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ON THE NILPOTENCY IN SEMIGROUPS

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This paper is an extension of the results of papers [3] and [5]. The first three theorems of this paper are extensions of the first three theorems of the paper [3] and the fourth theorem of this paper is an extension of the theorem of paper [5]. We introduce three kinds of nilpotency and consider instead of ideals (which are as a rule used to define various kinds of radicals) more generally subsets or subsemigroups of a given semigroup S.

Throughout the paper the empty subset $\emptyset$ is considered as a subsemigroup of the semigroup S. By an ideal we mean a two-sided ideal. If we consider a left ideal or a right ideal we shall indicate it explicitely. The $\cap$-semilattice of all left (right) [two-sided] ideals of the semigroup S contains always the empty set $\emptyset$.

We now introduce three kinds of nilpotency and shall study the relations among them.

Definition 1. *Let S be a semigroup and M a subset of S.*
a) *Let x be such an element of S for which there exists a positive integer $N(x)$ such that for each positive integer $n \geqslant N(x)$ (for almost all n) $x^n \in M$ holds. Then x will be called strongly nilpotent with respect to M.*
b) *Let x be an element of S such that for infinitely many positive integers n, $x^n \in M$ holds. Then x will be called weakly nilpotent with respect to M.*
c) *Let x be an element of S such that at least one power x^n is in M. Then x will be called almost nilpotent with respect to M.*

The set of all strongly nilpotent elements with respect to M will be denoted by $N_1(M)$, the set of all weakly nilpotent elements with respect to M will be denoted by $N_2(M)$ and the set of all almost nilpotent elements with respect to M will be denoted by $N_3(M)$.

Remark 1. From Definition 1 it is clear that every strongly nilpotent element with respect to M is weakly nilpotent with respect to M and every weakly nilpotent element with respect to M is almost nilpotent with respect to M. Therefore we have $N_1(M) \subseteq N_2(M) \subseteq N_3(M)$.

The following example shows that $N_1(M) \neq N_2(M)$ and $N_2(M) \neq N_3(M)$ can take place even if S is a commutative semigroup.

Example 1. Let $S = \langle 0, 1 \rangle$ with the ordinary multiplication as an operation in S. The element $x = \frac{1}{2}$ is almost nilpotent with respect to $M = \{\frac{1}{4}\}$ but it is not weakly nilpotent with respect to M.

In the same semigroup let us take $M = \left\{\frac{1}{2^{2k}} \mid k = 1, 2, \ldots\right\}$. M is a subsemigroup of S, the element $x = \frac{1}{2}$ is weakly nilpotent with respect to M, but it is not strongly nilpotent with respect to M.

The following lemmas are evident.

Lemma 1. *Let S be a semigroup and M a subsemigroup of S. Then every almost nilpotent element with respect to M is weakly nilpotent with respect to M, i. e. if M is a subsemigroup then $N_3(M) = N_2(M)$.*

Lemma 2. *Let S be a semigroup and M a left (right) [two-sided] ideal of S. Then every weakly nilpotent element with respect to M is also strongly nilpotent with respect to M. Thus we have $N_3(M) = N_2(M) = N_1(M)$.*

Lemma 3. *Let S be a semigroup and let M_1 and M_2 be subsets of S. Then $N_1(M_1 \cap M_2) = N_1(M_1) \cap N_1(M_2)$.*

Proof. a) If $A \subseteq B$, then $N_1(A) \subseteq N_1(B)$. Hence $N_1(M_1 \cap M_2) \subseteq N_1(M_1) \cap \cap N_1(M_2)$.

b) Let $x \in N_1(M_1) \cap N_1(M_2)$. Then almost all powers of the element x are in M_1 and almost all powers of the element x are in M_2, i. e. almost all powers of the element x are in $M_1 \cap M_2$. Therefore $x \in N_1(M_1 \cap M_2)$ and $N_1(M_1) \cap \cap N_1(M_2) \subseteq N_1(M_1 \cap M_2)$. Together with a) we have $N_1(M_1 \cap M_2) = = N_1(M_1) \cap N_1(M_2)$.

