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ON COMPLETE IDEALS IN SEMIGROUPS

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1.

R. Croisot introduced in [1] the following condition: *An element a of a semigroup S satisfies the Condition* (m, n) *if there exists an element* $x \in S$ *such that*

$$a = a^m x a^n .$$

Here m, n are non-negative integers, and a^0 means the void symbol. The set of all elements, satisfying the Condition (m, n) is called a class of regularity and will be denoted by $\mathscr{R}_s(m, n)$. (See [2]). By means of this notion some properties of semigroups have been studied. In this paper we show how these classes of regularity are connected with so-called complete ideals. For relations, which hold between the classes of regularity see [2] (p. p. 111—112). If all elements of a semigroup S satisfy the Condition (m, n) we shall write $S = \mathscr{R}_s(m, n)$.

2.

Definition 1. *We shall say that a left (right, two-sided) ideal* $L(R, M)$ *of a semigroup* S *is complete if* $SL = L$ $(RS = R,\ SM = MS = M)$.

In the following we shall treat only left complete ideals. The case of right complete ideals is analogous.

Remark 1. Evidently: A left ideal L of a semigroup S is complete if for any $a \in L$ there exist $x \in S$, $b \in L$ such that

$$xb = a. \tag{1}$$

Theorem 1. *The set union of two complete left ideals of a semigroup* S *is a complete left ideal of* S.

Proof. Let L_1, L_2 be two complete left ideals of S. Then $SL_1 = L_1$, $SL_2 = L_2$. Hence $S(L_1 \cup L_2) = SL_1 \cup SL_2 = L_1 \cup L_2$, which proves our assertion.

The question arises, whether the intersection of two left complete ideals is a left complete ideal. The next example gives a negative answer.

Example 1. Let $S = \{a, b, c, d\}$ be a semigroup with the multiplication table.

	a	b	c	d
a	a	a	a	a
b	a	a	b	b
c	a	a	c	d
d	a	a	c	d

$L_1 \quad \{a, b, c\}$, $L_2 = \{a, b, d\}$ are complete left ideals of S, but $L_1 \cap L_2 = L_3 = \{a, b\}$ is not a complete left ideal of S.

Example 2. A left ideal l of a semigroup S is called minimal if there exists no left ideal of S properly contained in l. Evidently, every minimal left ideal of a semigroup S is a complete left ideal of S.

Theorem 2. *Every left ideal of a semigroup S is a complete left ideal of S if and only if $S = \mathscr{R}_s(0, 1)$.*(1)

Proof. (a) Let $L = \bigcup_{a \in L} a$ be a left ideal of S. Then $SL \supset \{\bigcup_{a \in L} x_a\} . \{\bigcup_{a \in L} a\} \supset \bigcup_{a \in L} x_a . a = \bigcup_{a \in L} a = L$. On the other hand since L is a left ideal, $SL \subset L$. Hence $SL = L$.

(b) Let every left ideal of S be complete. Let $a \in S$ be any element of S. The left ideal $a \cup Sa$ satisfies $S(a \cup Sa) = a \cup Sa$, i.e. $Sa \cup S^2a = a \cup Sa$, hence $Sa = a \cup Sa$. Therefore $a \in Sa$, which proves that $S = \mathscr{R}_s (0, 1)$.

Remark 2. Clearly the following assertions hold.

(a) If S contains a left unit, then every left ideal is complete.

(b) $S = \mathscr{R}_s(0, 1) = \mathscr{R}_s(1, 0)$, if and only if every left, right and two-sided ideal of S is complete.

(c) If $S = \mathscr{R}_s(1, 1)$, then every ideal of S is complete.

(d) If all left ideals of S are complete, then $S^2 = S$.

The next example of a semigroup shows that the converse of the assertion (d) need not hold.

Example 3. Let S be an additive semigroup of positive numbers. Then $S^2 - S$. Let $L = \langle a, \infty)$ with $a > 0$. Then $SL = (a, \infty) \subset \langle a, \infty)$, so that L is not complete.

Remark 3. If not every left ideal of a semigroup S is complete, then essentialy less can be said about this semigroup. This statement holds:

If $L \subset \mathscr{R}_s(0, 1)$, where L is a left ideal of a semigroup S, then L is a complete left ideal of S.

