Mirko Navara

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DESCRIPTIONS OF STATE SPACES OF ORTHOMODULAR LATTICES (THE HYPERGRAPH APPROACH)

MIRKO NAVARA, Praha

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Summary. Using the general hypergraph technique developed in [7], we first give a much simpler proof of Shultz's theorem [10]: Each compact convex set is affinely homeomorphic to the state space of an orthomodular lattice. We also present partial solutions to open questions formulated in [10]—we show that not every compact convex set has to be a state space of a unital orthomodular lattice and that for unital orthomodular lattices the state space characterization can be obtained in the context of unital hypergraphs.

Keywords: hypergraph, orthomodular lattice, state space

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1. BASIC NOTIONS

Let us first recall the notions and facts on orthomodular lattices as we shall need them in the sequel (see [2, 4, 10]). The symbol oml will mean an orthomodular lattice. Also, if we deal with an oml $(L, 0, 1, \wedge, \vee, ^{\perp})$, we usually refer only to its domain L.

An element $a \in L$ is called an *atom* if there is no $b \in L$ such that $0 < b < a$. Let us say that L is *chain-finite* if all chains (=strictly monotonic sequences) in L are finite. In this case each element of L is a supremum of a finite set of atoms. A subset $M \subset L$ is called *compatible* if it is contained in a Boolean subalgebra of L. Maximal Boolean subalgebras of L are called *blocks* in L.

A *state* on an oml L is a function $s\colon L \to [0, 1]$ which is additive, i.e. s is such a function that $s(a \vee b) = s(a) + s(b)$ for all $a, b \in L$ with $a \leqslant b^{\perp}$. Let us denote by $S(L)$ the set of all states of L, and let us call $S(L)$ the *state space* of L.

Obviously, $S(L)$ can be viewed as a subset of $[0, 1]^L$. Due to the finite additivity of elements of $S(L)$ we can simply prove the following fact:

Proposition 1.1. *The state space $S(L)$ of an oml L is a compact convex subset of the set $[0,1]^L$ (here $[0,1]^L$ is understood with the product topology).*

2. Representations of state spaces of omls by means of hypergraphs

By a *hypergraph* we mean a couple $H = (\mathscr{V}(H), \mathscr{E}(H))$, where $\mathscr{V}(H)$ is a non-empty set (of *vertices*) and $\mathscr{E}(H) \subset \exp \mathscr{V}(H)$. The elements of $\mathscr{E}(H)$ are called *edges*.[1] In what follows, the letter H always denotes a hypergraph. A state on H is a mapping $s\colon \mathscr{V}(H) \to [0,1]$ such that $\sum_{v \in E} s(v) = 1$ for all $E \in \mathscr{E}(H)$ [3]. The set of all states on H (the state space of H) is denoted by $S(H)$.

Example 2.1. Consider a hypergraph U such that $\mathscr{V}(U) = \{u, v\}$ and $\mathscr{E}(U) = \{\{u\}, \{u, v\}\}$. One can easily see that there is only one state s on U. In fact, $s(u) = 1$, $s(v) = 0$.

A *subhypergraph* of H is a hypergraph G such that $\mathscr{V}(G) \subset \mathscr{V}(H)$ and $\mathscr{E}(G) = \mathscr{E}(H) \cap \exp G$. A hypergraph H is called *connected* if for each $u, v \in \mathscr{V}(H)$ there is a sequence of edges $E_1, \dots, E_n \subset \mathscr{E}(H)$ such that $u \in E_1$, $v \in E_n$, and $E_i \cap E_{i+1} \neq \emptyset$ for $i = 1, \dots, n-1$. A *component* of a hypergraph is a maximal connected subhypergraph.

2.1 Greechie diagrams.

Let us recall a hypergraph representation of chain-finite omls. Let L be a chain-finite oml and let $\mathscr{A}(L)$ be the set of its atoms. Let us define a hypergraph H such that $\mathscr{V}(H) = \mathscr{A}(L)$ and $\mathscr{E}(H)$ consists of all maximal orthogonal sets of atoms. The hypergraph H fully describes the structure of L. We call H the *Greechie diagram* of L. Obviously, the states on L are in a natural one-to-one correspondence with the states on H.

