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EQUIVALENCES ON MACHINE STATE SPACES

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0. Continuing our investigations of the structure of simple topological automata we concern ourselves herein with equivalences on the state spaces of such mathematical machines. Of particular interest are those equivalences which are „irreducible" and which, in the case of compact automata with totally disconnected state spaces, are sufficiently abundant to permit a topological imbedding of the state space in the product of, structurally speaking, more basic spaces, thus paralleling the results of Birkhoff [2], Numakura [5], and those reported in our paper [1]. In addition, and in keeping with our interest in totally disconnected state spaces, attention is directed to the „component" space, and a monotone-light factorization is obtained for compact automata.

As this paper is one of several in press on topological automata in the sense to be used here, we refrain from an extended bibliographic discussion referring the reader instead to Wallace [6] or Bednarek—Wallace [1] for related references.

Throughout this paper topological notions will take precedence and thus such words as „semigroup" will mean „topological semigroup", „morphisms" will be continuous, etc. The discrete topology may be used, and, futhermore, all finite sets are compact using any topology.

1. A *semigroup* is a nonvoid Hausdorff space togehter with a continuous associative multiplication, generally denoted by juxtaposition; that is, a semigroup is a function

$$m\colon S \times S \to S$$

such that (i) S is a nonvoid Hausdorff space (ii) m is continuous and (iii) m is associative, i. e. $m(m(x, y), z) = m(x, m(y, z))$ for all $x, y, z \in S$.

The type of topological automata of concern in this paper are those structures which, in the literature of semigroups, have been called *acts*, and we shall revert to this terminology in the sequel.

An *act* is such a continuous function

$$T \times X \to X$$

that (i) T is a semigroup (ii) X is a nonvoid Hausdorff space and (iii) denoting the value of the function at the place (t, x) by tx, we have

$$t(t'x) = (tt')x$$

for all t, $t' \in T$ and all $x \in X$. Here, T is the *input semigroup* and X is the *state space.* If both T and X are compact, we shall refer to the act as a *compact act.*

If T „acts" on X; that is, if $T \times X \to X$ is an act, then T acts on $X \times X$ and $T \times T$ acts on $X \times X$, with the functions defined coordinatewise. Quite often we shall be considering these associated acts in a single argument depending upon the context to make clear to which act we are referring.

If T acts on X, then a continuous function φ from X into the nonvoid Hausdorff space Y will be called a *T-morphism* (or just *morphism*) if T acts on Y and if the following diagram is analytic, where i denotes the identity map on T.

$$\begin{tikzcd}
T \times X \arrow[r] \arrow[d, "(i \times \varphi)"'] & X \arrow[d, "\varphi"] \\
T \times Y \arrow[r] & Y
\end{tikzcd}$$

If φ is a homeomorphism into as well as a morphism then we will call φ a *morphic imbedding* of X in Y.

2. If T acts on X, then an equivalence $\mathbf{E}$ on X is called a *T-equivalence* if $T\mathbf{E} \subset \mathbf{E}$; that is, if for any $(x, y) \in \mathbf{E}$ and any $t \in T$ it follows that $(tx, ty) \in \mathbf{E}$. Recalling that a *conqruence* $\mathbf{F}$ on a semigroup T is such an equivalence that $\Delta\, \mathbf{F} \subset \mathbf{F} \supset \mathbf{F}\, \Delta$, where Δ is the diagonal of T ($\Delta = \{(t, t) \mid t \in T\}$ and the multiplication is coordinatewise, we have

(2.1) *If $T \times X \xrightarrow{f} X$ is a compact act, and if $\mathbf{F}$ is a closed congruence on T and $\mathbf{E}$ is a closed T-equivalence on X satisfying $\mathbf{F}\,\mathbf{E} \subset \mathbf{E}$, then there exists one and only one continuous function α such that the following diagram is analytic. Thus $T/\mathbf{F}$ acts on $X/\mathbf{E}$, and α is called the canonical action.*

