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ON THE UNIQUENESS OF SOLUTIONS AND ON THE CONVERGENCE OF SUCCESSIVE APPROXIMATIONS FOR CERTAIN INITIAL PROBLEMS OF EQUATIONS OF THE HIGHER ORDERS

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1. **Introduction.** The problem of the convergence of successive approximations with respect to the ordinary differential equations and systems of ordinary differential equations was observed in F. Bauer's papers [1] and [2] under the conditions of the Krasnosielski and Krein type. B. Palczewski [4] showed that by the Krasnosielski-Krein conditions it is possible also to secure the uniqueness of solution of the Darboux problem and the convergence of competent successive approximations. In the present paper it will be our aim to determine conditions securing both the uniqueness of solution of the given initial problem (1), (2) and the convergence of the Picard successive approximations belonging to this problem.

2. **The uniqueness of solution.** In our considerations we shall use the following notations:

a) An arbitrary set of points $X(x_1, \ldots, x_m)$ from the m-dimensional Euclidean space ($m \geq 2$) for which $0 < x_i \leq A_i$, $0 \leq x_i \leq A_i$, $A_i > 0$, $i = 1, \ldots, m$ resp., will be denoted by R^0, R resp.

Further, let the symbols R_k, R_{lj} resp., for $1 \leq k, l, j \leq m$, $l < j$ denote the $m - 1$-dimensional, the $m - 2$-dimensional closed domain of the points $X_k(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m)$, $X_{lj}(x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m)$, resp., with the coordinates $0 \leq x_i \leq A_i$ for all $i \neq k$, $i \neq l, j$, resp.

b) Let us define the domains $E^0 = R^0 \times \{-\infty < u < +\infty\}$ and $E = R \times \{-\infty < u < +\infty\}$.

c) We denote the simplex in the ν-dimensional Euxlidean space E^ν with the $\nu+1$ linearly independent vertices $\Xi_0(0, \ldots, 0)$,$\Xi_1(\xi, 0, \ldots, 0)$, $\ldots$, $\Xi_{\nu-1}$ $(\xi, \ldots, \xi, 0)$, $\Xi_\nu(\xi, \ldots, \xi)$, $\xi > 0$ by Σ_ξ^ν, i.e. Σ_ξ^ν is the set of points $P \in E^\nu$ for which

$$P = \alpha_0 \Xi_0 + \alpha_1 \Xi_1 + \ldots + \alpha_\nu \Xi_\nu,$$

where $\alpha_0 + \alpha_1 + \ldots + \alpha_\nu = 1$, $\alpha_i \geq 0$ for $i = 0, 1, \ldots, \nu$. If $\xi = 0$, then $\Sigma_0^\nu = \Xi_0$.

d) Let us consider the set $M(R)$ of the function $v(X)$ which are defined and continuous in the domain R and have continuous derivatives $D_{x_1}^{t_1} \ldots D_{x_m}^{t_m} v(X)$ on R for $0 \leq t_j \leq k_j$, $j = 1, \ldots, m$ and $1 \leq \sum_{j=1}^m t_j \leq \sum_{j=1}^m k_j - 1$. By D_ξ^α we denote the differential operator $\dfrac{\partial^\alpha}{\partial \xi^\alpha}$ and $D_x^0 v(X) = v(X)$.

Now, we can define the initial problem and the notion of its solution as follows:

Under the solution of the initial problem

$$D_{x_1}^{k_1} \ldots D_{x_m}^{k_m} u = f(X, u), \; X \in R^0 \tag{1}$$

$$[D_{x_r}^{i_r} u(X)]_{x_r=0} = \sigma_r^{(i_r)}(X_r), \; X_r \in R_r; \; i_r = 0, 1, \ldots, k_r - 1; \; r = 1, \ldots, m \tag{2}$$

$$[D_{x_s}^{j_s} \sigma_r^{(i_r)}(X_r)]_{x_s=0} = [D_{x_r}^{i_r} \sigma_s^{(j_s)}(X_s)]_{x_r=0}, \; X_{rs} \in R_{rs}$$

$$r \neq s, \; i_r = 0, 1, \ldots, k_r - 1; \; j_s = 0, 1, \ldots, k_s - 1; \; r, s = 1, \ldots, m,$$

where the function $f(X, u)$ is continuous on E and $\sigma_r^{(i_r)}(X_r)$ has continuous derivatives $D_{x_1}^{l_1} \ldots D_{x_{r-1}}^{l_{r-1}} D_{x_{r+1}}^{l_{r+1}} \ldots D_{x_m}^{l_m} \sigma_r^{(i_r)}(X_r)$ on R_r for $0 \leq l_j \leq k_j$, $j = 1, \ldots, m$, $j \neq r$, we understand any function $u(X) \in M(R)$ with the continuous derivative $D_{x_1}^{k_1} \ldots D_{x_m}^{k_m} u$ in R^0 and satisfying the conditions (1), (2).