Lemma 4. *Let S be a semigroup and let M_1 and M_2 be subsemigroups of S. Then $N_3(M_1 \cap M_2) = N_3(M_1) \cap N_3(M_2)$ [hence $N_2(M_1 \cap M_2) = N_2(M_1) \cap \cap N_2(M_2)$].*

Proof. a) The relation $N_3(M_1 \cap M_2) \subseteq N_3(M_1) \cap N_3(M_2)$ can be proved in the same manner as in lemma 3.

b) If $x \in N_3(M_1) \cap N_3(M_2)$, then at least one power x^{n_1} is in M_1 and at least one power x^{n_2} is in M_2. But then $x^{n_1 n_2}$ is in $M_1 \cap M_2$ and $x \in N_3(M_1 \cap M_2)$. Thus $N_3(M_1) \cap N_3(M_2) \subseteq N_3(M_1 \cap M_2)$ and this together with a) implies $N_3(M_1 \cap M_2) = N_3(M_1) \cap N_3(M_2)$.

If M_1, M_2 and $M_1 \cap M_2$ are merely subsets of S (not subsemigroups), then neither $N_2(M_1 \cap M_2) = N_2(M_1) \cap N_2(M_2)$ nor $N_3(M_1 \cap M_2) = N_3(M_1) \cap \cap N_3(M_2)$ necessarily holds. This will be shown on the following examples.

Example 2. Let $S = \left\{\frac{1}{2^k} \mid k = 0, 1, 2, \ldots\right\}$ with the ordinary multiplication as operation. Let $M_1 = \{1, \frac{1}{2}\}$ and $M_2 = \{1, \frac{1}{4}\}$. Then $N_3(M_1) = \{1, \frac{1}{2}\}$, $N_3(M_2) = = \{1, \frac{1}{2}, \frac{1}{4}\}$ and $N_3(M_1) \cap N_3(M_2) = \{1, \frac{1}{2}\} \neq \{1\} = M_1 \cap M_2 = = N_3(M_1 \cap M_2)$.

Example 3. Let S be the semigroup from Example 2. Let $M_1 = \{1\} \cup \cup \left\{\frac{1}{2^{2k}} \mid k = 1, 2, \ldots\right\}$ and $M_2 = \{1\} \cup \left\{\frac{1}{2^{k-1}} \mid k = 1, 2, \ldots\right\}$. Then $N_2(M_1) = = S$, $N_2(M_2) = M_2$, but $N_2(M_1) \cap N_2(M_2) = M_2 = \{1\} \cup \left\{\frac{1}{2^{2k-1}} \mid k = 1, 2, \ldots\right\}$, $M_1 \cap M_2 = \{1\}$ and $N_2(M_1 \cap M_2) = \{1\} \neq N_2(M_1) \cap N_2(M_2)$.

Lemma 5. *Let S be a semigroup and $M_\varkappa$, $\varkappa \in K$, subsets of S. Then* $\bigcup_{\varkappa \in K} N_3(M_\varkappa) = = N_3(\bigcup_{\varkappa \in K} M_\varkappa)$.

Proof. a) For every $\varkappa \in K$ we have $M_\varkappa \subseteq \bigcup_{\varkappa \in K} M_\varkappa$ and therefore $\bigcup_{\varkappa \in K} N_3(M_\varkappa) \subseteq \subseteq N_3(\bigcup_{\varkappa \in K} M_\varkappa)$.

b) Let $x \in N_3(\bigcup_{\varkappa \in K} M_\varkappa)$. Then at least one power x^n is in $\bigcup_{\varkappa \in K} M_\varkappa$. Thus there exists a $\varkappa_0 \in K$ such that $x^n \in M_{\varkappa_0}$, i. e. $x \in N_3(M_{\varkappa_0}) \subseteq \bigcup_{\varkappa \in K} N_3(M_\varkappa)$. Therefore we have $N_3(\bigcup_{\varkappa \in K} M_\varkappa) \subseteq \bigcup_{\varkappa \in K} N_3(M_\varkappa)$ and this together with a) implies $\bigcup_{\varkappa \in K} N_3(M_\varkappa) = = N_3(\bigcup_{\varkappa \in K} M_\varkappa)$.

