r)$,

$$f(x) = f(0) + o(|x|^k), \quad k = \frac{N}{q} \text{ if } q \geq \frac{N(p-1)}{N-p}, \quad k = \frac{N-p}{p-1} \text{ if } q < \frac{N(p-1)}{N-p},$$

or

(f3)' for $x \in B(0, 2r)$,

$$f(x) = f(0) + o(|x|^\delta), \quad \delta = \frac{N-p}{p-1}.$$

Lemma 3.5. Assume (f0), (f1), (f2), (f3) or (f3)', (g0), (g1), (h0) and (h1) hold. Then for $\lambda > 0$, and $\varepsilon > 0$ small enough, we have

$$(3.7) \quad \sup_{t \geq 0} I_\lambda(tv_\varepsilon) < \frac{1}{N} S^{N/p} \|f\|_\infty^{(p-N)/p} = S_0,$$

where v_ε is given in the proof of Lemma 2.2.

Proof. Our proof is similar to that of Lemma 4.1 of [12]. Recall for $\varepsilon > 0$,

$$u_\varepsilon(x) = \frac{\psi(x)}{(\varepsilon + |x|^{p/(p-1)})^{(N-p)/p}}, \quad v_\varepsilon(x) = \frac{u_\varepsilon(x)}{\|u_\varepsilon(x)\|_{p^*}},$$

where $\psi \in C_0^\infty(B(0, 2r))$ is such that $0 \leq \psi(x) \leq 1$ and $\psi(x) \equiv 1$ on $B(0, r)$.

Calculations show that (cf. the proof of Lemma 5.6 in [5])

$$(3.8) \quad \int |v_\varepsilon|^t = \begin{cases} K \varepsilon^{(N(p-t)+t(p-1)(p-1))/p^2}, & \text{if } t > \frac{p^*}{p}, \\ K \varepsilon^{N(p-1)/p^2} |\ln \varepsilon|, & \text{if } t = \frac{p^*}{p}, \\ K \varepsilon^{t(N-p)/p^2}, & \text{if } t < \frac{p^*}{p}, \end{cases}$$

and

$$(3.9) \quad \int |\nabla v_\varepsilon|^t = \begin{cases} K' \varepsilon^{N(p-t)(p-1)/p^2}, & \text{if } t > \frac{N(p-1)}{N-1}, \\ K' \varepsilon^{t(N-p)/p^2} |\ln \varepsilon|, & \text{if } t = \frac{N(p-1)}{N-1}, \\ K' \varepsilon^{t(N-p)/p^2}, & \text{if } t < \frac{N(p-1)}{N-1}. \end{cases}$$

In particular, we have

$$(3.10) \quad \int |v_\varepsilon| = \begin{cases} K \varepsilon^{(N-p)/p^2}, & \text{if } p > \frac{2N}{N+1}, \\ K \varepsilon^{(N-p)/p^2} |\ln \varepsilon|, & \text{if } p = \frac{2N}{N+1}, \\ K \varepsilon^{(N(p-1)+p)(p-1)/p^2}, & \text{if } p < \frac{2N}{N+1}, \end{cases}$$

$$(3.11) \quad \int |v_\varepsilon|^p = \begin{cases} K \varepsilon^{p-1} & p^2 < N, \\ K \varepsilon^{p-1} |\ln \varepsilon| & p^2 = N, \\ K \varepsilon^{(N-p)/p} & p^2 > N. \end{cases}$$

$$\int v_\varepsilon^{p^*} = 1,$$

$$(3.12) \quad \int |\nabla v_\varepsilon| = \begin{cases} K' \varepsilon^{(N-p)/p^2}, & \text{if } p > \frac{2N-1}{N-1}, \\ K' \varepsilon^{(N-p)/p^2} |\ln \varepsilon|, & \text{if } p = \frac{2N-1}{N-1}, \\ K' \varepsilon^{N(p-1)^2/p^2}, & \text{if } p < \frac{2N-1}{N-1}, \end{cases}$$

$$(3.13) \quad \int |\nabla v_\varepsilon|^{p-1} = K' \varepsilon^{(N-p)(p-1)/p^2},$$

and

$$(3.14) \quad \int |\nabla v_\varepsilon|^p = \frac{\int |\nabla u_\varepsilon|^p}{\|u_\varepsilon\|_{p^*}^{p^*}} = \frac{K_1}{K_2} + O(\varepsilon^{(N-p)/p}),$$

where $K_1/K_2 = S$.

