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A THEOREM ON THE EXTENSION OF MEASURES IN UNIFORM SPACES

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There are well-known methods of extension of measures from a ring R to the smallest σ-ring S containing R. But there are some unsolved problems in the case when R is not a ring.

We use the terminology of measure theory according to [1] and the terminology of general topology according to [2]. Recall only that a set E is Baire iff it belongs to the smallest σ-ring over the family of all compact G_δ sets.

Theorem. *Let X be a uniform space, $\mathscr{B}$ a base for the uniformity of X, K a family of subsets of X containing $\emptyset$. Let ν be a set-function defined on K, $\nu(\emptyset) = 0$, non-negative, σ-subadditive and fulfilling the following condition:*

$$(1) \qquad \nu(E) = \inf \{\sum_{i=1}^{\infty} \nu(E_i) : E \subset \bigcup_{i=1}^{\infty} E_i, E_i \in K, E_i \times E_i \subset V\},$$

for any $V \in \mathscr{B}$ and $E \in K$.

Then the function

$$(2) \qquad \nu^*(F) = \inf \{\sum_{i=1}^{\infty} \nu(E_i) : F \subset \bigcup_{i=1}^{\infty} E_i, E_i \in K\}$$

is a measure on the family of all Baire subsets of X. $\nu^*(E) = \nu(E)$ for any Baire set of K.

Proof. Clearly ν^* is an outer measure and $\nu^*(E) = \nu(E)$ for any Baire set E of K. We have to prove that all compact G_δ sets (and hence all Baire sets too) are measurable. According to a theorem of [3] (Theorem 5; also [4], Theorem 3) it suffices to prove that ν^* satisfies the following property:

$$(3) \qquad \nu^*(A \cup B) = \nu^*(A) + \nu^*(B), \text{ whenever } A \times B \subset X \times X - V$$

for some $V \in \mathscr{B}$.

Let $A, B \subset X$, $A \times B \subset X \times X - V$, $V \in \mathscr{B}$. (3) holds if either $\nu^*(A) = \infty$ or $\nu^*(B) = \infty$. Let $\nu^*(A \cup B) < \infty$. Then by definition (2) there exists for any $\delta > 0$ a sequence $\{E_i\}$ of sets of K such that

$$(4) \qquad A \cup B \subset \bigcup_{i=1}^{\infty} E_i, \; \nu^*(A \cup B) + \delta > \sum_{i=1}^{\infty} \nu(E_i).$$

By the condition (1) it follows that for any positive integer i there exists a sequence $\{E_i^k\}_{k=1}^{\infty}$ of sets of K such that

$$(5) \qquad E_i \subset \bigcup E_i^k, \; E_i^k \times E_i^k \subset V, \; \nu(E_i) > \sum_{k=1}^{\infty} \nu(E_i^k) - \frac{\delta}{2^i}.$$

Let $E \subset X$ be any set for which $E \times E \subset V$. Then either $E \cap A = \emptyset$ or $E \cap B = \emptyset$. Really, if $E \cap A \neq \emptyset$ and $E \cap B \neq \emptyset$, then $(E \times E) \cap (A \times B) \neq \emptyset$, which is a contradiction, since $A \times B \subset X \times X - V$, but $E \times E \subset V$. Hence for any i and k either $E_i^k \cap A = \emptyset$ or $E_i^k \cap B = \emptyset$. Therefore

$$(6) \qquad \sum_{k,i\ 1}^{\infty} \nu(E_i^k) \geqq \sum\{\nu(E_i^k) : E_i^k \cap A \neq \emptyset\} + \sum\{\nu(E_i^k) : E_i^k \cap B \neq \emptyset\}.$$

By (4) and (5) we have $\cup \{E_i^k : E_i^k \cap A \neq \emptyset\} \supset A$, $\cup \{E_i^k : E_i^k \cap B \neq \emptyset\} \supset B$, hence

$$(7) \qquad \sum\{\nu(E_i^k) : E_i^k \cap A \neq \emptyset\} \geqq \nu^*(A), \; \sum\{\nu(E_i^k) : E_i^k \cap B \neq \emptyset\} \geqq \nu^*(B).$$

From (4)—(7) it follows that

$$\nu^*(A \cup B) + \delta > \nu^*(A) + \nu^*(B) - \delta$$

for any $\delta > 0$. Now (3) follows from the subadditivity of ν^* and the preceding relation.

Corollary 1. *Let X be a metric space, K be the system of all closed spheres in X, ν be a set-function on K, non-negative, $\nu(\emptyset) = 0$, σ-subadditive and satisfying the following condition:*

$$(1') \qquad \nu(E) = \inf \{\sum_{i=1}^{\infty} \nu(E_i) : E \subset \bigcup_{i=1}^{\infty} E_i, \; E_i \in K, \; \operatorname{diam} E_i < r\}$$

for any $r > 0$ and $E \in K$.
Then the function ν^* defined by (2) *is a measure on the σ-ring S of all Baire subsets of X.* (1)

Corollary 2. *Let X be a topological group, K be a system of subsets of X, $\emptyset \in K$, ν be a set function defined on K, non negative, $\nu(\emptyset) = 0$, σ-subadditive and fulfilling the following condition:*

(1) Cf. [5], Theorem 1.

$$(1'') \quad \nu(E) = \inf \{\sum_{i=1}^{\infty} \nu(E_i) : E \subset \bigcup_{i=1}^{\infty} E_i, E_i \in K, x_i E_i \subset U \textit{ for some } x_i \in X\},$$

for any $E \in K$ and any neighbourhood U of zero element.

Then the function ν^* defined by (2) *is a measure on the system of all Baire sets.*

Example. Let X be a σ-compact and locally compact topological group, ν be the Haar measure, K be any regular Vitali covering of X (see [6]) by Baire sets, satisfying the following condition: If $\nu(E) = 0$, then for any $\delta > 0$ and any neighbourhood V of zero element there are $E_i \in K$ $(i = 1, 2, \ldots)$ such that $\sum_{i=1}^{\infty} \nu(E_i) < \delta$ and $x_i E_i \subset V$ for some x_i. In this example all assumptions of Corollary 2 are satisfied.

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