It can be shown by an elementary calculation that the problem (1), (2) and that one of solving the following integral equation

$$u(X) = G_0(X) + \int_{\Sigma_{x_1}^{k_1}} d\mu_1 \ldots \int_{\Sigma_{x_m}^{k_m}} f[\Xi, u(\Xi)] \, d\mu_m \tag{3}$$

are mutually equivalent for $X \in R$, where $\Xi = (\xi_1, \ldots, \xi_m)$ and μ_i, $i = 1, \ldots, m$ denotes the Lebesgue measure defined in the space E^{k_i}. The function $G_0(X)$ can be explicitly expressed by the initial functions $\sigma_r^{(i_r)}(X_r)$. This assertion follows directly from the relation

$$G_0(X) = \sum_{j=0}^{m} \sum_{i_1, \ldots, i_j} \sum_{l_1, \ldots, l_j} (-1)^{j-1} \frac{x_{i_1}^{l_1} \ldots x_{i_j}^{l_j}}{l_1! \ldots l_j!} [D_{x_{i_1}}^{l_1} \ldots D_{x_{i_j}}^{l_j} u(X)]_{\substack{x_{i_1}=0 \\ \vdots \\ x_{i_j}=0}},$$

where $0 \leq l_1 \leq k_{i_1} - 1, \ldots, 0 \leq l_j \leq k_{i_j} - 1$ and $(i_1, \ldots, i_j)$ is an arbitrary combination of the m natural numbers $(1, \ldots, m)$ j at a time for $i_1 < \ldots < i_j$.

By means of (3) we define the Picard sequence of successive approximations for the problem (1), (2)

$$u_{n+1}(X) = G(X) + \int_{\Sigma_{x1}^{k1}} \mathrm{d}\,\mu_1 \ldots \int_{\Sigma_{xm}^{km}} f[\Xi, u_n(\Xi)]\mathrm{d}\,\mu_m, \tag{4}$$

where $n = 0, 1, \ldots, u_0(X) \in C(R)$ and the function $G(X) \in M(R)$ satisfies conditions (2) and $D_{x_1}^{k_1} \ldots D_{x_m}^{k_m} G(X) = 0$ on R.

Before we approach the convergence of the successive approximations (4) we must prove the following:

Lemma 1. *Let the function $v(X) \geq 0$ be continuous in the domain R and the integral inequalities*

$$v(X) \leq \int_{\Sigma_{x1}^{k1}} \mathrm{d}\mu_1 \ldots \int_{\Sigma_{xm}^{km}} \frac{K}{\xi_1^{k_1} \ldots \xi_m^{k_m}} v(\Xi)\mathrm{d}\mu_m, \; K > 0 \tag{5}$$

$$v(X) \leq \int_{\Sigma_{x1}^{k1}} \mathrm{d}\mu_1 \ldots \int_{\Sigma_{xm}^{km}} C v^{\alpha}(\Xi)\mathrm{d}\mu_m, \; C > 0, \; 0 < \alpha < 1 \tag{6}$$

hold on R, where

$$k_j \sqrt[m]{K}(1 - \alpha) < k_j - (k_j - 1)(1 - \alpha), \; k_j \geq 1, \; j = 1, \ldots, m; \tag{7}$$

then $v(X) \equiv 0$ in the domain R.

Proof. Let us denote $M = \sup_R v(X)$. From the assumption (6) we get

$$v(X) \leq C^{1+\alpha+\ldots+\alpha^n} M^{\alpha^{n+1}} (x_1^{k_1} \ldots x_m^{k_m})^{1+\alpha+\ldots+\alpha^{n+1}}.$$

Hence we obtain the estimate

$$0 \leq v(X) \leq c^{1/1-\alpha} (x_1^{k_1} \ldots x_m^{k_m})^{1/1-\alpha}. \tag{8}$$