From now on, we shall assume that all hypergraphs satisfy the following condition (which is fulfilled for all Greechie diagrams of chain-finite omls).

Assumption 2.2. In this paper we shall deal only with those hypergraphs which are *chain-finite*, i.e. hypergraphs satisfying the following condition:

there is no infinite set of vertices such that each of its finite subsets is contained in an edge.

(In particular, all edges are assumed to be finite.)

[1] Unlike the standard definition, we do not require that $\mathscr{E}(H)$ be a covering of $\mathscr{V}(H)$.

2.2 State space representations "up to a state isomorphism".

If we are interested only in the state space of an oml and not in the intrinsic structure of the oml in question, we can employ the representation by hypergraphs, which we are going to describe in this subsection. In this representation one may identify vertices (atoms) which are indistinguishable at every state.

Definition 2.3 [5]. Vertices $u, v \in \mathcal{V}(H)$ are called *state-equivalent* if $s(u) = s(v)$ for each state $s \in S(H)$.

Example 2.4. Suppose that $j, k \in N$, $j \leqslant k$. Define a hypergraph $H_{j,k}$ with vertices x, x', y, y', v, w, u_1, ..., u_k, u' and with edges $\{x, x'\}$, $\{y, y'\}$, $\{u_1, \ldots, u_k, x'\}$, $\{u_1, \ldots, u_{j-1}, w, y'\}$, $\{y, v, u_{j+1}, \ldots, u_k, x'\}$ and $\{u', u_i\}$, $i = 1, \ldots, k$ (see Fig. 1). It can be easily seen that each $s \in S(H_{j,k})$ satisfies

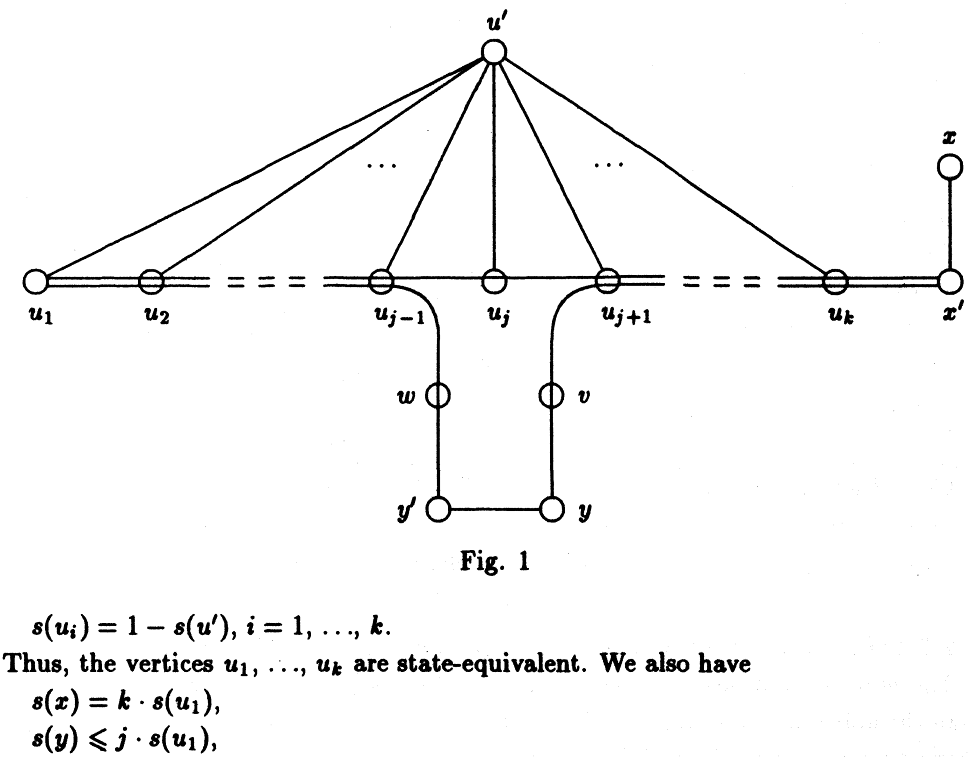

Fig. 1

$s(u_i) = 1 - s(u')$, $i = 1, \ldots, k$.