Proof. Let $\alpha\,(\bar{t}, \bar{x}) = (\overline{tx})$ which is well defined in view of the assumption that $\mathbf{F}\,\mathbf{E} \subset \mathbf{E}$. Now $T/\mathbf{F}$ is a semigroup and the space $X/\mathbf{E}$ is also Hausdorff, while the analyticity follows directly from the definition of α. If $M \subset X/\mathbf{E}$, then $\alpha^{-1}(M) = (\psi \times \varphi) f^{-1} \varphi^{-1}(M)$, and since T and X are compact and $T/\mathbf{F}$ and $X/\mathbf{E}$ are Hausdorff, the mapping $(\psi \times \varphi)$ is closed thus yielding $\alpha^{-1}(M)$ closed for every closed set $M \subset X/\mathbf{E}$. The completion of the proof that α is an act depends only upon the assumption that f is an act.

$$\begin{tikzcd} T \times X \arrow[r, "f"] \arrow[d, "(\psi \times \varphi)"'] & X \arrow[d, "\varphi"] \\ T/\mathbf{F} \times X/\mathbf{E} \arrow[r, "\alpha"'] & X/\mathbf{E} \end{tikzcd}$$

Although not concerned directly with acts, proposition (2.2) is stated here without proof (the proof being rather immediate) and is used in a later section in which we consider totally disconnected spaces, in particular the so-called component spaces.

To state (2.2) we define, in accordance with Bourbaki [3, p. 122], the *product* $\mathbf{E} \times \mathbf{F}$ of two equivalence relations, where $\mathbf{E}$ is an equivalence on X and $\mathbf{F}$ is an equivalence on Y, to be the equivalence on $X \times Y$ defined by $((x_1, y_1), (x_2, y_2)) \in \mathbf{E} \times \mathbf{F}$ iff $(x_1, x_2) \in \mathbf{E}$ and $(y_1, y_2) \in \mathbf{F}$. We then have

(2.2) *If X and Y are compact Hausdorff spaces, and if $\mathbf{E}$ and $\mathbf{F}$ are closed equivalences on X and Y respectively, then there exists a homeomorphism h such that the following diagram is analytic, where φ, ψ and γ are the natural maps.*

$$\begin{tikzcd} X \times Y \arrow[r, "\gamma"] \arrow[d, "(\varphi \times \psi)"'] & X \times Y/\mathbf{E} \times \mathbf{F} \arrow[dl, "h"] \\ X/\mathbf{E} \times Y/\mathbf{F} & \end{tikzcd}$$

3. Throughout this section we assume that $T \times X \to X$ is an act, any additional assumptions on the input semigroup or state space being stated specifically. A T-equivalence is said to be irreducible iff whenever it is an intersection of T-equivalences it is one of them; that is, if $\mathbf{E}$ is a T-equivalence and if $\mathbf{E} = \cap \{E_\lambda \mid \lambda \in \Lambda\}$, where each $\mathbf{E}_\lambda$ is a T-equivalence, then $\mathbf{E} = \mathbf{E}_{\lambda_0}$ for some $\lambda_0 \in \Lambda$.

(3.1) *The T-equivalence $\mathbf{E}$ is irreducible iff there is a T-equivalence $\mathbf{F} \supset \supset \mathbf{E} \neq \mathbf{F}$ such that $\mathbf{E}' \supset \mathbf{E} \neq \mathbf{E}'$ implies $\mathbf{E}' \supset \mathbf{F}$ for each T-equivalence $\mathbf{E}'$.*

Proof. If $\mathbf{E}$ is an irreducible T-equivalence, we let $\mathbf{M}$ be the family of T-equivalences each of which contains $\mathbf{E}$ properly. If $\mathbf{M}$ is empty then $\mathbf{E} = \mathbf{F}$, otherwise let $\mathbf{F} = \cap \{M \mid M \in \mathbf{M}\}$ which is easily seen to have the desired properties.

