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ON α-CONTINUOUS FUNCTIONS

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Summary. Classes of functions continuous in various senses, in particular θ-continuous, α-continuous, feebly continuous a.o., and relations between the classes, are studied.

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1. Introduction

The notion of a θ-continuous function between topological spaces was introduced by S. Fomin in his study of extensions of Hausdorff spaces [6]. Since then this concept has been frequently used in investigations of nonregular Hausdorff spaces. A function $f\colon X \to Y$ is called θ-continuous if for every $x \in X$ and every open neighborhood V of $f(x)$ there exists an open neighborhood U of x such that $f(\operatorname{Cl} U) \subset \operatorname{Cl} V$. It is clear that continuous functions are θ-continuous and that θ-continuous functions into regular spaces are continuous. Although θ-continuous functions behave, in general, nicely, they may cause some unexpected problems. For instance, as was pointed out in [24], if $f\colon X \to Y$ is θ-continuous, then $f\colon X \to f(X)$ is not necessarily θ-continuous. The possible bad behavior of a θ-continuous function $f\colon X \to Y$ is caused by the following possible deficiency: there may exist a point $x \in X$ and open neighborhoods V of $f(x)$ and U of x such that $f(\operatorname{Cl} U) \subset \operatorname{Cl} V$ and $f(U - \{x\}) \subset \operatorname{Cl} V - V$. In [24] θ-continuous functions with this property at $x \in X$ are called defective at x. In order to overcome the possible defectiveness of θ-continuous functions L. Rudolf [24] introduced two subclasses of θ-continuous functions having better categorical properties, namely, weakly continuous functions and s-maps. A function $f\colon X \to Y$ is weakly continuous (in the sense of Rudolf) if for every $x \in X$

and every open neighborhood V of $f(x)$ there exists an open neighborhood U of x such that $f(\mathrm{Cl}\,U) \subset \mathrm{Cl}\,V$ and $(U - \{x\}) \cap f^{-1}(V)$ contains a set which is open and dense in U. A basic fact about a weakly continuous function is that it becomes continuous if its domain is appropriately retopologized. Namely, $f\colon (X, \tau) \to Y$ is weakly continuous iff $f\colon (X, s(\tau)) \to Y$ is continuous, where $s(\tau)$ is the topology $\{A \subset X : U \subset A \subset \mathrm{Int}\,\mathrm{Cl}\,U$ for some $U \in \tau\}$, called the standard r.o. extension of τ [24]. Further strengthening of θ-continuity is obtained in a similar way: a function $f\colon (X, \tau) \to (Y, \sigma)$ is called an s-map if $f\colon (X, s(\tau)) \to (Y, s(\sigma))$ is continuous. These two new classes of noncontinuous functions were used in [24] for obtaining Taĭmanov type theorems about extensions of θ-continuous functions.

Recently, T. Noiri [19] and A. S. Mashour et al. [15] defined strongly semi-continuous or α-continuous functions. A function $f\colon (X, \tau) \to Y$ is called α-continuous if $f\colon (X, \tau^\alpha) \to Y$ is continuous, where τ^α is the topology $\{A \subset X : A \subset \mathrm{Int}\,\mathrm{Cl}\,\mathrm{Int}\,A\}$ of α-open sets introduced by O. Njåstad [18]. Noticing that $\tau^\alpha = s(\tau)$ we see that the α-continuity is precisely the weak continuity of L. Rudolf. Similarly, the α-irresolute functions considered in [14] are precisely the s-maps. In this paper we continue the study of α-continuous and α-irresolute functions.

In Section 3 we introduce nearly feebly open functions and establish that a semi-continuous function is nearly feebly open iff it inversely preserves nowhere dense sets. This result enables us to improve Z. Frolík's theorem on preservation of Baire spaces.

In Section 4 we consider a class of generalized open sets in a space, called almost locally dense sets, and their connection with the problem of preservation of properties of sets under α-continuous functions. Making use of nearly feebly open functions we show that an almost locally dense set A in a space X has an important property that nowhere denseness and category of a subset B of A are the same relative to X and to A.