Thus, the vertices $u_1, \ldots, u_k$ are state-equivalent. We also have

$s(x) = k \cdot s(u_1)$,

$s(y) \leqslant j \cdot s(u_1)$,

$s(y) \geqslant (j-1) \cdot s(u_1)$,

which implies the inequality

$$\frac{j-1}{k} s(x) \leqslant s(y) \leqslant \frac{j}{k} s(x).$$

(Notice that each state on $H_{j,k}$ is uniquely determined by its values on y and x, and for each $r \in [0, 1]$ there is a state attaining the value r at x.)

For $v \in \mathscr{V}(H)$ we denote by $[v]$ the class of vertices of H state-equivalent to v. Put $\mathscr{P}(H) = \{[v] : v \in \mathscr{V}(H)\}$. Obviously, the state space of H is fully determined by the partition $\mathscr{P}(H)$ of $\mathscr{V}(H)$ within the state equivalence and by the values of states on these classes.

Definition 2.5 [5]. Let H_1, H_2 be hypergraphs. A state isomorphism of H_1 and H_2 is a bijection $f : \mathscr{P}(H_1) \to \mathscr{P}(H_2)$ such that

1. for each state s_2 on H_2 the mapping $v \mapsto s_2\big(f([v])\big)$ is a state on H_1,
2. for each state s_1 on H_1 the mapping $v \mapsto s_1(f^{-1}([v]))$ is a state on H_2.

The notion of state isomorphism can be naturally extended to a state isomorphism of an oml and a hypergraph—instead of an oml we consider its Greechie diagram. The characterization of hypergraphs which are state-isomorphic to omls is then as follows. (By a graph we mean a hypergraph G such that $\mathscr{E}(G)$ is a covering of $\mathscr{V}(G)$ and $\operatorname{card} E = 2$ for all $E \in \mathscr{E}(G)$. Let us recall that an *even graph* is a graph containing no cycle of an odd length.)

Theorem 2.6 [7, 6]. *Let H be a chain-finite hypergraph such that not all edges are singletons and such that $\mathscr{E}(H)$ is a covering of $\mathscr{V}(H)$. Then H is state-isomorphic to a (chain-finite) oml if and only if for each $v \in \mathscr{V}(H)$ one of the following conditions holds:*

1. *the component of H containing v is an even graph,*
2. *there are vertices $u_1, \ldots, u_m \in \mathscr{V}(H)$, $m \geqslant 2$ (not necessarily different from v and from each other) such that each state $s \in S(H)$ satisfies the equality $s(v) + \sum_{i \leqslant m} s(u_i) = 1$.*

Corollary 2.7 [7]. *Let H be a chain-finite hypergraph such that $\mathscr{E}(H)$ is a covering of $\mathscr{V}(H)$. Suppose that there is a vertex $v \in \mathscr{V}(H)$ such that $s(v) = 0$ for all $s \in S(H)$. Then H is state-isomorphic to a chain-finite oml.*

2.3 State space representations "up to an affine homeomorphism".

The state isomorphism of hypergraphs is obviously a strictly stronger condition than the affine homeomorphism of their state spaces. (Observe that the state isomorphism preserves the values of the states, not only the convex structure. For instance, two hypergraphs whose state spaces are singletons need not be state-isomorphic.) In order to represent state spaces of omls up to affine homeomorphisms we are allowed to deal with all hypergraphs as the following proposition states.

Proposition 2.8. *Let H be a (chain-finite) hypergraph. Then $S(H)$ is affinely homeomorphic to the state space of an oml.*

Proof. For each $v \in \mathscr{V}(H) \setminus \bigcup \mathscr{E}(H)$, let us add to H a new vertex w and the edge $\{v, w\}$. We obtain a hypergraph F with $\bigcup \mathscr{E}(F) = \mathscr{V}(F)$. Now, we shall construct a hypergraph G by adding to F a new component which is isomorphic to the hypergraph U from Ex. 2.1. More precisely, $\mathscr{V}(G) = \mathscr{V}(F) \cup \{u_0, v_0\}$, $\mathscr{E}(G) = \mathscr{E}(F) \cup \{\{u_0\}, \{u_0, v_0\}\}$. Each state on H has a unique extension to G and G satisfies the assumption of Cor. 2.7. □

3. An alternative proof of Shultz's theorem

The main result of [10] is the following theorem.