Note that for $\varepsilon > 0$ small enough, $J_\lambda(v_\varepsilon) > 0$, so $I_\lambda(tv_\varepsilon)$ attains its maximum at some $t_\varepsilon \in (0, \infty)$ with $s'(t_\varepsilon) = 0$, where $s(t) = I_\lambda(tv_\varepsilon)$. That is,

$$0 = s'(t_\varepsilon) = t_\varepsilon^{p^*-1} \left(\int (|\nabla v_\varepsilon|^p - \lambda |v_\varepsilon|^p) - \alpha t_\varepsilon^{q-p} \int h |v_\varepsilon|^q - t_\varepsilon^{p^*-p} \int f |v_\varepsilon|^{p^*} \right).$$

Thus, by (g1) and (h1),

$$t_\varepsilon^{p^*-p} \leq \frac{\int |\nabla v_\varepsilon|^p}{f(x^*) \int |v_\varepsilon|^{p^*}},$$

where $f(x^*) = \inf_{x \in B(0,r)} f(x) > 0$. It then follows that t_ε is bounded from above. We may also assume that t_ε is bounded from below, otherwise $I_\lambda(t_\varepsilon v_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Now,

$$(3.15) \quad I_\lambda(t_\varepsilon v_\varepsilon) = \sup_{t \geq 0} I_\lambda(tv_\varepsilon) = E(\varepsilon) - F(\varepsilon) + V(\varepsilon),$$

where

$$\begin{aligned} E(\varepsilon) &= \frac{t_\varepsilon^p}{p} \int |\nabla v_\varepsilon|^p - \frac{f(0)t_\varepsilon^{p^*}}{p^*} \int v_\varepsilon^{p^*}, \\ F(\varepsilon) &= \frac{\lambda t_\varepsilon^p}{p} \int g v_\varepsilon^p + \alpha \frac{t_\varepsilon^q}{q} \int h v_\varepsilon^q, \\ V(\varepsilon) &= \frac{t_\varepsilon^{p^*}}{p^*} \int (f(0) - f(x)) v_\varepsilon^{p^*}. \end{aligned}$$

The maximum of $ap^{-1}t^p - b(p^*)^{-1}t^{p^*}$ is achieved at $t = (a/b)^{(N-p)/p^2}$ for positive a, b , so

$$E(\varepsilon) \leq \left(\frac{1}{p} - \frac{1}{p^*} \right) [f(0)]^{(p-N)/p} \left[\int |\nabla v_\varepsilon|^p \right]^{N/p} \left[\int v_\varepsilon^{p^*} \right]^{-N/p^*} = \frac{1}{N} S^{N/p} \|f\|_{\infty}^{(p-N)/p}.$$

We also have, for k and δ given in (f3) and (f3)' respectively,

$$V_\varepsilon = O(\varepsilon^{k(p-1)/p}), \quad O(\varepsilon^{\delta(p-1)/p}).$$

Assuming (f3) holds, we estimate, using the fact that t_ε is bounded from below,

$$F(\varepsilon) \geq \alpha h_0 \int v_\varepsilon^q = \begin{cases} K \varepsilon^{[N(p-q)+qp](p-1)/p^2}, & \text{if } q > \frac{N(p-1)}{N-p}, \\ K \varepsilon^{N(p-1)/p^2} |\ln \varepsilon|, & \text{if } q = \frac{N(p-1)}{N-p}, \\ K \varepsilon^{q(N-p)/p^2}, & \text{if } q < \frac{N(p-1)}{N-p}. \end{cases}$$