In the last Section of this paper we investigate several classes of functions between the weak continuity in the sense of Levine and the α-irresoluteness. The main result of this section is that θ-continuous feebly open feebly continuous irreducible surjections are α-irresolute. There are a few interesting consequences of this result. First, a Hausdorff space X is quasiregular iff $k_X\colon EX \to X$ is α-irresolute, where (EX, k_X) is the Iliadis absolute. Second, if a Hausdorff space X is quasiregular, then X is a Baire space iff EX is a Baire space. Third, spaces (X, τ) and (Y, σ) are semi-homeomorphic iff (X, τ^α) and (Y, σ^α) are homeomorphic.

2. Preliminary definitions and notation

Throughout the present paper (X, τ) and (Y, σ) (or X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X, $\mathrm{Cl}\,A$, $\mathrm{Int}\,A$ and $\mathrm{Bd}\,A$ denote the closure, interior and boundary of A in X, respectively. Recall that a set A in a space (X, τ) is regular open if $A = \mathrm{Int}\,\mathrm{Cl}\,A$ and that the family of regular open sets of (X, τ) is a base for a topology τ_S on X coarser that τ. The space (X, τ_S) is called the semiregularization of (X, τ). Also, recall that a set A in a space X is α-open [18] (resp. locally dense [2], semi-open [13]) if $A \subset \mathrm{Int}\,\mathrm{Cl}\,\mathrm{Int}\,A$ (resp. $A \subset \mathrm{Int}\,\mathrm{Cl}\,A$, $A \subset \mathrm{Cl}\,\mathrm{Int}\,A$). The family of all α-open sets in (X, τ) is a topology on X, denoted as τ^α. It is shown in [18] that $\tau^\alpha = \{U - N \mid U \in \tau$ and N is nowhere dense in $(X, \tau)\}$ (a set $A \subset X$ is nowhere dense (codense) if $\mathrm{Int}\,\mathrm{Cl}\,A = \emptyset$ ($\mathrm{Int}\,A = \emptyset$)). Also, $A \in \tau^\alpha$ iff A is locally dense and semi-open. A function $f : X \to Y$ is called α-continuous [15] (resp. nearly continuous [22], semi-continuous [13]) if $f^{-1}(V)$ is α-open (resp. locally dense, semi-open) for every open set V in Y. Clearly, f is α-continuous iff it is nearly continuous and semi-continuous. Another class of noncontinuous functions is introduced in [7]. A function $f : X \to Y$ is feebly continuous if $\mathrm{Int}\,f^{-1}(V) \neq \emptyset$ for every nonempty open set V is Y. Evidently, semi-continuous surjections are feebly continuous and the converse does not hold in general. We also need the following weak forms of openness of functions. A function $f : X \to Y$ is called nearly open [22] (semi-open [13]) if $f(U)$ is locally dense (semi-open) for every open set U in X. Finally, a function $f : X \to Y$ is said to be feebly open [7] if $\mathrm{Int}\,f(U) \neq \emptyset$ for every nonempty open set U in X.

3. Nearly feebly open functions

The following generalization of feebly open functions will be very useful. We say that a function $f : X \to Y$ is nearly feebly open if $\mathrm{Int}\,\mathrm{Cl}\,f(U) \neq \emptyset$ for every nonempty open set U in X. In our first result we offer several characterizations of nearly feebly open functions. The straightforward proof is omitted.

Lemma 3.1. *The following are equivalent for a function $f : X \to Y$.*

(a) *f is nearly feebly open.*

(b) *The inverse images under f of open dense sets in Y are dense in X.*

(c) *The inverse images under f of (closed) nowhere dense sets in Y are codense in X.*

(d) *For every open set V in Y, $f^{-1}(\mathrm{Bd}\,V)$ is codense in X.*

Our next result improves and extends Corollary 4.5 of [9].

Lemma 3.2. *Let $f: X \to Y$ be a semi-continuous function. Then the following are equivalent.*

(a) *f is nearly feebly open.*

(b) *The inverse images under f of nowhere dense sets in Y are nowhere dense in X.*

(c) $f(U) \subset \operatorname{Cl}\operatorname{Int}\operatorname{Cl} f(U)$ *for every open set U in X.*

(d) $f^{-1}(\operatorname{Int}\operatorname{Cl} V) \subset \operatorname{Cl} f^{-1}(V)$ *for every open set V in Y.*

Proof. (a)⇒(b): It is enough to show that the inverse images of closed nowhere dense sets are nowhere dense. Let N be a closed nowhere dense set in Y. Since f is semi-continuous, $f^{-1}(Y - N)$ is semi-open and hence $\operatorname{Int}\operatorname{Cl} f^{-1}(N) \subset f^{-1}(N)$. By Lemma 3.1, $\operatorname{Int} f^{-1}(N) = \emptyset$ and consequently, $\operatorname{Int}\operatorname{Cl} f^{-1}(N) = \emptyset$.