Theorem 3.1. *Let C be a compact convex subset of a locally convex Hausdorff topological linear space. Then there is a chain-finite oml whose state space is affinely homeomorphic to C.*

We are now going to present a simple proof of this theorem. Let us first formulate two lemmas.

Lemma 3.2. *Suppose that $r \in R$, $r > 0$. Then there is a hypergraph G_r and vertices $x, y \in \mathscr{V}(G_r)$ such that*

1. *each state s on G_r satisfies $s(y) = r \cdot s(x)$,*
2. *each state on G_r is uniquely determined by its value on x,*
3. *for each q, $0 \leqslant q \leqslant \min(1/r, 1)$, there is a state on G_r attaining the value q at x.*

Proof. For $r \leqslant 1$ we take sequences of integers $\{j_i\}_{i \in N}$, $\{k_i\}_{i \in N}$ such that

$$r = \bigcap_{i \in N} [(j_i - 1)/k_i, j_i/k_i].$$

For each $i \in N$ we take the hypergraph H_{j_i,k_i} from Ex. 2.4. In the hypergraphs H_{j_i,k_i}, $i \in N$, we identify all vertices corresponding to the vertex x from Ex. 2.4 and we also identify the vertices corresponding to y. Each state s on the resulting hypergraph G_r satisfies $s(y) = r \cdot s(x)$. For $r > 1$ we only interchange the role of x and y. □

Remark 3.3. If $x' \in \mathscr{V}(G_r)$ is a vertex corresponding to $x' \in H_{j_i,k_i}$ for some i, we obtain $s(y) = r \cdot (1 - s(x'))$ for each $s \in S(G_r)$.

Lemma 3.4. *Let G be a chain-finite hypergraph. Let C be the set of all states s on G satisfying the inequality*

$$\text{(F)} \qquad \sum_{i\leqslant n} p_i s(x_i) \leqslant q,$$

where $n \in N$, $q, p_i \in R$ and $x_i \in \mathscr{V}(G)$ for $i = 1, \ldots, n$. Then there is a chain-finite hypergraph H and a vertex $v \in \mathscr{V}(H)$ such that

1. *G is a subhypergraph of H,*
2. *each state $s \in C$ has a unique extension to a state on H,*
3. *each $s \in S(H)$ satisfies* (F) *and the equality in* (F) *occurs if and only if $s(v) = 0$.*

In particular, $S(H)$ is affinely homeomorphic to C.[2]

Proof. We may suppose that $p_1, \ldots, p_n \neq 0$. The inequality (F) is equivalent to the following inequality:

$$\text{(F}'\text{)} \qquad \sum_{i\leqslant n, p_i>0} p_i s(x_i) + \sum_{i\leqslant n, p_i<0} |p_i|(1 - s(x_i)) \leqslant q + \sum_{i\leqslant n, p_i<0} |p_i| \,.$$

Put

$$Q = q + \sum_{i\leqslant n, p_i<0} |p_i| \,.$$

If $Q < 0$ then $C = \emptyset$. In this case it suffices to add to G a new component which consists of the stateless hypergraph with vertices u, v and edges $\{u\}$, $\{v\}$, $\{u, v\}$.

If $Q = 0$ then (F') reduces to the equations $s(x_i) = 0$, $i = 1, \ldots, n$. For each $i \leqslant n$ we add to G a vertex y_i and the edges $\{y_i\}$, $\{x_i, y_i\}$. It remains to take any x_i for v.