Conversely suppose that $\mathbf{E} = \cap \{\mathbf{E}_\lambda \mid \lambda \in \Lambda\}$, where each $\mathbf{E}_\lambda$ is a T-equivalence containing $\mathbf{E}$ properly, and that there exists a T-equivalence $\mathbf{F}$ with the above-mentioned properties. Then $\mathbf{F} \supset \mathbf{E}_\lambda$ for each λ and so $\mathbf{F} \neq \mathbf{E} \subset \mathbf{F} \subset \subset \cap \{\mathbf{E}_\lambda \mid \lambda \in \Lambda\}$ contradicting our assumption that $\mathbf{E} = \cap \{\mathbf{E}_\lambda \mid \lambda \in \Lambda\}$.

As an immediate corollary to the above we have:

(3.2) *If the diagonal Δ of X is an irreducible T-equivalence then there exists a T-equivalence $\mathbf{E}_0 \neq \Delta$ contained in all T-equivalences that are not equal to Δ.*

A way of generating state spaces for which the diagonal is an irreducible T-equivalence is given below.

(3.3) *If $\mathbf{E}$ is an irreducible T-equivalence then the diagonal of $X/\mathbf{E}$ is an irreducible T-equivalence provided that the natural map is a morphism.*

Proof. Denoting the diagonal of $X/\mathbf{E}$ by diag $(X/\mathbf{E})$, suppose diag $(X/\mathbf{E}) = \cap \{F_\lambda \mid \lambda \in \Lambda\}$ where each $\mathbf{F}_\lambda$ is a T-equivalence on $X/\mathbf{E}$. Letting φ denote the natural map from X onto $X/\mathbf{E}$, we note that

$$E = (\varphi \times \varphi)^{-1} \operatorname{diag} (X/\mathbf{E}) = \cap \{(\varphi \times \varphi)^{-1} F_\lambda \mid \lambda \in \Lambda\}.$$

For each λ, $(\varphi \times \varphi)^{-1} \mathbf{F}_\lambda$ is a T-equivalence and since $\mathbf{E}$ is irreducible, there is a λ_0 such that $\mathbf{E} = (\varphi \times \varphi)^{-1} \mathbf{F}_{\lambda_0}$. From this it follows that diag $(X/\mathbf{E}) = (\varphi \times \varphi)\, \mathbf{E} = \mathbf{F}_{\lambda_0}$ and, therefore, that diag $(X/\mathbf{E})$ is irreducible.

We now direct our attention to the case of a compact act with totally disconnected state space and *for the remainder of this section we assume that $T \times X \to X$ is a compact act and that X is totally disconnected.* In our paper [1] we proved that under these assumptions every open subset of $X \times X$ containing the diagonal of X contains an open T-equivalence, and we use this result to obtain the following:

(3.4) *The diagonal of X is the intersection of a family of irreducible T-equivalences each of which is open.*

Proof. If $a, b \in X$ and $a \neq b$ then by [1] there is an open T-equivalence $\mathbf{E}_0$ such that $(a, b) \notin \mathbf{E}_0$. Let $\mathbf{M}$ be the family of open T-equivalences each not containing (a, b), and let $\mathbf{E} = \cup \{M \mid M \in \mathbf{M}\}$. Clearly $\mathbf{E}$ is an open T-equivalence which does not contain (a, b). Suppose that $\mathbf{E} = \cap \{\mathbf{E}_\lambda \mid \lambda \in \Lambda\}$ where each $\mathbf{E}_\lambda$ is a T-equivalence. Since $(a, b) \notin \mathbf{E}$ there is a $\lambda_0 \in \Lambda$ such that $(a, b) \notin$ $\notin \mathbf{E}_{\lambda_0}$, and so $\mathbf{E}_{\lambda_0} = \mathbf{E}$ from which we conclude that $\mathbf{E}$ is irreducible.