(b)⇒(c): Let U be open in X and $y \notin \operatorname{Cl}\operatorname{Int}\operatorname{Cl} f(U)$. There exists an open set V in Y such that $y \in V$ and $V \cap f(U)$ is nowhere dense [11, Theorem 2, p. 72]. Therefore $f^{-1}(V \cap f(U)) \supset f^{-1}(V) \cap U$ and $f^{-1}(V) \cap U$ is nowhere dense in X. This implies $\operatorname{Int}(f^{-1}(V) \cap U) = \operatorname{Int} f^{-1}(V) \cap U = \emptyset$ and hence $\operatorname{Cl}\operatorname{Int} f^{-1}(V) \cap U = \emptyset$. Since f is semi-continuous, $f^{-1}(V) \subset \operatorname{Cl}\operatorname{Int} f^{-1}(V)$ and consequently, $f^{-1}(V) \cap U = \emptyset$. So, $y \notin f(U)$ and $f(U) \subset \operatorname{Cl}\operatorname{Int}\operatorname{Cl} f(U)$.

(c)⇒(a): Obvious.

(c)⇒(d): Let V be open in Y and $x \notin \operatorname{Cl} f^{-1}(V)$. There exists an open set U in X with $x \in U$ and $U \cap f^{-1}(V) = \emptyset$. This gives $f(U) \cap V = \emptyset$ and hence $\operatorname{Cl}\operatorname{Int}\operatorname{Cl} f(U) \cap \operatorname{Int}\operatorname{Cl} V = \emptyset$. Since $f(U) \subset \operatorname{Cl}\operatorname{Int}\operatorname{Cl} f(U)$, $f(U) \cap \operatorname{Int}\operatorname{Cl} V = \emptyset$. Therefore $U \cap f^{-1}(\operatorname{Int}\operatorname{Cl} V) = \emptyset$. So, $x \notin f^{-1}(\operatorname{Int}\operatorname{Cl} V)$.

(d)⇒(c): Let U be an open set in X and $y \notin \operatorname{Cl}\operatorname{Int}\operatorname{Cl} f(U)$. There exists an open set V in Y with $y \in V$ and $V \cap \operatorname{Int}\operatorname{Cl} f(U) = \emptyset$. This implies $V \cap \operatorname{Cl}\operatorname{Int}\operatorname{Cl} f(U) = \emptyset$ and hence, $f^{-1}(V) \subset f^{-1}(Y - \operatorname{Cl}\operatorname{Int}\operatorname{Cl} f(U))$. Since $Y - \operatorname{Cl}\operatorname{Int}\operatorname{Cl} f(U) = \operatorname{Int}(Y - \operatorname{Int}\operatorname{Cl} f(U)) = \operatorname{Int}\operatorname{Cl}(Y - \operatorname{Cl} f(U))$, $f^{-1}(V) \subset \operatorname{Cl} f^{-1}(Y - \operatorname{Cl} f(U))$ by (d). But $\operatorname{Cl} f^{-1}(Y - \operatorname{Cl} f(U)) \subset \operatorname{Cl} f^{-1}(Y - f(U)) \subset \operatorname{Cl}(X - U) = X - U$ so that $f^{-1}(V) \cap U = \emptyset$. Therefore $V \cap f(U) = \emptyset$ and $y \notin f(U)$. □

We remark that in the proof of (c)⇔(d) the assumption that f is semi-continuous is not used.

Recall that a space X is Baire if no nonempty open set in X is meager. (A subset of X is meager if it is a countable union of nowhere dense sets). In [9, Theorem 4.7] (see also [5]) it is shown that if $f: X \to Y$ is a feebly continuous surjection satisfying condition (b) of Lemma 3.2 and X is a Baire space, then Y is a Baire space. This result is then used to obtain Z. Frolík's result [7] that if $f: X \to Y$ is a semi-continuous feebly open surjection and X is a Baire space, then Y is a Baire space. By Lemma 3.2 and Theorem 4.7 in [9] we have the following theorem.