Suppose finally that $Q > 0$. For each $i \leqslant n$ such that $p_i > 0$ we add to G a vertex y_i and a copy of the hypergraph G_r from Lemma 3.2, where we take p_i/Q for r, and identify $x, y \in G_r$ with x_i, y_i, respectively. For each $i \leqslant n$ such that $p_i < 0$ we add a vertex y_i and a copy of the hypergraph G_r from Lemma 3.2, where we take $|p_i|/Q$ for r, and identify $x', y \in G_r$ with x_i, y_i, respectively (cf. Rem. 3.3). Finally, we add a vertex v and the edge $\{y_1, \ldots, y_n, v\}$. In the resulting hypergraph H we have

$$\sum_{i\leqslant n, p_i>0} p_i s(x_i) + \sum_{i\leqslant n, p_i<0} |p_i|(1 - s(x_i)) = Q \cdot \sum_{i\leqslant n} s(y_i) = Q - Qs(v) \leqslant Q$$

for each state s. One can easily check that H has the desired properties. □

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[2] If we require a relation between C and $S(H)$ analogous to the state isomorphism, a similar result is given in [Th. 4.6, 7] for the case when (F) is replaced by a family of equalities of the form $x_\alpha = y_\alpha$, $x_\alpha, y_\alpha \in \mathscr{V}(G)$, $\alpha \in I$.

Proof of Theorem 3.1. Without any loss of generality we may suppose that C is a subset of $[0,1]^X$ for some set X. The set C can be described as the set of all $s \in [0,1]^X$ which satisfy certain family of inequalities of the form (F).

Let G be a hypergraph with $\mathcal{V}(G) = X$ and $\mathcal{E}(G) = \emptyset$. Then $S(G) = [0,1]^X$. We apply Lemma 3.4 to the hypergraph G and to all inequalities of the form (F) determining the set C. We obtain a hypergraph H with $S(H)$ affinely homeomorphic to C. Prop. 2.8 applied to H gives the desired oml. The proof is complete. □

4. State spaces of unital omls

For the application of omls in quantum physics and other fields it is reasonable to require that the omls have "reasonably large" state spaces. Conditions used most frequently are the following (see [2, 9]): An oml L is called

- *unital* if $\forall a \in L$, $a \neq 0$ $\exists s \in S(L)$: $s(a) = 1$,
- *full* (or *order determining*) if $\forall a, b \in L$, $a \not\leq b$ $\exists s \in S(L)$: $s(a) > s(b)$,
- *rich* (or *strongly order determining*) if $\forall a, b \in L$, $a \not\leq b$ $\exists s \in S(L)$: $1 = s(a) > s(b)$.

Shultz [10] posed a question of what sets can be represented as state spaces of full or rich omls. Unfortunately, the technique presented in Section 2.2 is of little use here since the constructions used in this paper (as well as the technique of Shultz) lead to omls which are not full. Only the unitality is preserved by state isomorphisms (of omls). Let us call a hypergraph H *unital* if $\forall v \in \mathcal{V}(H)$ $\exists s \in S(H)$: $s(v) = 1$.

In this section we show that the characterization of state spaces is the same for unital omls and for unital hypergraphs. We also show that not all convex compacts are state spaces of unital omls. As all rich logics are unital, this result restricts also the class of state spaces of rich logics. Thus, this gives a partial answer to the question posed in [10].

Theorem 4.1. *Let H be a unital chain-finite hypergraph. There is a unital oml L such that $S(L)$ is affinely homeomorphic to $S(H)$.*

Proof. The case when all edges of H are singletons is trivial (H is state-isomorphic to the two-element Boolean algebra).

If H contains singleton edges, these must be disjoint to all other edges of H (because of the unitality). We omit all singleton edges and their vertices in H and denote the resulting hypergraph by H_1. This procedure does not influence the state space (up to an affine homeomorphism) and H_1 is again unital.

Let G be a maximal subhypergraph of H_1 such that G is a graph and G is connected. Due to the unitality, G is even and there are vertices $u, v \in \mathcal{V}(G)$ such that

$\mathscr{V}(G) = [u] \cup [v]$. Each edge of H_1 contains at most one vertex from $[u]$ and at most one vertex from $[v]$ (because of the unitality of H_1). We identify all elements of $[u]$ in H_1 and also all elements of $[v]$. Then G reduces to a single edge $\{u, v\}$. Repeating this procedure for all maximal subgraphs of H_1 we obtain a hypergraph H_2 which is state-isomorphic to H_1 (and therefore unital).