Let us define now the function $Q(X)$ as follows:

$$Q(X) = \begin{cases} \left[\prod_{j=1}^{m} x_j^{-k_j\sqrt[m]{K} - k_j + 1}\right] v(X) & \text{for} \quad X \in R^0 \\ 0 & \text{for} \quad X \in R - R^0. \end{cases}$$

From (8) we get for $X \in R$

$$0 \leq Q(X) \leq C^{1/1-\alpha} \prod_{j=1}^{m} x_j^{[k_j - (k_j - 1)(1-\alpha) - k_j\sqrt[m]{K}(1-\alpha)]/(1-\alpha)}$$

and with respect to (7) we have $\lim_{R^0 \ni X \to X_0 \in R - R^0} Q(X) = 0$. Consequently, the above defined function $Q(X)$ is continuous in the domain R. We assert that $Q(X) \equiv 0$ on R. If this were not the case, there would exist a point $Y(y_1, \ldots, y_m) \in R^0$ for which $0 < Q(Y) = \sup_R Q(X)$. This assertion leads us to the following conclusion

$$Q(Y) \leq \prod_{j=1}^{m} y^{-k_j \sqrt[m]{K} - k_j + 1} \int_{\Sigma_{y^1}^{k_1}} \mathrm{d}\mu_1 \ldots \int_{\Sigma_{y^m}^{k_m}} \frac{K}{\xi_1^{k_1} \ldots \xi_m^{k_m}} v(\Xi) \mathrm{d}\mu_m =$$

$$= \prod_{j=1}^{m} y_j^{-k_j \sqrt[m]{K} - k_j + 1} \int_{\Sigma_{y^1}^{k_1}} \mathrm{d}\mu_1 \ldots \int_{\Sigma_{y^m}^{k_m}} K \prod_{i=1}^{m} \xi_i^{k_i \sqrt[m]{K} - 1} Q(\Xi) \mathrm{d}\mu_m < \frac{Q(Y)}{(\prod_{l=1}^{m} k_l)[\prod_{i=1}^{m} \prod_{j=1}^{k_i - 1} (k_i \sqrt[m]{K} + j)]}$$

which is impossible. This completes the proof.

Remark 1. If

$$\text{(5')} \qquad v(X) \leq \int_{\Sigma_{x^1}^{k_1}} \mathrm{d}\mu_1 \ldots \int_{\Sigma_{x^m}^{k_m}} \frac{\prod_{i=1}^{m} \prod_{j=0}^{k_i - 1} (k_i \sqrt[m]{K} + j)}{\xi_1^{k_1} \ldots \xi_m^{k_m}} v(\Xi) \mathrm{d}\mu_m, \; K > 0$$

is considered instead of the inequality (5), then Lemma 1 remains true.

Remark 2. According to Lemma 1 we have immediately the following theorem on the uniqueness for the initial problem (1), (2).

Theorem 1. *Let $f(X, u)$ be a function defined and continuous on E and satisfying*

$$\text{(9)} \qquad |f(X, u) - f(X, v)| \leq \frac{K}{x_1^{k_1} \ldots x_m^{k_m}} |u - v|, \; K > 0$$

for (X, u), $(X, v) \in E^0$ *and*

$$\text{(10)} \qquad |f(X, u) - f(X, v)| \leq C|u - v|^{\alpha}, \; C > 0, \; 0 < \alpha < 1$$

in the domain E, where

$$\text{(11)} \qquad k_j \sqrt[m]{K}(1 - \alpha) < k_j - (k_j - 1)(1 - \alpha), \; k_j \geq 1, \; j = 1, \ldots, m.$$

Then the initial problem (1), (2) *has at most one solution.*

Remark 3. The following example shows that if none of the conditions from (11) are satisfied Theorem 1 fails. Let us consider the initial problem

(I) $$D_x^m D_y^n u = f_1(x, y, u), \ (x, y) \in (0, 1\rangle \times (0, 1\rangle$$