Theorem 3.3. *If $f\colon X \to Y$ is a semi-continuous nearly feebly-open surjection and X is a Baire space, then Y is a Baire space.*

The following example shows that Theorem 3.3 is an actual improvement of Frolík's result.

Example 3.4. Let $(\mathbf{R}, \tau)$ be the space of the reals with the usual topology and let σ be the simple extension of τ by the set of irrationals $\mathbf{P}$, i.e., $\sigma = \{U \cup (V \cap \mathbf{P})\colon U, V \in \tau\}$. Note that both $(\mathbf{R}, \tau)$ and $(\mathbf{R}, \sigma)$ are Baire spaces. The identity function $i\colon (\mathbf{R}, \sigma) \to (\mathbf{R}, \tau)$ is continuous and nearly feebly open (moreover, nearly open) while it is not feebly open since $\operatorname{Int} i(\mathbf{P}) = \emptyset$.

4. Almost locally dense sets

We say that a set A in a space X is almost locally dense if $A \subset \operatorname{Cl}\operatorname{Int}\operatorname{Cl} A$. This class of generalized open sets was considered in [1] under the name of semi-preopen sets. It is not difficult to show that A is almost locally dense iff $A = B \cap C$, where B is semi-open and C is locally dense in X iff $A = \operatorname{Cl} U \cap D$ where U is open and D is dense in X iff A is dense in a semi-open set. By using the fact that $\operatorname{Cl}\operatorname{Int}\operatorname{Cl} A = \{x \in X : U \cap A$ is not nowhere dense in X for every open neighborhood U of $x\}$ [11, Theorem 2, p. 72], an almost locally dense set may be described as the set being not locally nowhere dense at each of its points. It is easy to show that an arbitrary union of almost locally dense sets is almost locally dense and that the intersection of an open set and an almost locally dense set is almost locally dense. A simple example of an almost locally dense set which is neither semi-open nor locally dense is the set $[0, 1) \cap \mathbf{Q}$ ($\mathbf{Q}$ denotes the set of rationals) on the real line with the usual topology.

We are now ready to justify the introduction of almost local denseness of a set in a space.

Theorem 4.1. *The following are equivalent for a set A in a space (X, τ).*

(a) *A is almost locally dense.*

(b) *The inclusion function $i\colon A \to X$ is nearly feebly open.*

(c) *For every nowhere dense set N in X, $A \cap N$ is nowhere dense in A.*

(d) *If $U \in \tau|A$, then U is almost locally dense in X.*

(e) $\operatorname{Int}\operatorname{Cl} V \cap A \subset \operatorname{Cl}_A(V \cap A)$ *for every open set V in X.*

Proof. The inclusion function $i\colon A \to X$ is continuous so that (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) follow by Lemma 3.2. Since the intersection of an almost locally dense set and an open set is almost locally dense (a) $\Rightarrow$ (d). Finally (d) $\Rightarrow$ (a), because $A \in \tau|A$. □

Since it is known that if $B \subset A \subset X$ and B is nowhere dense in A, then B is nowhere dense in X, we conclude that a set A in X is almost locally dense iff nowhere denseness (and category) of any subset $B \subset A$ is the same relative to X and to A.

Remark 4.2. One may characterize semi-open and locally dense sets in a space in a way similar to the characterization of almost locally dense sets in Theorem 4.1. Namely, a set A in a space X is semi-open (locally dense) iff the inclusion function $i: A \to X$ is semi-open (nearly-open). In the previous statement it is possible to replace "semi-open" with "feebly-open" since continuous (moreover, semi-continuous) feebly open functions are semi-open. It is stated in 6.3 of [16] that "nearly-open" can be replaced by "skeletal", where a function $f: X \to Y$ is defined to be skeletal if $\operatorname{Int} f^{-1}(\operatorname{Cl} V) \subset \operatorname{Cl} f^{-1}(V)$ for every open set V in Y. Since it is not difficult to show that a continuous function f is skeletal iff f satisfies one of the equivalent conditions in Lemma 3.2, 6.3 of [16] gives a characterization of almost locally dense sets.