From (f3) we derive that for $\varepsilon > 0$ small enough, $F(\varepsilon)$ dominates $V(\varepsilon)$. Thus we conclude from the above that, for $\varepsilon > 0$ small enough and $K > 0$,

$$(3.16) \quad I_\lambda(t_\varepsilon v_\varepsilon) \leq \begin{cases} S_0 - K\varepsilon^{[N(p-q)+qp](p-1)/p^2}, & \text{if } q > \frac{N(p-1)}{N-p}, \\ S_0 - K\varepsilon^{N(p-1)/p^2} |\ln \varepsilon|, & \text{if } q = \frac{N(p-1)}{N-p}, \\ S_0 - K\varepsilon^{q(N-p)/p^2}, & \text{if } q < \frac{N(p-1)}{N-p}. \end{cases}$$

On the other hand, assume (f3)' holds. We have

$$F(\varepsilon) \geq g_0 \int v_\varepsilon^p = \begin{cases} K\varepsilon^{p-1}, & \text{if } p^2 < N, \\ K\varepsilon^{p-1} |\ln \varepsilon|, & \text{if } p^2 = N, \\ K\varepsilon^{(N-p)/p}, & \text{if } p^2 > N. \end{cases}$$

Since $p-1 \leq (N-p)/p$ for $p^2 \leq N$ and $\delta(p-1)/p > (N-p)/p$ for $p^2 > N$ by (f3)', $F(\varepsilon)$ dominates $V(\varepsilon)$. Again we have

$$(3.16)' \quad I_\lambda(t_\varepsilon v_\varepsilon) \leq \begin{cases} S_0 - K\varepsilon^{p-1}, & \text{if } p^2 < N, \\ S_0 - K\varepsilon^{p-1} |\ln \varepsilon|, & \text{if } p^2 = N, \\ S_0 - K\varepsilon^{(N-p)/p}, & \text{if } p^2 > N. \end{cases}$$

The lemma then follows. \square

Lemma 3.6. Assume (f0), (f1), (f2), (f3) or (f3)', (g0), (g1), (h0), (h1), (h2), and λ and α as in Lemma 2.5. Then $c_0 = \inf_{\Lambda_\lambda^-} I_\lambda(u) < S_0$.

This lemma follows from Lemma 3.5 and the fact that $t_\varepsilon v_\varepsilon \in \Lambda_\lambda^-$ for some $t_\varepsilon > 0$ (cf. the proof of Lemma 2.2). Thus we have proved, via Lemmas 3.4 and 3.6, the existence of a nonnegative solution. The next result shows that the solution is actually positive.

Proposition 3.7. Let u be a nonnegative solution of (1.1) $_\lambda$ with $q \leq p^*$. Then $u > 0$ in \mathbb{R}^N .

The proof is essentially as that of Lemma 4.3 of [12] and is omitted.

Now we can state our main result.

Theorem 3.8. Assume that (f0), (f1), (f2), (f3) or (f3)', (g0), (g1), (h0), (h1), and (h2) hold. Then there exist $\lambda^* > \lambda_1^+$ and $\alpha^* > 0$, so that the problem

$$-\Delta_p u = \lambda g(x)|u|^{p-2}u + \alpha h(x)|u|^{q-2}u + f(x)|u|^{p^*-2}u$$

has at least one positive solution in V for any $\lambda \in [\lambda_1^+, \lambda^*)$ and $\alpha \in (0, \alpha^*)$.

4. THE CASE $p < q < p^*$ AND SOME REMARKS

For this case, the set Λ_λ^- is defined as in (2.3). We first have the following result.

Lemma 4.1. *Assume $\text{supp } f^+ \cap \text{supp } h^+$ contains an open set. Then $\Lambda_\lambda^- \neq \emptyset$ for $\lambda > 0$.*

Proof. Suppose $\text{supp } f^+ \cap \text{supp } h^+$ contains an open set B and let $\varphi > 0$ be such that $\text{supp } \varphi \subset B$ with $J_\lambda(\varphi) > 0$. Such φ exists as explained in the proof of Lemma 2.2. For $t > 0$, we have

$$\Psi(t\varphi) = t^p J_\lambda(\varphi) - \alpha t^q \int h\varphi^q - t^{p^*} \int f\varphi^{p^*}.$$