(II) $$[D_x^i u(x, y)]_{x=0} = [D_y^j u(x, y)]_{y=0} = 0, \ x \in \langle 0, 1\rangle, \ y \in \langle 0, 1\rangle$$
$$i = 1, \ldots, m-1; \ j = 1, \ldots, n-1,$$

where

$$f_1(x, y, u) = \begin{cases} 0 & \text{for } -\infty < u \leq 0 \\ \dfrac{\prod_{i=0}^{m-1}(m\sqrt{K}+i)\prod_{j=0}^{n-1}(n\sqrt{K}+j)}{x^m y^n} u & \text{for } 0 < u \leq \\ \leq \left[\dfrac{x^m y^n}{\prod_{i=0}^{m-1}(m\sqrt{K}+i)\prod_{j=0}^{n-1}(n\sqrt{K}+j)}\right]^{1/1-\alpha} & \\ \left[\dfrac{x^m y^n}{\prod_{i=0}^{m-1}(m\sqrt{K}+i)\prod_{j=0}^{n-1}(n\sqrt{K}+j)}\right]^{\alpha/1-\alpha} & \text{for } u > \\ > \dfrac{x^m y^n}{\prod_{i=0}^{m-1}(m\sqrt{K}+i)\prod_{j=0}^{n-1}(n\sqrt{K}+j)} & \end{cases}$$

for $(x, y) \in D = \langle 0, 1\rangle \times \langle 0, 1\rangle$ and $K > 0$, $0 < \alpha < 1$. By easy calculations we can find out that the above defined function f_1 is continuous and bounded in the domain $E = D \times \{-\infty < u < +\infty\}$ and satisfies the conditions:

$$|f_1(x, y, u) - f_1(x, y, v)| \leq \frac{\prod_{i=0}^{m-1}(m\sqrt{K}+i)\prod_{j=0}^{n-1}(n\sqrt{K}+j)}{x^m y^n} |u - v|$$

on $\{0 < x \leq 1\} \times \{0 < y \leq 1\} \times \{-\infty < u, v < +\infty\}$ and

$$|f_1(x, y, u) - f_1(x, y, v)| \leq |u - v|^\alpha$$

in the domain E. If furthermore $m\sqrt{K}(1-\alpha) < m - (m-1)(1-\alpha)$ and $n\sqrt{K}(1-\alpha) < n - (n-1)(1-\alpha)$, then there exists one and only one solution $u(x, y) = 0$ of the given initial problem (I), (II).

If $m\sqrt{K}(1-\alpha) \geq m - (m-1)(1-\alpha)$ and $n\sqrt{K}(1-\alpha) \geq n - (n-1)(1-\alpha)$, then it is also obvious that the function

$$u_1(x, y) = [\prod_{i=0}^{m-1}(m\sqrt{K} + i)\prod_{j=0}^{n-1}(n\sqrt{K} + j)]^{-1/1-\alpha} x^{m\sqrt{K}+m-1} y^{n\sqrt{K}+n-1}$$

satisfies equation (I) and zero conditions (II). In this case we have at least two solutions of the given initial problem: $u(x, y) = 0$ and $u(x, y) = u_1(x, y)$.

3. The convergence of successive approximations.

Theorem 2. *If the function $f(X, u)$ defined, continuous and bounded on E fulfils the conditions* (9), (10), (11) *from Theorem* 1, *then the Picard sequence of successive approximations* (4) *uniformly converges to the unique solution of the problem* (1), (2) *in the domain R.*

Proof. Let us put $M = \sup |f(X, u)|$. Then, in view of (4)

$$|u_{n+1}(X) - u_{n+1}(Y)| \leq |G(X) - G(Y)| +$$

$$+ |\sum_{i=1}^{m} \int_{\Sigma_{y_1}^{k_1}} d\mu_1 \dots \int_{\Sigma_{y_{i-1}}^{k_{i-1}}} d\mu_{i-1} \int_{y_i}^{x_i} d\varphi_1 \int_0^{\varphi_1} d\varphi_2 \dots \int_0^{\varphi_{k_i-1}} d\xi_i \times \int_{\Sigma_{x_{i+1}}^{k_{i+1}}} d\mu_{i+1} \dots \int_{\Sigma_{x_m}^{k_m}} f(\Xi, u_n) d\mu_m| \leq$$

$$\leq |G(X) - G(Y)| + M\sum_{i=1}^{m}\left\{|x_i - y_i| \frac{A_i^{k_i-1}}{(k_i - 1)!} \prod_{\substack{j=1 \\ j\neq i}}^{m} \frac{A_j^{k_j}}{k_j!}\right\}$$

for $Y = (y_1, \dots, y_m)$, $X \in R$, $n = 0, 1, \dots$, where $u_0(X) \in C(R)$. Moreover

$$|u_{n+1}(X)| \leq \max_{X\in R} |G(X)| + M \prod_{i=1}^{m} \frac{A_j^{k_j}}{k_j!}, \; X \in R.$$