Lemma 6. *Let S be a semigroup and M_1 and M_2 subsets of S. Then $N_2(M_1) \cup \cup N_2(M_2) = N_2(M_1 \cup M_2)$.*

Proof. a) The relation $N_2(M_1) \cup N_2(M_2) \subseteq N_2(M_1 \cup M_2)$ is evident.

b) Let $x \in N_2(M_1 \cup M_2)$. Then infinitely many powers x^n are in $M_1 \cup M_2$. Thus infinitely many powers x^n are either in M_1 or in M_2. Therefore x is either in $N_2(M_1)$ or in $N_2(M_2)$ and $N_2(M_1 \cup M_2) \subseteq N_2(M_1) \cup N_2(M_2)$. This together with a) implies $N_2(M_1) \cup N_2(M_2) = N_2(M_1 \cup M_2)$.

Lemma 6 cannot be extended to the case of infinitely many subsets $M_\varkappa$, $\varkappa \in \in K$. This is clear from the following example.

Example 4. Let S be the set of all positive integers with ordinary addition as operation. Let $M_n = \{2n + 1\}$, where $n = 1, 2, 3, \ldots$. Then $\bigcup_{n=1}^{\infty} M_n = = \{2n + 1 \mid n = 1, 2, 3, \ldots\}$ and $1 \in N_2(\bigcup_{n=1}^{\infty} M_n)$. On the other hand $N_2(M_n) = = \emptyset$ for n = 1, 2, 3, … and therefore also $\bigcup_{n=1}^{\infty} N_2(M_n) = \emptyset$. This implies that $N_2(\bigcup_{n=1}^{\infty} M_n) \neq \bigcup_{n=1}^{\infty} N_2(M_n)$.

The next example shows that $N_1(M_1 \cup M_2) = N_1(M_1) \cup N_1(M_2)$ need not hold.

Example 5. Let S be the set of all positive integers with the ordinary addition as operation. Let $M_1 = \{2k | k = 1, 2, \dots\}$ and $M_2 = \{2k + 1 | k = 1, 2, \dots\}$. Then $1 \in N_1(M_1 \cup M_2)$ but $1 \notin N_1(M_1) \cup N_1(M_2)$.

Lemma 7. *Let S be a semigroup and M_1 and M_2 subsemigroups of S. Then $N_1(M_1 \cup M_2) = N_1(M_1) \cup N_1(M_2)$.*

Proof. a) It follows from $M_1 \subseteq M_1 \cup M_2$ and $M_2 \subseteq M_1 \cup M_2$ that $N_1(M_1) \cup N_1(M_2) \subseteq N_1(M_1 \cup M_2)$.

b) Let $x \in N_1(M_1 \cup M_2)$. Then there exists a positive integer N such that for every integer $n \geqslant N$ we have $x^n \in M_1 \cup M_2$. Let $X = \{x^n | n \geqslant N\}$. Note that $X \cap M_1$, $X \cap M_2$ are semigroups and $(X \cap M_1) \cup (X \cap M_2) = X$.

We now show that at least one of the semigroups M_1 and M_2 contains two consecutive powers of the element x. If it were not so, then one of the semigroups M_1 and M_2 would contain all even and the other all odd powers x^n of the element x for $n \geqslant N$. If for example $X \cap M_1$ were the set of all even powers x^n, $n \geqslant N$, then $X \cap M_2$ would be the set of all odd powers x^n, $n \geqslant N$. This contradicts the fact that $X \cap M_2$ is a semigroup.

Suppose that M_1 contains two consecutive powers of the element x. Then it can be easily verified that M_1 contains all powers x^n for $n \geqslant N_0 \geqslant N$. Therefore $x \in N_1(M_1)$ and hence $x \in N_1(M_1) \cup N_1(M_2)$.

We proved that $N_1(M_1 \cup M_2) \subseteq N_1(M_1) \cup N_1(M_2)$. This together with a) implies $N_1(M_1 \cup M_2) = N_1(M_1) \cup N_1(M_2)$.

The results we obtained can be arranged into two tables (see Table 1 and 2)

Table 1

$\cap$	M_1 and M_2 are:		
	subsets	subsemigroups	left (right) [two-sided] ideals
$N_1(M_1 \cap M_2) = N_1(M_1) \cap N_1(M_2)$	+ (L3)	+	+
$N_2(M_1 \cap M_2) = N_2(M_1) \cap N_2(M_2)$	— (E3)	+ (L4)	
$N_3(M_1 \cap M_2) = N_3(M_1) \cap N_3(M_2)$	— (E2)		

Table 2

$\cup$	M_1 and M_2 are:		
	subsets	subsemigroups	left (right) [two-sided] ideals
$N_1(M_1 \cup M_2) = N_1(M_1) \cup N_1(M_2)$	— (E4)	+ (L7)	+
$N_2(M_1 \cup M_2) = N_2(M_1) \cup N_2(M_2)$	+ (L6)	+	
$N_3(M_1 \cup M_2) = N_3(M_1) \cup N_3(M_2)$	+ (L5)		

in which the signs + and − have an apparent meaning. In parentheses a reference to the corresponding Lemma or Example is given.