Theorem 4.3. *Let A be an almost locally dense set in a space (X, τ). Then $(\tau|A)^\alpha = \tau^\alpha|A$.*

Proof. We need only to show that $\tau^\alpha|A \subset (\tau|A)^\alpha$ since $(\tau|A)^\alpha \subset \tau^\alpha|A$ for any $A \subset X$. Let $U \in \tau^\alpha|A$. Then $U = (V - N) \cap A$ where $V \in \tau$ and N is nowhere dense in X. Clearly, $U = (V \cap A) - (N \cap A)$. Since $V \cap A \in \tau|A$ and $N \cap A$ is nowhere dense in A by Theorem 4.1, $U \in (\tau|A)^\alpha$. □

The converse of Theorem 4.3 is not true as the following example shows. Let $(\mathbf{R}, \tau)$ be the reals with the usual topology and $\mathbf{N}$ the set of all natural numbers. Since $\mathbf{N}$ is closed discrete and nowhere dense in $(\mathbf{R}, \tau)$ both $(\tau|\mathbf{N})^\alpha$ and $\tau^\alpha|\mathbf{N}$ are discrete. Clearly, $\mathbf{N}$ is not almost locally dense in $(\mathbf{R}, \tau)$.

We now improve and generalize the result due to T. Noiri [20] that the image of a locally dense connected set under an α-continuous function is connected, as well as the result of I. L. Reilly and M. K. Vamanamurthy [23] that the same holds in the case of semi-open connected sets. We say that a property P of a space is semitopological if a space (X, τ) has P iff (X, τ^α) has P (See Remark 5.11). We also say that a property P is a continuous (α-continuous) invariant if it is preserved under continuous (α-continuous) surjections. The proof of the following lemma is left to the reader.

Lemma 4.4. *A property P of a space is an α-continuous invariant iff P is a continuous invariant and semitopological.*

Theorem 4.5. *Let P be an α-continuous invariant property, let $f\colon (X,\tau) \to (Y,\sigma)$ be an α-continuous function and let A be an almost locally dense set in X such that $(A,\tau|A)$ has P. Then $(f(A),\sigma|f(A))$ has P.*

Proof. Since $f\colon (X,\tau^\alpha) \to (Y,\sigma)$ is continuous, $f|A\colon (A,\tau^\alpha|A) \to (f(A),\sigma|f(A))$ is continuous. By Theorem 4.3, $\tau^\alpha|A = (\tau|A)^\alpha$ since A is almost locally dense. Therefore, $f|A\colon (A,(\tau|A)^\alpha) \to (f(A),\sigma|f(A))$ is continuous. This means $f|A\colon (A,\tau|A) \to (f(A),\sigma|f(A))$ is α-continuous. Since $(A,\tau|A)$ has P, P is an α-continuous invariant and $f|A$ is an α-continuous surjection, $(f(A),\sigma|f(A))$ has P. □

Since connectedness is a semitopological and a continuous invariant property, Theorem 4.5 gives an improvement of the above mentioned results. We remark that there are some other important properties for which Theorem 4.5 and Lemma 4.4 apply. For instance, pseudocompactness and feeble compactness are α-continuous invariants.

5. Between weak continuity and α-irresoluteness

It is well known that a function $f\colon X \to Y$ is semi-continuous iff for every $x \in X$ and any open neighborhoods U and V of x and $f(x)$, respectively, there exists a nonempty open set $G \subset U$ such that $f(G) \subset V$, iff $f(U \cap D)$ is dense in $f(U)$ for every open set U and every dense set D in X [17]. Along these lines one can obtain the following characterizations of α-continuity. The proof is left to the reader.

Theorem 5.1. *The following are equivalent for a function $f\colon X \to Y$.*

(a) *f is α-continuous.*

(b) *For every $x \in X$, every semi-open set U containing x and every open set V containing $f(x)$ there exists a nonempty open set $G \subset U$ such that $f(G) \subset V$.*

(c) *$f(\operatorname{Cl} U) \subset \operatorname{Cl} f(U \cap D)$ for every open set U and every dense set D in X.*

(d) *$f(\operatorname{Cl} U) \subset \operatorname{Cl} f(\operatorname{Cl} U \cap D)$ for every open set U and every dense set D in X.*

(e) *$f(\operatorname{Cl} A) \subset \operatorname{Cl} f(A)$ for every (almost) locally dense set A in X.*

We now turn our attention to sufficient conditions for α-irresoluteness of functions. Recall that a function $f\colon (X,\tau) \to (Y,\sigma)$ is α-irresolute [14] if $f\colon (X,\tau^\alpha) \to (Y,\sigma^\alpha)$ is continuous. In [20] ([15]) it is established that if $f\colon X \to Y$ is α-continuous and semi-open (nearly open) then f is α-irresolute. Our next result generalizes these results.