Let again $s(t) = at^p - \alpha bt^q - ct^{p^*}$, with $a = J_\lambda(\varphi) > 0$, $b = \int h\varphi^q > 0$ and $c = \int f\varphi^{p^*} > 0$. Obviously $s(t) > 0$ for $t > 0$ small and $s(t) \rightarrow -\infty$ as $t \rightarrow \infty$. Suppose $s(t_0) = 0$. Then

$$\begin{aligned} s'(t_0) &= t_0^{q-1}(pat_0^{p-q} - \alpha qb - p^* ct_0^{p^*-q}) \\ &= t_0^{q-1}[\alpha(p-q)b - (p^* - p)ct_0^{p^*-q}] < 0 \end{aligned}$$

since $p < q$. That is, $t_0\varphi \in \Lambda_\lambda^-$. This concludes the proof. \square

Remark 4.2. Note that (f1) and (h1) imply that $\text{supp } f^+ \cap \text{supp } h^+$ contains an open set. So we will assume for simplicity in the sequel that (f1) and (h1) hold. We also note that Lemma 4.1 holds if $h \equiv 0$.

Instead of (h2), we need

$$(h2)' \int h|u_1^+|^p < 0.$$

Lemma 4.3. *Assume (f0), (f1), (f2), (g0), (g1), (h0), (h1) and (h2)' hold. Then there exist $\lambda_1^+ > \lambda_1^+$ and $\alpha_1 > 0$ such that for any $\tilde{\lambda} \in (0, \lambda_1^+)$, there exists $\sigma > 0$, such that for any $\lambda \in [\tilde{\lambda}, \lambda_1^+)$ and $\alpha \in (0, \alpha_1)$, $J_\lambda(u) \geq \sigma \|u\|^p$ for any $u \in \Lambda_\lambda^-$.*

Proof. If the conclusion were false, there would exist λ_n , α_n and $u_n \in \Lambda_{\lambda_n}^-$ such that

$$\alpha_n \rightarrow 0, \quad \lambda_n \rightarrow \tilde{\lambda} \in [\tilde{\lambda}, \lambda_1^+], \quad J_{\lambda_n}(u_n) < \frac{1}{n} \|u_n\|^p.$$

As in the proof of Lemma 2.3, we conclude that $v_n = u_n / \|u_n\| \rightarrow ku_1^+$ for some $k \neq 0$. Then instead of (2.7) we have

$$(4.1) \quad \|u_n\|^{p-p^*} J_{\lambda_n}(v_n) > \frac{p^* - q}{p - q} \int f|v_n|^{p^*} \rightarrow \frac{p^* - q}{p - q} \int f|ku_1^+|^{p^*} > 0,$$

since $p < q$ and $\int f|ku_1^+|^{p^*} < 0$ by (f2). If $\|u_n\| \not\rightarrow 0$, then (4.1) contradicts the fact that $J_{\lambda_n}(v_n) \rightarrow 0$. If $\|u_n\| \rightarrow 0$, (4.1) implies that $J_{\lambda_n}(v_n) > 0$. We then have

$$(4.2) \quad \alpha_n \int h|v_n|^q + \int f|v_n|^{p^*} \cdot \|u_n\|^{p^*-q} = J_{\lambda_n}(v_n) \cdot \|u_n\|^{p-q} > 0.$$

Note that $\int f|v_n|^{p^*} \cdot \|u_n\|^{p^*-q} \rightarrow 0$ since $\|u_n\| \rightarrow 0$. Inequality (4.2) then implies $\int h|v_n|^q > 0$, contradicting (h2)'. Thus the lemma is proved. \square

Remark 4.4. We point out that Lemma 4.3 holds if $h \equiv 0$.

The reason is that instead of (4.2), we now have

$$(4.2)' \quad \int f|v_n|^{p^*} \cdot \|u_n\|^{p^*-p} = J_{\lambda_n}(v_n) > 0.$$

This leads to a contradiction again.