The above results imply:

Theorem 1. *Let S be a semigroup. Then the mapping $M \to N_1(M)$ is:*

a) *a homomorphism of the lattice of all left (right) [two-sided] ideals of S into the lattice of all subsets of S,*

b) *a homomorphism of the $\cap$-semilattice of all subsemigroups of S into the $\cap$-semilattice of all subsets of S,*

c) *an endomorphism of the $\cap$-semilattice of all subsets of S.*

The mapping $M \to N_2(M)$ is:

a) *a homomorphism of the $\cap$-semilattice of all subsemigroups of S into the $\cap$-semilattice of all subsets of S,*

b) *an endomorphism of the $\cup$-semilattice of all subsets of S.*

The mapping $M \to N_3(M)$ is an endomorphism of the $\cup$-semilattice of all subsets of S.

We now introduce some further notions which are generalizations of the notions of Clifford's, Schwarz's and Ševrin's radicals from the papers [3] and [5].

Definition 2. *Let S be a semigroup and M a subset of S. An ideal I, each element of which is strongly nilpotent with respect to M, is called a strong nilideal with respect to M.*

An ideal I, each element of which is weakly nilpotent with respect to M, is called a weak nilideal with respect to M.

The union of all strong nilideals with respect to M will be denoted by $R_1^*(M)$.

The union of all weak nilideals with respect to M will be denoted by $R_2^*(M)$.

Definition 3. *Let S be a semigroup and M a subset of S. An ideal (a subsemigroup) I, for which there exists a positive integer N such that for all integers $n \geqslant N$ (for almost all n) $I^n \subseteq M$ holds, will be called a nilpotent ideal (a nilpotent subsemigroup) with respect to M.*

The union of all nilpotent ideals with respect to M will be denoted by $R(M)$.

Definition 4. *Let S be a semigroup and M a subset of S. An ideal I, every subsemigroup of which generated by a finite number of elements is nilpotent with respect to M, will be called a locally nilpotent ideal with respect to M.*

The union of all locally nilpotent ideals with respect to M will be denoted by $L(M)$.

Lemma 8. *An ideal I is a weak nilideal with respect to M if and only if every element $x \in I$ is almost nilpotent with respect to M.*

Proof. a) If the ideal I is a weak nilideal with respect to M, then clearly each element $x \in I$ is an almost nilpotent element with respect to M.

b) Let every element x of the ideal I be an almost nilpotent element with respect to M. Then $x \in I$ implies that $\{x, x^2, x^3, \ldots, x^n, \ldots\} \subseteq I$. In addition to this for some power x^{n_1} we have $x^{n_1} \in M$. But since $x^{n_1} \in I$, there exists again a positive integer $n_2 > n_1$ for which $x^{n_2} \in M$. Thus there exists a sequence $x^{n_1}, x^{n_2}, \ldots, x^{n_k}, \ldots, n_1 < n_2 < n_3 < \ldots < n_k < \ldots$ of powers of the element x, the members of which are in M. This means that x is a weakly nilpotent element with respect to M. Since x is any element of I, I is a weak nilideal with respect to M.

The following example shows that $R_1^*(M)$ and $R_2^*(M)$ may be distinct even if M is a subsemigroup of S.

Example 6. Let S be the set of all positive integers with the ordinary addition as operation. Let M be the subsemigroup of all even integers. Every odd positive integer is weakly nilpotent with respect to M but it is not strongly nilpotent with respect to M. Every even positive integer is strongly nilpotent with respect to M. Note that every ideal contains together with each integer $a > 0$ all integers $\geqslant a$. Hence $R_1^*(M) = \emptyset \neq S = R_2^*(M)$.

