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ON THE EMBEDDING OF SEMIGROUPS INTO NILPOTENT GROUPS

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To provide some background, Neumann—Taylor [2] characterized a subsemigroup of a nilpotent group in terms of a certain non-tautological semigroup law. Let S be a semigroup satisfying the right and left cancellation laws. A sequence $\varphi_1, \varphi_2, \ldots$ of words in variables x, y, $z_1, z_2, \ldots$, or simply, x, y, z, where z stands for the sequence of variables z_1, z_2, ... is defined in the following manner: $\varphi_1(x, y; z) = xy$ and define inductively $\varphi_{i+1}(x, y; z) = \varphi_i(x, y; z)z_i\Phi_i(y, x; z)$. The L_n law is defined as follows, $L_n : \varphi_n(x, y; z) = \varphi_n(y, x; z)$. Thus for example, L_1 defines the commutative law and L_2 is the law $xyz_1yx = yxz_1xy$. Neumann—Taylor 2] proved the following

(*) **Theorem.** *The semigroup S can be embedded in a nilpotent group of class n if, and only if, it is cancellative and satisfies the Law L_n.*

In this paper, another characterization of subsemigroups of nilpotent groups is given which relies on a simple congruence. This characterization exhibits the similarity between subsemigroups of nilpotent groups and their group counterparts in that it is reminiscent of the way the lower central series terminates in the identity element in a nilpotent group. A congruence, ζ, will be defined on a cancellative semigroup S with the corresponding factor semigroup denoted by S/ζ. Let $T_0 = S$, $T_1 = T_0/\zeta$ and, inductively, $T_i = T_{i-1}/\zeta$. The purpose of this paper is to prove the following

Theorem. *A cancellative semigroup S can be embedded in a nilpotent group of class n if and only if T_n consists of a single element.*

Proof. (1) For $a, b \in S$, a is said to be related to b or, $a\zeta b$, if $asb = bsa$ for every s belonging to S.

If $a\zeta b$, then a and b commute. This is true since $a(ba)b = bbaa = abba$ which implies the conclusion.

(2) $a\zeta b$ if and only if $ab^{-1} \in Z(gp\{S\})$, the center of the group generated by S.

Since $a\zeta b$, $b^{-1}as = sab^{-1}$ for every s belonging to S. By (1), a and b commute, so that $ab^{-1}s = sab^{-1}$ and $ab^{-1} \in Z(gp\{S\})$. Conversely, if $ab^{-1} \in Z(gp\{S\})$, then a and b commute and $ab^{-1}s = sab^{-1}$. It follows that $b^{-1}as = sab^{-1}$ is true for every s in S and the conclusion follows.

(3) ζ is a congruence on S.

It will first be demonstrated that ζ is an equivalence relation. Clearly, $a\zeta a$; and $a\zeta b$ implies $b\zeta a$ by definition of ζ. Suppose that $a\zeta b$ and $b\zeta c$ for some a, b, c belonging to S. Then by (2) ab^{-1} and bc^{-1} belong to $Z(gp\{S\})$ and $ab^{-1} . bc^{-1} = = ac^{-1} \in Z(gp\{S\})$. Again, by (2), $a\zeta c$.

ζ is also two-sidedly stable, that is, if $a\zeta b$, then $ta\zeta tb$ and $at\zeta bt$ for any t belonging to S. Since $a\zeta b$ implies $ab^{-1} \in Z(gp\{S\})$, $ta(tb)^{-1} = t . ab^{-1} . t^{-1} = ab^{-1}$ and $at(bt)^{-1} = ab^{-1}$. By (2), $ta\zeta tb$ and $at\zeta bt$ are valid for any $t \in S$.

(4) Let S/ζ denote the set of equivalence classes of S with respect to the relation ζ and S_a denote the ζ—class of S containing the element a as representative.

By (3), S/ζ is a semigroup and $S_aS_b = S_{ab}$ for $a, b \in S$.

(5) A cancellative semigroup S is embeddable in a nilpotent group of class n if and only if S/ζ is embeddable in a nilpotent group of class $n - 1$.

Suppose the factor semigroup $S/\zeta = \{S_a | a \in S\}$ is embeddable in a nilpotent group of class $n - 1$. Then, $\varphi_{n-1}(S_a, S_b; S_c) = \varphi_{n-1}(S_b, S_a; S_c)$ by the hypothesis and Theorem (*). Hence, by (4), $S_{\Phi_{n-1}(a,b;c)} = S_{\Phi_{n-1}(b,a;c)}$. Thus, $\varphi_{n-1}(a, b; c)\zeta\varphi_{n-1}(b, a; c)$ so that $\varphi_{n-1}(a, b; c)s\varphi_{n-1}(b, a; c) = \varphi_{n-1}(b, a; c)s\varphi_{n-1}$ $(a, b; c)$ for all $a, b, c_1, \ldots, s \in S$. But, by (*), this says that S is embeddable in a nilpotent group of class n.

On the other hand, suppose S may be embedded in a group which is nilpotent of class n. Then, $\varphi_{n-1}(a, b; c)s\varphi_{n-1}(b, a; c) = \varphi_{n-1}(b, a; c)s\varphi_{n-1}(a, b; c)$ for all $a, b, c_1, \ldots, s \in S$. This states that $\varphi_{n-1}(a, b; c)\zeta\varphi_{n-1}(b, a; c)$ so that $S_{\varphi_{n-1}(a,b;c)} = S_{\Phi_{n-1}(b,a;c)}$. From (4), $\varphi_{n-1}(S_a, S_b; S_c) = \varphi_{n-1}(S_b, S_a; S_c)$ for all $S_a, S_b, S_{c_1}, \ldots, S_{c_{n-2}} \in S/\zeta$. Hence, S/ζ satisfies law L_{n-1} and the conclusion follows by (*).

Successive applications of ζ will now be considered. Let $T_0 = S, T_1 = T_0/\zeta$ and, inductively, $T_i = T_{i-1}/\zeta$.

(6) A cancellative semigroup S can be embedded in a nilpotent group of class n if and only if T_n consists of a single element.

Suppose T_n consists of a single element. Then, all the elements of T_{n-1} must be related under ζ and, by (1), commute. T_{n-1} is embeddable in a class 1 group and, according to (5), T_{n-2} may be embedded in a class 2 nilpotent group. Proceeding by induction it is finally concluded that $S = T_{n-n} = T_0$ is embeddable in a nilpotent group of class n.

Conversely, suppose S is embeddable in a class n group. Then by (5), T_1 may be embedded in a group which is nilpotent of class $n - 1$. Proceeding by finite induction it is found that T_{n-1} is commutative and, thus, T_n consists of one element.

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