Lemma 4.5. Assume that (f0), (f1), (f2), (g0), (g1), (h0), (h1) and (h2)' hold. Then for $\lambda \in (0, \lambda^*)$ and $\alpha \in (0, \alpha_1)$,

- (i) $I_\lambda(u) > 0$ for any $u \in \Lambda_\lambda^-$,
- (ii) any minimizing sequence of I_λ on Λ_λ^- is uniformly bounded.

Proof. We observe that, for $u \in \Lambda_\lambda^-$, from (2.4) and Lemma 4.3,

$$(4.3) \quad \begin{aligned} I_\lambda(u) &= \frac{1}{N} J_\lambda(u) - \alpha \frac{p^* - q}{p^* q} \int h|u|^q \\ &> \left[\frac{1}{N} - \frac{p^* - p}{qp^*} \right] J_\lambda(u) = \frac{q - p}{Nq} J_\lambda(u) \geq \frac{q - p}{Nq} \sigma \|u\|^p. \end{aligned}$$

Since σ only depends on α_1 and λ^* , the conclusions then follow directly. This completes the proof. \square

Now, by Lemma 4.5, there exists $R > 0$, so that for any $\alpha \in (0, \alpha_1)$ and $\lambda \in (0, \lambda^*)$, for any minimizing sequence $\{u_n\} \subset \Lambda_\lambda^-$ of I_λ (here I_λ also depends on α), we can assume that, by taking a subsequence if necessary, $\|u_n\| \leq R$. Define $U_R = \{u \in V : \|u\| \leq R\}$.

Lemma 4.6. Assume (f0), (f1), (f2), (g0), (g1), (h0), (h1) and (h2)' hold. For any $\bar{\lambda} \in (0, \lambda_1^+)$, there exist $\varrho > 0$ and α^* with $\alpha^* \leq \alpha_1$, such that for any $\lambda \in [\bar{\lambda}, \lambda^*)$, $\alpha \in (0, \alpha^*)$ and $u \in \Lambda_\lambda^- \cap U_{2R}$, $-\langle \Psi_\lambda'(u), u \rangle \geq \varrho$.

Proof. We first show that for some $\eta > 0$, depending only on α_1 and λ^* , $\|u\| \geq \eta$ for $u \in \Lambda_\lambda^-$. Indeed, if for some $u_n \in \Lambda_\lambda^-$, $u_n \rightarrow 0$, then we have, by

Lemma 4.3,

$$0 < \sigma \leq J_\lambda(v_n) = \alpha \int h|v_n|^q \cdot \|u_n\|^{q-p} + \int f|v_n|^{p^*} \cdot \|u_n\|^{p^*-p} \rightarrow 0,$$

a contradiction, where $v_n = u_n/\|u_n\|$.

Using Lemma 4.3 we get

$$\begin{aligned} -(\Psi'_\lambda(u), u) &= (p^* - p)J_\lambda(u) - \alpha(p^* - q) \int h|u|^q \\ &\geq (p^* - p)J_\lambda(u) - \alpha(p^* - q)\|h\|_Q \cdot \|u\|^q \\ &\geq (p^* - p)\sigma\eta^p - \alpha(p^* - q)\|h\|_Q(2R)^q > c > 0, \end{aligned}$$

for α small enough. The lemma is proved. \square

Lemma 4.6 implies that $\Lambda_\lambda^- \cap U_{2R}$ is a closed set (in fact one can prove that Λ_λ^- is a closed set). Replacing Λ_λ^- by $\Lambda_\lambda^- \cap U_{2R}$, and noting that any minimizing sequence in $\Lambda_\lambda^- \cap U_{2R}$ will be a positive distance from the boundary $\|u\| = 2R$, we can check straightforwardly that the proofs of Lemmas 3.2, 3.4, 3.5, and 3.6 remain valid. So we can state our result.

Theorem 4.7. *Assume that (f0), (f1), (f2), (f3) or (f3)', (g0), (g1), (h0), (h1), and (h2)' hold. Then there exist $\lambda^* > \lambda_1^+$ and $\alpha^* > 0$, such that for any $\lambda \in [\lambda_1^+, \lambda^*)$ and $\alpha \in (0, \alpha^*)$, the problem*

$$-\Delta_p u = \lambda g(x)|u|^{p-2}u + \alpha h(x)|u|^{q-2}u + f(x)|u|^{p^*-2}u$$

has at least one positive solution in V .