Consider an edge $E = \{u, v\} \in \mathscr{E}(H_2)$. If E forms a component of H_2, v and u satisfy condition 1 of Th. 2.6. If both v and u are covered by edges of H_2 with more than two elements, condition 2 of Th. 2.6 is fulfilled. Suppose finally that exactly one of the two vertices, say v, is covered by edges of H_2 with more than two elements. Then v satisfies condition 2 of Th. 2.6, while u does not. However, when we omit the vertex u and the edge $\{u, v\}$ in H_2, the state space remains the same (up to an affine homeomorphism). Doing this for all two-element edges such that exactly one of their vertices is covered by edges with more than two elements, we obtain a hypergraph H_3 with $S(H_3)$ affinely homeomorphic to $S(H_2)$. One can easily verify that H_3 is unital and satisfies the assumption of Th. 2.6. Thus there is an oml L state-isomorphic to H_3, which is unital, and $S(L)$ is affinely homeomorphic to $S(H)$. The proof is complete. □

Th. 4.1 is an analogue of Prop. 2.8 for unital hypergraphs and omls. As we believe, this theorem may help toward the characterization of state spaces of unital omls. Though we do not know such a characterization, we can show that it is substantially different from the case of general omls considered in the previous section. To do so, let us first recall a notion from the convexity theory.

A subset F of a convex set C is called a *face* if for each $s, t \in C$ and $\alpha \in (0,1)$ the relation $\alpha s + (1-\alpha)t \in F$ implies $s, t \in F$. We denote by $\mathscr{F}(C)$ the lattice of all faces of C (the ordering in $\mathscr{F}(C)$ is given by inclusion).

Proposition 4.2. *A compact convex set C of an affine dimension 2 is affinely homeomorphic to the state space of a unital oml (or rich oml) if and only if C is either a triangle or a parallelogram.*

Proof. Suppose that L is an oml such that the affine dimension of $S(L)$ equals 2. For each $a \in L$, the set $F(a) = \{s \in S(L) : s(a) = 1\}$ is a face of $S(L)$. Obviously, the mapping $F: L \to \mathscr{F}(S(L))$ preserves the ordering.

If a, b are compatible elements of L and $a \not\leq b$ then $F(a) \not\subset F(b)$, because there is a state $t \in S(L)$ with $s(a \wedge b^{\perp}) = 1$. Thus F restricted to any compatible subset of L becomes an order isomorphism.

The maximal length of a chain in $\mathscr{F}(S(L))$ is 4 and the same holds for chains in L, hence L contains no block larger than 2^3.

Suppose that L contains a block B with 3 atoms, say a, b, c. As a compatible set, B is isomorphic (under the isomorphism $F|B$) to a sublattice of $\mathcal{F}(S(L))$. The atoms of B correspond to faces of dimension 0 (i.e., they correspond to extreme points) and coatoms of B correspond to faces of dimension 1 (i.e, they correspond to edges of the polytope $S(L)$). The edge $F(a^\perp)$ contains the extreme points $F(b)$, $F(c)$; these must be the endpoints of $F(a^\perp)$. Analogous relations for $F(b^\perp)$ and $F(c^\perp)$ yield that $S(L)$ is the convex hull of $F(a)$, $F(b)$ and $F(c)$. However, this is a triangle. Thus, in this case L is state-isomorphic to 2^3.

Suppose now that $S(L)$ is not a triangle. So, L contains no block larger than 2^2. This may occur only if L is a horizontal sum (see [4]) of two-atomic Boolean algebras. The state space of the horizontal sum of a family of ω two-atomic Boolean algebras is affinely homeomorphic to $[0,1]^\omega$. In the case of L, we have $\omega = 2$ and L is thus the oml MO2 (see [4]). Only parallelograms are affinely homeomorphic to $[0,1]^2$.

The omls which appeared in the proof (2^3 and MO2) are not only unital, but also rich, hence Prop. 4.2 holds for rich omls, too. The proof is complete. □

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Author's address: Dept. of Mathematics, Faculty of Electrical Engineering, Technical University of Prague, Technická 2, 166 27 Prague, Czechoslovakia.