Theorem 5.2. *If $f\colon (X,\tau) \to (Y,\sigma)$ is α-continuous and nearly feebly open, then f is α-irresolute.*

P r o o f. Let $V \in \sigma^\alpha$. Then $V = W - N$, where $W \in \sigma$ and N is nowhere dense in Y. Since $f^{-1}(V) = f^{-1}(W) \cap (X - f^{-1}(N))$ and $f^{-1}(W) \in \tau^\alpha$, the result follows if we show that $X - f^{-1}(N) \in \tau^\alpha$. But this is clear, since $f^{-1}(N)$ is nowhere dense in X by Lemma 3.2. □

The following interesting class of functions was considered in [4] in connection with completely regular absolutes of arbitrary spaces. A function $f: X \to (Y, \sigma)$ is called η-continuous if (i) $f: X \to (Y, \sigma_S)$ is nearly continuous and (ii) $\operatorname{Int}\operatorname{Cl} f^{-1}(U) \cap \operatorname{Int}\operatorname{Cl} f^{-1}(V) = \operatorname{Int}\operatorname{Cl} f^{-1}(U \cap V)$ for any regular open sets U and V in Y. A stronger form of η-continuity is defined in [20]. A function $f: X \to Y$ is strongly η-continuous if (i) f is nearly continuous and (ii) $\operatorname{Int}\operatorname{Cl} f^{-1}(U) \cap \operatorname{Int}\operatorname{Cl} f^{-1}(V) = \operatorname{Int}\operatorname{Cl} f^{-1}(U \cap V)$ for any open sets U and V in Y.Recall also that a function $f: X \to Y$ is weakly continuous [12] (in the sense of Levine) if for every $x \in X$ and every open neighborhood V of $f(x)$ there exists an open neighborhood U of x such that $f(U) \subset \operatorname{Cl} V$. The following chain of implications holds: α-irresoluteness $\Rightarrow$ α-continuity $\Rightarrow$ strong η-continuity $\Rightarrow$ η-continuity $\Rightarrow$ θ-continuity $\Rightarrow$ weak-continuity. The reverse implications are not true in general. It is shown in [4] that weakly continuous closed irreducible surjections and weakly continuous open functions are η-continuous. Since both classes of η-continuous functions in these results are contained in the class of feebly open functions the following theorem generalizes both results.

Theorem 5.3. *If $f: X \to Y$ is weakly continuous and nearly feebly open, then f is strongly η-continuous.*

P r o o f. First we show that f is nearly continuous. Let V be an open set in Y. Since $f^{-1}(V) = f^{-1}(\operatorname{Cl} V) \cap (X - f^{-1}(\operatorname{Bd} V))$ and since the weak continuity of f gives that $f^{-1}(V) \subset \operatorname{Int} f^{-1}(\operatorname{Cl} V)$, we have

$$f^{-1}(V) = \operatorname{Int} f^{-1}(\operatorname{Cl} V) \cap (X - f^{-1}(\operatorname{Bd} V)).$$

By Lemma 3.1, $X - f^{-1}(\operatorname{Bd} V)$ is dense. Since the intersection of an open set and a dense set is locally dense, $f^{-1}(V)$ is locally dense and hence f is nearly continuous. Now, let U and V be open sets in Y. We use the fact that f is weakly continuous iff $\operatorname{Cl} f^{-1}(V) \subset f^{-1}(\operatorname{Cl} V)$. Therefore

$$\operatorname{Int}\operatorname{Cl} f^{-1}(U) \cap \operatorname{Int}\operatorname{Cl} f^{-1}(V) \subset \operatorname{Int} f^{-1}(\operatorname{Cl} U) \cap \operatorname{Int} f^{-1}(\operatorname{Cl} V) = \operatorname{Int} f^{-1}(\operatorname{Cl} U \cap \operatorname{Cl} V)$$
$$= \operatorname{Int} f^{-1}((U \cap V) \cup (U \cap \operatorname{Bd} V) \cup (V \cap \operatorname{Bd} U) \cup (\operatorname{Bd} U \cap \operatorname{Bd} V)).$$