Remark 4.8. As we remarked earlier, for $\lambda \in (0, \lambda_1^+)$, Theorems 3.8 and 4.7 hold without the integral conditions (f2), (h2) and (h2)', and can be proved via Mountain Pass argument. Cf. [1], [2], [7] and [12].

Remark 4.9. We note that the proofs are applicable to Dirichlet problems on bounded domains and similar results hold. We can also deal with

$$-\Delta_p u + a(x)|u|^{p-2}u = \lambda g(x)|u|^{p-2}u + \alpha h(x)|u|^{q-2}u + f(x)|u|^{p^*-2}u$$

in \mathbb{R}^N , where $a(x) \in L^\infty_{loc}(\mathbb{R}^N)$, $a(x) \geq 0$.

Remark 4.10. Similarly, one can consider the negative principal eigenvalue $\lambda_1^- < 0$ given by

$$-\Delta_p u = \lambda g(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \quad \int g(x)|u|^p < 0.$$

Existence of positive solutions of (1.1) $_\lambda$ for $\lambda < 0$ can be obtained provided conditions similar to (f2), (h2) and (h2)' hold.

References

- [1] *C. O. Alves*: Multiple positive solutions for equations involving critical Sobolev exponent in \mathbb{R}^N . *Electron. J. Differential Equations* 13 (1997), 1–10.
- [2] *C. O. Alves, J. V. Gonçalves, O. H. Miyagaki*: Remarks on multiplicity of positive solutions for nonlinear elliptic equations in \mathbb{R}^N with critical growth. Preprint.
- [3] *H. Brezis, L. Nirenberg*: Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.* 36 (1983), 437–477.
- [4] *P. Drábek, Y. X. Huang*: Multiple positive solutions of quasilinear elliptic equations in \mathbb{R}^N . *Nonlinear Anal.* To appear.
- [5] *P. Drábek, Y. X. Huang*: Multiplicity of positive solutions for some quasilinear elliptic equation in \mathbb{R}^N with critical Sobolev exponent. *J. Differential Equations* 140 (1997), 106–132.
- [6] *P. Drábek, Y. X. Huang*: Bifurcation problems for the p -Laplacian in \mathbb{R}^N . *Trans. Amer. Math. Soc.* 349 (1997), 171–188.
- [7] *J. V. Gonçalves, C. O. Alves*: Existence of positive solutions for m -Laplacian equations in \mathbb{R}^N involving critical Sobolev exponents. *Nonlinear Anal.* 32 (1998), 53–70.
- [8] *P. L. Lions*: The concentration-compactness principle in the calculus of variations, the limit case, Part I, II. *Rev. Mat. Iberoamericana* 1 (1985), no. 2, 3, 109–145, 45–121.
- [9] *J. Mawhin, M. Willem*: *Critical Point Theory and Hamiltonian Systems*. *Appl. Math. Sci.* Vol. 74, Springer-Verlag, New York, 1989.
- [10] *E. S. Noussair, C. A. Swanson*: Multiple finite energy solutions of critical semilinear field equations. *J. Math. Anal. Appl.* 195 (1995), 278–293.
- [11] *E. S. Noussair, C. A. Swanson, J. Yang*: Quasilinear elliptic problems with critical exponents. *Nonlinear Anal.* 20 (1993), 285–301.
- [12] *C. A. Swanson, L. S. Yu*: Critical p -Laplacian problems in \mathbb{R}^N . *Ann. Mat. Pura Appl.* 169 (1995), 233–250.
- [13] *G. Tarantello*: On nonhomogeneous elliptic equations involving critical Sobolev exponent. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 9 (1992), 281–304.
- [14] *J. Yang*: Positive solutions of quasilinear elliptic obstacle problems with critical exponents. *Nonlinear Anal.* 25 (1995), 1283–1306.

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