Lemma 9. *Let S be a semigroup, M a subset and A a subsemigroup of S. Then the following there statements are equivalent:*

a) *The subsemigroup A is nilpotent with respect to M.*

b) *There exist infinitely many positive integers n such that $A^n \subseteq M$.*

c) *There exists a positive integer n such that $A^n \subseteq M$.*

Proof. It is clear from definition 3 that a) implies b) and b) implies c). It remains only to prove that c) implies a). Let n be a positive integer such that $A^n \subseteq M$. Since A is a subsemigroup we have $A^{n+1} \subseteq A^n \subseteq M$, $A^{n+2} \subseteq \subseteq A^n \subseteq M, \ldots$ and therefore A is a nilpotent subsemigroup with respect to M.

Remark 2. Lemma 9 evidently holds also in the case where A is a left (right) [two-sided] ideal.

Lemma 10. *Let S be a semigroup and let M_1 and M_2 be subsets of S. Then $R_1^*(M_1 \cap M_2) = R_1^*(M_1) \cap R_1^*(M_2)$.*

Proof. a) Evidently $R_1^*(M_1 \cap M_2) \subseteq R_1^*(M_1) \cap R_1^*(M_2)$.

b) Let $x \in R_1^*(M_1) \cap R_1^*(M_2)$. Then $x \in R_1^*(M_1)$ and $x \in R_1^*(M_2)$, i. e. $x \in I_1$ and $x \in I_2$, where I_1 is a strong nilideal with respect to M_1 and I_2 is a strong nilideal with respect to M_2. We show that $I_1 \cap I_2$ is a strong nilideal with respect to $M_1 \cap M_2$. Let $y \in I_1 \cap I_2$. Then $y \in I_1$, $y \in I_2$, i. e. there exists a positive integer N such that for every integer $n \geqslant N$ we have $y^n \in M_1$ and $y^n \in M_2$. Hence for all integers $n \geqslant N$ we have $y^n \in M_1 \cap M_2$. This means that $I_1 \cap I_2$ is a strong nilideal with respect to $M_1 \cap M_2$.

Since $I_1 \cap I_2$ is a strong nilideal with respect to $M_1 \cap M_2$ and $x \in I_1 \cap I_2$, we have $x \in R_1^*(M_1 \cap M_2)$. Thus $R_1^*(M_1) \cap R_1^*(M_2) \subseteq R_1^*(M_1 \cap M_2)$ and this together with a) proves $R_1^*(M_1 \cap M_2) = R_1^*(M_1) \cap R_1^*(M_2)$.

Lemma 11. *Let S be a semigroup and let M_1 and M_2 be subsets of S. Then* $R\ (M_1 \cap M_2) = R\ (M_1) \cap R\ (M_2)$.

Proof. a) Evidently $R(M_1 \cap M_2) \subseteq R(M_1) \cap R(M_2)$.

b) Let $x \in R(M_1) \cap R(M_2)$. Then $x \in R(M_1)$ and $x \in R(M_2)$, i. e. $x \in I_1$ and $x \in I_2$, where I_1 is a nilpotent ideal with respect to M_1 and I_2 is a nilpotent ideal with respect to M_2. We shov that $I_1 \cap I_2$ is a nilpotent ideal with respect to $M_1 \cap M_2$. As a matter of fact for almost all n we have $I_1^n \subseteq M_1$ and $I_2^n \subseteq M_2$, thus $(I_1 \cap I_2)^n \subseteq M_1 \cap M_2$. Since $x \in I_1 \cap I_2$, we obtain $R(M_1) \cap R(M_2) \subseteq$ $\subseteq R(M_1 \cap M_2)$ and this together with a) proves $R(M_1) \cap R(M_2) = R(M_1 \cap$ $\cap M_2)$.

Lemma 12. *Let S be a semigroup and let M_1 and M_2 be subsets of S. Then* $L(M_1 \cap M_2) = L(M_1) \cap L(M_2)$.

Proof. a) Evidently $L(M_1 \cap M_2) \subseteq L(M_1) \cap L(M_2)$.

b) Let $x \in L(M_1) \cap L(M_2)$. Then $x \in L(M_1)$ and $x \in L(M_2)$, i. e. $x \in I_1$, where I_1 is a locally nilpotent ideal with respect to M_1 and $x \in I_2$, where I_2 is a locally nilpotent ideal with respect to M_2. We show that $I_1 \cap I_2$ is a locally nilpotent ideal with respect to $M_1 \cap M_2$.