Clearly $N = (U \cap \operatorname{Bd} V) \cup (V \cap \operatorname{Bd} U) \cup (\operatorname{Bd} U \cap \operatorname{Bd} V)$ is nowhere dense and by Lemma 3.1, $\operatorname{Int} f^{-1}(N) = \emptyset$. Since $\operatorname{Int} f^{-1}((U \cap V) \cup N) = \operatorname{Int}(f^{-1}(U \cap V) \cup f^{-1}(N)) \subset \operatorname{Int}(\operatorname{Cl} f^{-1}(U \cap V) \cup f^{-1}(N)) \subset \operatorname{Int}\operatorname{Cl} f^{-1}(U \cap V) \cup \operatorname{Int} f^{-1}(N) = \operatorname{Int}\operatorname{Cl} f^{-1}(U \cap V)$, we have that $\operatorname{Int}\operatorname{Cl} f^{-1}(U) \cap \operatorname{Int}\operatorname{Cl} f^{-1}(V) \subset \operatorname{Int}\operatorname{Cl} f^{-1}(U \cap V)$ and the result follows. □

To see that weakly continuous nearly feebly open functions are not necessarily α-continuous consider the identity function $i\colon (\mathbf{R},\tau) \to (\mathbf{R},\sigma)$ where $(\mathbf{R},\tau)$ and $(\mathbf{R},\sigma)$ are the spaces in the Example 3.4. It is not difficult to see that i is open and weakly continuous and hence strongly η-continuous. But i is not feebly continuous and hence is not α-continuous. In the light of this example, one may conjecture that weakly continuous open feebly continuous functions are α-continuous. This is not the case.

Example 5.4. Let (X,τ) be the set of real numbers with the usual topology, and let (Y,σ) be the Sierpiński space. So $Y = \{0,1\}$ and $\sigma = \{\emptyset,\{0\},Y\}$. Let now τ^* be the topology on X generated by τ and the set $\{0\}$. Define a function $f\colon (X,\tau^*) \to (Y,\sigma)$ by $f(x) = 0$ if $x \in \mathbf{Q}$ and $f(x) = 1$ if $x \notin \mathbf{Q}$, where $\mathbf{Q}$ is the set of rationals. It is not difficult to check that f is weakly continuous, open and feebly continuous. But f is not semi-continuous, and hence not α-continuous, since $\operatorname{Int} f^{-1}(\{0\}) = \{0\}$ is obviously not dense in $\mathbf{Q} = f^{-1}(\{0\})$.

However, in case of irreducible surjections, the situation is different. Recall that a surjection $f\colon X \to Y$ is irreducible if $f(F) \neq Y$ for every proper closed set F in X, or equivalently, for every nonempty open set U in X there exists $y \in Y$ such that $f^{-1}(y) \subset U$.

Theorem 5.5. *If $f\colon X \to Y$ is a weakly continuous feebly open feebly continuous irreducible surjection, then f is α-irresolute.*

Proof. As we have noted in Section 2 a function is α-continuous iff it is semi-continuous and nearly continuous. So if we show that f is semi-continuous the result will follow by Theorem 5.3 and Theorem 5.2. Let V be an open set in Y, let $x \in f^{-1}(V)$ and let G be an open neighborhood of x. We claim that $G \cap \operatorname{Int} f^{-1}(V) \neq \emptyset$. Since f is weakly continuous there exists an open neighborhood G' of x such that $f(G') \subset \operatorname{Cl} V$. Let $U = G \cap G'$. Evidently, $f(U) \subset \operatorname{Cl} V$. Since f is feebly open $\operatorname{Int} f(U) \neq \emptyset$ and hence $\operatorname{Int} f(U) \cap V \neq \emptyset$. Let $W = \operatorname{Int} f(U) \cap V$. The feeble continuity of f implies that $\operatorname{Int} f^{-1}(W) \neq \emptyset$. Since f is irreducible there exists $y \in Y$ such that $f^{-1}(y) \subset \operatorname{Int} f^{-1}(W)$. Evidently, $y \in W \subset f(U)$ and hence $f^{-1}(y) \cap U \neq \emptyset$. Therefore, $U \cap \operatorname{Int} f^{-1}(W) \neq \emptyset$ and consequently, $U \cap \operatorname{Int} f^{-1}(V) \neq \emptyset$. So $G \cap \operatorname{Int} f^{-1}(V) \neq \emptyset$ and $x \in \operatorname{Cl} \operatorname{Int} f^{-1}(V)$. This shows that f is semi-continuous and the result follows. □