(1) Similar questions are studied in [4].

Proof. The statement follows from the assumption and from part (a) of the proof of Theorem 2.

The semigroup in example 3 shows that the converse is not true. It is sufficient to take $L = (a, \infty)$. It can be easily shown that L is complete but $L \subset \mathscr{R}_s(0, 1)$ does not hold.

3.

Let $\{S_i\}$, $i \in I$ be an arbitrary system of semigroups. Denote by S the set of all functions ξ, defined on I such that $\xi(i) \in S_i$. Introduce in S a multiplication in this way: If $\alpha, \beta \in S$ are arbitrary elements of S, then the product $\gamma = \alpha \,.\, \beta$ is given by $\gamma(i) = \alpha(i) \,.\, \beta(i)$ (for every $i \in I$). The set S with this multiplication is a semigroup, which is called a direct product of semigroups $\{S_i\}$, $i \in I$, and is denoted by $S = \prod_{i \in I} S$.

If L_i is a left ideal of the semigroup S_i, $i \in I$, then $L = \prod_{i \in I} L_i$ is a left ideal of the semigroup $S = \prod_{i \in I} S_i$. (See [3]).

Let us put the question, whether the completeness of left ideals L_i in S_i, $i \in I$, implies the completeness of a left ideal $L = \prod_{i \in I} L_i$ in $S = \prod_{i \in I} S_i$.

Theorem 3. *Let L_i be for every $i \in I$ a complete left ideal of the semigroup S_i. Then $L = \prod_{i \in I} L_i$ is a complete left ideal of $S = \prod_{i \in I} S_i$.*

Proof. Let L_i be a complete left ideal of a semigroup S_i, hence $S_i L_i = L_i$. We have to prove that for any $\mu \in L$, there exist $\nu \in L$ and $\alpha \in S$ such that

$$\alpha \,.\, \nu = \mu \,.$$

Since L_i is a complete left ideal of S_i, there exist for every $\mu(i) = a_i \in L_i$ two elements $b_i \in L_i$ and $x_i \in S_i$ such that

$$x_i b_i = a_i \,.$$

The functions ν, α defined by $\nu(i) = b_i$, $\alpha(i) = x_i$ satisfy

$$\alpha \,.\, \nu = \mu \,.$$

This proves our statement.

Let $N \subseteq S = \prod_{\in Ii} S_i$. The set of all elements $x_i \in S_i$ for which there exists at least one element $\xi \in N$ such that $\xi(i) = x_i$ will be denoted by $P_i(N)$ and called the projection of the set N into the semigroup S_i.

Theorem 4. *Let L be a complete left ideal of a semigroup $S = \prod_{i \in I} S_i$. Then*

(a) $P_i(L)$ *is a complete left ideal of* S_i.
(b) $\prod_{i \in I} P_i(L)$ *is a complete left ideal of* S.

Proof. (a) Let L be a complete left ideal of $S = \prod_{i \in I} S_i$. The fact that the $P_i(L)$ is a left ideal of S_i is known from [3]. It is only necessary to prove that it is complete. Let $a_i \in P_i(L)$. To prove that $P_i(L)$ is a complete left ideal, it is sufficient to show that there exist $b_i \in P_i(L)$, and $x_i \in S_i$ such that

$$x_i b_i = a_i.$$

Since $a_i \in P_i(L)$, it follows that there exists an element $\mu \in L$ such that $\mu(i) = a_i$. Since L is a complete left ideal of $S = \prod_{i \in I} S_i$, there exist elements $\nu \in L$, and $\alpha \in L$, and $\alpha \in S$ such that

$$\alpha \,.\, \nu = \mu.$$

This means that for every $i \in I$, we have $\alpha(i) \,.\, \nu(i) = \mu(i)$, where $\mu(i) = a_i$, $\nu(i) = b_i \in P_i(L)$ and $\alpha(i) = x_i \in S_i$. Therefore, we have

$$x_i b_i = a_i.$$

This proves (a).

(b) The statement (b) follows from (a) and Theorem 3.