Let A be a subsemigroup generated by a finite number of elements of $I_1 \cap I_2$. Since $A \subseteq I_1$ and $A \subseteq I_2$ for almost all positive integers n, $A^n \subseteq M_1$ and $A^n \subseteq M_2$ holds. Thus $A^n \subseteq M_1 \cap M_2$ and $I_1 \cap I_2$ is a locally nilpotent ideal with respect to $M_1 \cap M_2$.

As $x \in I_1 \cap I_2$, we obtain $x \in L(M_1 \cap M_2)$. Hence $L(M_1) \cap L(M_2) \subseteq$ $\subseteq L(M_1 \cap M_2)$ and this together with a) gives $L(M_1) \cap L(M_2) = L(M_1 \cap$ $\cap M_2)$.

Lemma 13. *Let S be a semigroup and M_1 and M_2 subsemigroups of S. Then* $R_2^*(M_1 \cap M_2) = R_2^*(M_1) \cap R_2^*(M_2)$.

Proof. a) Evidently $R_2^*(M_1 \cap M_2) \subseteq R_2^*(M_1) \cap R_2^*(M_2)$.

b) Let $x \in R_2^*(M_1) \cap R_2^*(M_2)$. Then $x \in R_2^*(M_1)$ and $x \in R_2^*(M_2)$, i. e. $x \in I_1$, where I_1 is a weak nilideal with respect to M_1 and $x \in I_2$ where I_2 is a weak nilideal with respect to M_2. Therefore $x \in I_1 \cap I_2$.

We now show that every element $y \in I_1 \cap I_2$ is weakly nilpotent with respect to $M_1 \cap M_2$, i. e. that $I_1 \cap I_2$ is a weak nilideal with respect to $M_1 \cap M_2$. Since $y \in I_1 \cap I_2$, there exist positive integers n_1 and n_2 such that $y^{n_1} \in M_1$ and $y^{n_2} \in M_2$. As M_1 and M_2 are subsemigroups of S we have for the cyclic semigroups generated by the elements y^{n_1} and y^{n_2}: $\{y^{n_1}, y^{2n_1}, \ldots\} \subseteq M_1$ and $\{y^{n_2}, y^{2n_2}, \ldots\} \subseteq M_2$. But then for the cyclic semigroup generated by the element $y^{n_1 n_2}$ we have $\{y^{n_1 n_2}, y^{2n_1 n_2}, \ldots\} \subseteq M_1 \cap M_2$. Hence y is a weakly nilpotent element with respect to $M_1 \cap M_2$, thus $I_1 \cap I_2$ is a weak nilideal with respect to $M_1 \cap M_2$.

Since $x \in I_1 \cap I_2$ we have $x \in R_2^*(M_1 \cap M_2)$. We proved that $R_2^*(M_1) \cap \cap R_2^*(M_2) \subseteq R_2^*(M_1 \cap M_2)$ and this together with a) gives $R_2^*(M_1 \cap M_2) = = R_2^*(M_1) \cap R_2^*(M_2)$.

The following example shows that $R_2^*(M_1) \cap R_2^*(M_2) = R_2^*(M_1 \cap M_2)$ need not hold.

Example 7. Let S be the set of all positive integers with the ordinary addition as operation. Let M_1 contain the number 1 and those integers $n > 1$ whose factorization into primes has either an even number of factors equal to the number 2 or it has no factor equal to 2. Let M_2 contain the number 1 and those integers $n > 1$ whose factorization into primes has an odd number of factors equal to 2. Clearly $M_1 \cap M_2 = \{1\}$ and $R_2^*(M_1 \cap M_2) = \emptyset$. Further $R_2^*(M_1) = S$ and $R_2^*(M_2) = S$ and therefore $R_2^*(M_1) \cap R_2^*(M_2) = S \neq \emptyset = = R_2^*(M_1 \cap M_2)$.

The results we obtained are arranged into tables. (See Tables 3, 4 and 5.