The preceding proposition guaranteeing that it is possible to „separate points“ by open irreducible T-equivalences enables us to assert:

(3.5) *The state space can be morphically imbedded in the product of finite discrete spaces each having the property that its diagonal is an irreducible T-equivalence.*

Proof. Let $\mathbf{M} = \{\mathbf{E}_\lambda \mid \lambda \in \Lambda\}$ be the family of proper open T-equivalences and note that for each $\mathbf{E}_\lambda$ the space $x/\mathbf{E}_\lambda$ is finite and discrete and, by (3.3), has the property that its diagonal is an irreducible T-equivalence. For each $\lambda \in \Lambda$ let φ_λ denote the natural map from X onto $X/\mathbf{E}_\lambda$ and define the map ψ from X into the cartesian product $P\,\{X/\mathbf{E}_\lambda \mid \lambda \in \Lambda\}$ by

$$(\psi\,(x))_\lambda = \varphi_\lambda(x)$$

for each $\lambda \in \Lambda$. It is immediate that ψ is a homeomorphism of X into $P\{X/\mathbf{E}_\lambda \mid \lambda \in \Lambda\}$, since X is compact, ψ is one-to-one and $P\{X/\mathbf{E}_\lambda \mid \lambda \in \Lambda\}$ is Hausdorff.

Defining the act $T \times P\{X/\mathbf{E}_\lambda \ \lambda \in \Lambda\} \to P\{X/\mathbf{E}_\lambda \mid \lambda \in \Lambda\}$ by $(tp)_\lambda = tp_\lambda$ it follows immediately that ψ is a T-morphism as well as a homeomorphism.

4. We now revert to the assumption that $T \times X \to X$ is a compact act and direct our attention to the *component spaces* of T and X, (denoted by compo T and compo X respectively), that is, to the quotient spaces obtained by identifying points lying in the same component. These spaces are totally disconnected and, as we shall see, inherit an action from $T \times X \to X$.

(4.1) *If $T \times X \to X$ is a compact act then*

(i) compo $(T \times X) \stackrel{t}{=}$ compo $T \times$ compo X

(ii) *the following diagram is analytic*

$$\begin{tikzcd}
T \times X \arrow[r] \arrow[d, "(\psi \times \varphi)"'] & X \arrow[d, "\varphi"] \\
\text{compo } T \times \text{compo } X \arrow[r] & \text{compo } X
\end{tikzcd}$$

and thus compo T *acts on* compo X.

Proof. Let $\mathbf{E}$ and $\mathbf{F}$ be the equivalences on X and T respectively corresponding to the component decompositions of these spaces, and let $\mathbf{G} = \mathbf{E} \times \mathbf{F}$

be their *product.* Although derivable via other means we sketch here the proof that the equivalences $\mathbf{E}$ and $\mathbf{F}$, and thus $\mathbf{G}$, are closed.

Suppose that $(x_1, x_2) \in \mathbf{E}$ and that C_1 and C_2 are the components of X containing x_1 and x_2 respectively. Now C_1 and C_2 are disjoint closed sets in the compact Hausdorff space X such that no continuum in X meets both C_1 and C_2. Therefore there exists a clopen (closed and open) set O which contains C_1 and does not meet C_2. Clearly $(x_1, x_2) \in O \times X \backslash O \subset X \times X \backslash \mathbf{E}$ so that $\mathbf{E}$ is closed. In a similar way we prove that $\mathbf{F}$ is closed and then $\mathbf{G}$ being the homeomorphic image of the cartesian product of $\mathbf{E}$ and $\mathbf{F}$ is also closed.