We now derive a few consequences of Theorem 5.5. First, we need the concept of a quasiregular space. A space X is called quasiregular [21] if for every nonempty open set V in X there exists a nonempty open set U in X such that $\operatorname{Cl} U \subset V$. It is clear that weakly continuous surjections onto quasiregular spaces are feebly continuous. Combining this fact and Theorem 5.5 we obtain the following corollary.

Corollary 5.6. *Weakly continuous feebly open irreducible surjections onto quasiregular spaces are α-irresolute.*

Recall now that with every Hausdorff space X we associate the space EX, called the Iliadis absolute, which is unique (up to homeomorphism) with respect to having these properties: EX is Tichonov extremally disconnected and zero-dimensional and there exists a perfect (i.e., closed with compact fibers) θ-continuous irreducible surjection $k_X : EX \to X$ (see [21] for definitions and properties). The surjection k_X is not necessarily continuous. In fact, k_X is continuous iff X is regular [21]. Since k_X is feebly open (being closed and irreducible), it follows from Corollary 5.6 that k_X is α-irresolute if X is quasiregular. Our next result implies that the converse is also true.

Lemma 5.7. *Quasiregularity is preserved under feebly open feebly continuous closed surjections.*

Proof. Let $f : X \to Y$ be a feebly open feebly continuous closed surjection, X a quasiregular space and V a nonempty open set in Y. Since f is feebly continuous, $U = \operatorname{Int} f^{-1}(V)$ is nonempty. There exists a nonempty open set W in X such that $\operatorname{Cl} W \subset U$ since X is quasiregular. The feeble openness and closedness of f give $\emptyset \neq \operatorname{Int} f(W) \subset \operatorname{Cl} \operatorname{Int} f(W) \subset \operatorname{Cl} f(W) \subset f(\operatorname{Cl} W) \subset f(U) \subset V$. Therefore Y is quasiregular. □

We can now state the promised result.

Theorem 5.8. *Let X be a Hausdorff space. Then X is quasiregular iff $k_X : EX \to X$ is α-irresolute (feebly continuous).*

Theorem 5.9. *Let X be a Hausdorff quasiregular space. Then X is a Baire space iff EX is a Baire space.*

Proof. The sufficiency follows by Theorem 5.8 and Theorem 3.3. To establish necessity suppose that EX is not a Baire space. Then there exists a nonempty open meager set U in EX. Since nowhere dense sets are preserved under closed θ-continuous irreducible surjections [Theorem 6.5 (d), 21], $k_X(U)$ is meager in X. The feeble openness of k_X gives $\operatorname{Int} k_X(U) \neq \emptyset$ which contradicts the assumption that X is a Baire space, and the result follows. □

Another consequence of Theorem 5.5 involves the concept of a semi-homeomorphism of spaces. A bijection $f : X \to Y$ is a semi-homeomorphism [3] if f and f^{-1} preserve semi-open sets and then X and Y are said to be semi-homeomorphic spaces.

It was shown in [10] that a bijection $f: X \to Y$ is a semi-homeomorphism iff f is a θ-homeomorphism (i.e., f and f^{-1} are θ-continuous) and f is a feeble homeomorphism (i.e., f and f^{-1} are feebly continuous). Since bijections are irreducible, the previous result and Theorem 5.5 give at once the following result from [8].

Theorem 5.10. *Spaces (X,τ) and (Y,σ) are semi-homeomorphic iff (X,τ^α) and (Y,σ^α) are homeomorphic.*

Remark 5.11. In Section 4 we have defined a property to be semitopological if it is shared by a space (X,τ) and the space (X,τ^α). Originally, semitopological properties were defined in [3] as properties preserved under semi-homeomorphisms. Theorem 5.10 justifies our definition. For a list of semitopological properties see [10].

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