Table 3

$\cap$	M_1 and M_2 are:		
	subsets	subsemigroups	left (right) [two-sided] ideals
$R_1^*(M_1 \cap M_2) = R_1^*(M_1) \cap R_1^*(M_2)$	+ (L10)	+	+
$R_2^*(M_1 \cap M_2) = R_2^*(M_1) \cap R_2^*(M_2)$	— (E6)	+ (L13)	

Table 4

$\cap$	M_1 and M_2 are:		
	subsets	subsemigroups	left (right) [two-sided] ideals
$R(M_1 \cap M_2) = R(M_1) \cap R(M_2)$	+ (L11)	+	+

Table 5

$\cap$	M_1 and M_2 are:		
	subsets	subsemigroups	left (right) [two-sided] ideals
$L(M_1 \cap M_2) = L(M_1) \cap L(M_2)$	+ (L12)	+	+

Remark 3. For the unions the relations $R_1^*(M_1 \cup M_2) = R_1^*(M_1) \cup \cup R_1^*(M_2)$, $R_2^*(M_1 \cup M_2) = R_2^*(M_1) \cup R_2^*(M_2)$, $R(M_1 \cup M_2) = R(M_1) \cup R(M_2)$ and $L(M_1 \cup M_2) = L(M_1) \cup L(M_2)$ need not hold. This follows from an example in paper [3], p. 213, even if M_1 and M_2 are two-sided ideals. (See also [5].)

Remark 4. Lemmas 3, 4, 6, 7, 10, 11, 12 and 13 can be extended by induction from two subsets M_1 and M_2 to any finite number of subsets $M_\varkappa$, $\varkappa \in K$. But the following example shows that Lemmas 3, 4, 10, 11, 12 and 13 cannot be extended to an infinite number of subsets.

Example 8. The closed interval $S = \langle 0, \frac{2}{3} \rangle$ with the ordinary multiplication as operation is a semigroup. The closed intervals $J_n = \left\langle 0, \frac{1}{n} \right\rangle$, $n = 2, 3, \ldots$ are ideals of S. $N_1(J_n) = S$ for $n = 2, 3, \ldots$ and therefore $\bigcap_{n=2}^{\infty} N_1(J_n) = S$. But $\bigcap_{n=2}^{\infty} J_n = \{0\}$ and $N_1(\bigcap_{n=2}^{\infty} J_n) = \{0\} \neq S$.

Since S is a commutative semigroup and J_n, $n = 2, 3, \ldots$ are ideals of S, the foregoing sets of strongly nilpotent elements are at the same time sets of weakly nilpotent elements and also sets of almost nilpotent elements. By [3] and [1] they are clearly radicals with respect to these ideals.

The above lemmas imply the following theorems:

Theorem 2. *Let S be a semigroup. Then the mapping $M \to R_1^*(M)$ is:*

a) *a homomorphism of the $\cap$-semilattice of all subsets of S into the $\cap$-semilattice of all (two-sided) ideals of S,*

b) *a homomorphism of the $\cap$-semilattice of all subsemigroups of S into the $\cap$-semilattice of all (two-sided) ideals of S,*

c) *a homomorphism of the $\cap$-semilattice of all left (right) [two-sided] ideals of S into the $\cap$-semilattice of all (two-sided) ideals of S.*

The mapping $M \to R_2^*(M)$ is a homomorphism of the $\cap$-semilattice of all subsemigroups of S into the $\cap$-semilattice of all (two-sided) ideals of S.

Theorem 3. *Let S be a semigroup. Then the mapping $M \to R(M)$ is:*

a) *a homomorphism of the $\cap$-semilattice of all subsets of S into the $\cap$-semilattice of all (two-sided) ideals of S,*

b) *a homomorphism of the $\cap$-semilattice of all subsemigroups of S into the $\cap$-semilattice of all (two-sided) ideals of S,*

c) *a homomorphism of the $\cap$-semilattice of all left (right) [two-sided] ideals of S into the $\cap$-semilattice of all (two-sided) ideals of S.*

Theorem 4. *Let S be a semigroup. Then the mapping $M \to L(M)$ is:*

a) *a homomorphism of the $\cap$-semilattice of all subsets of S into the $\cap$-semilattice of all (two-sided) ideals of S,*

b) *a homomorphism of the $\cap$-semilattice of all subsemigroups of S into the $\cap$-semilattice of all (two-sided) ideals of S,*

c) *a homomorphism of the $\cap$-semilattice of all left (right) [two-sided] ideals of S into the $\cap$-semilattice of all (two-sided) ideals of S.*

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