Theorem 5. *A semigroup* $S = \prod_{i \in I} S_i$ *satisfies the Condition* (m, n) *if and only if each of the semigroups* S_i *satisfies this Condition.*

Proof. (a) Let us assume that every semigroup S_i satisfies Condition (m, n).

Let $\alpha \in S$ be an arbitrary element. Then $\alpha(i) = a_i \in S_i$ for every $i \in I$. Since S_i satisfies Condition (m, n), there exists an $x_i \in S_i$ such that

$$(*) \qquad a_i = a_i^m x_i a_i^n.$$

Define $\eta \in S$ by the requirement that $\eta(i) = x_i$, for every $i \in I$. The relation (*) can be writeen in the form $\alpha(i) = [\alpha(i)]^m \,.\, \eta(i) \,.\, [\alpha(i)]^n$, for every $i \in I$. This means

$$\alpha = \alpha^m \,.\, \eta \,.\, \alpha^n.$$

But the last relation says that $S = \prod_{i \in I} S_i$ satisfies Condition (m, n).

(b) Let $S = \prod_{i \in I} S_i$ satisfy Condition (m, n). Let $a_i \in S_i$ be an arbitrary element. Then there exists at least one element $\alpha \in S$ such that $\alpha(i) = a_i$. Since S satisfies Condition (m, n), there exists an element $\eta \in S$ such that

$$\alpha = \alpha^m \,.\, \eta \,.\, \alpha^n.$$

Hence for our i

$$a_i = a_i^m x_i a_i^n.$$

This means that S_i satisfies Condition (m, n).

Theorems 2 and 5 imply:

Corollary 1. *Every left ideal of the semigroup* $S = \prod_{i \in I} S_i$ *is complete if and only if every left ideal of the semigroup* $S_i (i \in I)$ *is complete.*

Corollary 2. *The following statements are equivalent:*

(a) *Each of the semigroups* S_i $(i = I)$ *satisfies Condition* (0,1).

(b) *The semigroup* $S = \prod_{i \in I} S_i$ *satisfies Condition* (0,1).

(c) *Every left ideal of* S_i $(i \in I)$ *is complete.*

(d) *Every left ideal of* $S = \prod_{i \in I} S_i$ *is complete.*

Proof. (a) $\Rightarrow$ (b) according to Theorem 5. (b) $\Rightarrow$ (c) according to Theorem 5 and Theorem 2. (c) $\Rightarrow$ (d) according to Corollary 1. (d) $\Rightarrow$ (a) according to Corollary 1 and Theorem 2.

A left ideal L of a semigroup S is called semiprime if for every element $a \in S$ and an arbitrary integer n the relation $a^n \in L$ implies $a \in L$.

In [2] (p. 241) it is proved that every left ideal of a semigroup S is a semiprime ideal if and only if S satisfies Condition (0,2).

Theorem 6. *Let* L_i *be a left semiprime ideal of* S_i *for every* $i \in I$. *Then* $L \quad \prod_{i \in I} L_i$ *is a left semiprime ideal of* $S = \prod_{i \in I} S_i$.

Proof. Let $\alpha \in S = \prod_{i \in I} S_i$ be an arbitrary element and let $\alpha^n \in L = \prod_{i \in I} L_i$. Then $[\alpha(i)]^n \in L_i$ for every $i \in I$. Since L_i is a semiprime ideal of S_i, we have $\alpha(i) \in L_i$ for every $i \in I$. Hence $\alpha \in L = \prod_{i \in I} L_i$.

Corollary. *Let every left ideal of a semigroup* S_i *be a semiprime ideal of* S_i *for every* $i \in I$. *Then:*

(a) *Every left ideal of* $S = \prod_{i \in I} S_i$ *is a semiprime ideal of* S.

(b) *Every left ideal of* $S = \prod_{i \in I} S_i$ *is a complete left ideal of* S.

Proof. The statement (a) follows from Theorem 5 and the Remark preceding Theorem 6. The statement (b) follows from the relation: $\mathcal{R}_s(m_1, n_1) \subseteq \subseteq \mathcal{R}_s(m_2, n_2)$ if $m_1 \geqslant m_2, n_1 \geqslant n_2$ (see pp. 111—112 in [2]) and from Theorem 5.

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