Since $\mathbf{E}$ and $\mathbf{F}$ are closed the spaces compo T, compo X and compo $(T \times X)$ are compact Hausdorff spaces, thus to establish (i) we need only observe that $\mathbf{G} = \mathbf{E} \times \mathbf{E}$ is the equivalence corresponding to the component decomposition of $T \times X$ and then apply (2.2). In an equally direct way it follows from the continuity of the action and of the multiplication on T that $\mathbf{E}$ is a T-equivalence and $\mathbf{F}$ is a congruence which togehter satisfy $\mathbf{F}\,\mathbf{E} \subset \mathbf{E}$, therefore yeilding (ii) as an immediate consequent of (2.1).

A consideration of totally disconnected spaces leads one quite directly to the classical Eilenberg — Whyburn *monotone-light factorization* of a continous mapping of a compact space. As such a factorization is available for a homomorphism of a compact semigroup it is natural to ask whether or not this is also the cas for morphisms on the state spaces of compact acts, and the final theorem of this note shows that this is indeed so.

Recall that a *factorization* of a continuous function $\varphi\colon X \to Y$ is any representation of φ in the form $\varphi(x) = \beta\alpha(x)$, whree $\alpha\colon X \to M$ and $\beta\colon M \to Y$ are continuous. A mapping $f\colon X \to Y$ is said to be *monotone* provided that for each point $y \in Y$, the inverse image $f^{-1}(y)$ is connected, the mapping is called *light* provided that $f^{-1}(y)$ is totally disconnected for each $y \in Y$.

(4.2) *If $T \times X \to X$ is a compact act and if φ is a T-morphism of X into the compact space Y then there exists a unique factorization of φ, $\varphi(x) = \beta\alpha(x)$, where $\alpha\colon X \to M$ and $\beta\colon M \to Y$ are monotone and light morphisms respectively and, moreover, the following diagram is analytic (i* denotes the identity map on T).

$$\begin{tikzcd}
T \times X \arrow[rrr] \arrow[dd, "(i \times \varphi)"'] \arrow[dr, "(i \times \alpha)"] & & & X \arrow[dl, "\alpha"'] \arrow[dd, "\varphi"] \\
& T \times M \arrow[r] \arrow[dl, "(i \times \beta)"] & M \arrow[dr, "\beta"'] & \\
T \times Y \arrow[rrr] & & & Y
\end{tikzcd}$$

Proof: It is proved in Hocking-Young [4] (p. 137) that if we let M (the *middle space*) be the quotient space corresponding to the decomposition of X into components of point inverses $\varphi^{-1}(y)$ then the natural map $\alpha: X \to M$ and the map $\beta: M \to Y$ defined by $\beta(p) = \varphi(\alpha^{-1}(p))$ are the mappings that effect the unique (up to homeomorphisms) monotone-light factorization of φ and, moreover, M is Hausdorff. For our purposes it remains only to show that α and β are T-morphisms.

To prove that α is a T-morphism we first observe that the T-morphism φ is characterized by the property that $t\varphi^{-1}(y) \subset \varphi^{-1}(ty)$ for each $t \in T$ and each $y \in Y$. Now if $\alpha(x_1) = \alpha(x_2)$ then x_1 and x_2 are contained in the same component, call it C, of $\varphi^{-1}(y)$ for some $y \in Y$. Thus for any $t \in T$, tx_1 and tx_2 are elements of tC which is connected and contained in $t\varphi^{-1}(y)$ and, therefore, in $\varphi^{-1}(ty)$. This implies $\alpha(tx_1) = \alpha(tx_2)$ from which it follows that α is a T-morphism, the act $T \times M \to M$ being definde by $tm = \alpha(t\, \alpha^{-1}(m))$ for each $t \in T$ and each $m \in M$.

To show that β is a T-morphism we let $p \in M$, so $p = \alpha(x)$ for some $x \in X$, and observe that

$$\beta(tp) = \beta(t\, \alpha(x)) = \beta\alpha(tx) = \varphi(tx) = t\varphi(x) = t\, \beta\, \alpha(x) = t\, \beta(p).$$

The middle space M is, of course, compact and totally disconnected and so $T \times M \to M$ is a compact act with totally disconnected state space.

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