The continuity of $G(X)$ and the above inequalities guarantee the equicontinuity and uniform boundedness of the sequence $\{u_n(X)\}_{n=1}^{\infty}$ defined by formula (4). We shall show that this sequence is uniformly convergent. The sequence of function $\{z_n(X)\}_{n=1}^{\infty}$ as

$$z_n(X) = \sup_{j=1,2,\dots} |u_{n+j}(X) - u_n(X)| \tag{12}$$

for $X \in R$ and $n = 0, 1, \dots$ is again a sequence of equicontinuous and uniformly bounded functions. In any point $X \in R$ there exists

$$\limsup_{n\to\infty} z_n(X) = z(X) \tag{13}$$

and furthermore, from the equicontinuity of the sequence $\{z_n(X)\}_{n=1}^{\infty}$ it follows that the function $z(X)$ is continuous in the domain R. The hypotheses (9) and (10) enable us to make the following statements:

$$|u_{n+m+1}(X) - u_{n+1}(X)| \leq \int\limits_{\Sigma_{x_1}^{k_1}} d\mu_1 \ldots \int\limits_{\Sigma_{x_m}^{k_m}} |f(\Xi, u_{n+m}) - f(\Xi, u_n)|\, d\mu_m \leq$$

$$\leq \int\limits_{\Sigma_{x_1}^{k_1}} d\mu_1 \ldots \int\limits_{\Sigma_{k_m}^{x_m}} \frac{K}{\xi_1^{k_1} \ldots \xi_m^{k_m}} |u_{n+m}(\Xi) - u_n(\Xi)|\, d\mu_m$$

and

$$|u_{n+m+1}(X) - u_{n+1}(X)| \leq \int\limits_{\Sigma_{x_1}^{k_1}} d\mu_1 \ldots \int\limits_{\Sigma_{x_m}^{k_m}} C|u_{n+m}(\Xi) - u_n(\Xi)|\, d\mu_m .$$

Hence in view of (12) and (13) we obtain

$$z(X) \leq \int\limits_{\Sigma_{x_1}^{k}} d\mu_1 \ldots \int\limits_{\Sigma_{x_m}^{k_m}} \frac{K}{\xi_1^{k_1} \ldots \xi_m^{k_m}} z(\Xi) d\mu_m$$

$$z(X) \leq \int\limits_{\Sigma_{x_1}^{k_1}} d\mu_1 \ldots \int\limits_{\Sigma_{x_m}^{k_m}} C z^{\alpha}(\Xi) d\mu_m .$$

Since the function $z(X)$ is continuous and $z(X) \geq 0$ on R the assumptions of Lemma 1 are satisfied. Hence $z(X) \equiv 0$ in the domain R. This fact guarantees the validity of the equalities

$$\limsup_{n\to\infty} z_n(X) = \liminf_{n\to\infty} z_n(X) = \lim_{n\to\infty} z_n(X) = 0 \tag{14}$$

in every point $X \in R$. The equicontinuity of the sequence of functions $\{z_n(X)\}_{n=1}^{\infty}$ and the equality (14) secure the uniform convergence of this sequence to zero in the domain R. We may conclude from (12) and (14) that sequence $\{u_n(X)\}_{n=1}^{\infty}$ is also uniformly convergent on R. Denoting $u(X) = \lim\limits_{n\to\infty} u_n(X)$ we get

$$u(X) = G(X) + \int\limits_{\Sigma_{k_1}^{x_1}} d\mu_1 \ldots \int\limits_{\Sigma_{x_m}^{k_m}} f[\Xi, u(\Xi)] d\mu_m .$$

The proof is given.

Remark 4. In this remark an example showing that the assertion of Theorem 2 need not hold if $k_j \sqrt[m]{K}(1-\alpha) \geq k_j - (k_j - 1)(1-\alpha)$, $k_j \geq 1$ for $j = 1, \ldots, m$, is given.

In fact, it is sufficient to take the initial problem

$$D_{x_1}^{k_1} \ldots D_{k_m}^{x_m} u = f_2(X, u),\ X \in R^0,\ k_i \geq 1,\ i = 1, \ldots, m, \tag{1'}$$

$$[D_{x_r}^{i_r} u(X)]_{x_r=0} = 0,\ X_r \in R_r,\ i_r = 0,1, \ldots, k_r - 1,\ r = 1, \ldots, m , \tag{2'}$$

where

$$
f_2(X,u)=\begin{cases}
\left[\dfrac{x_{x_1}^{k_1}\ldots x_{x_m}^{k_m}}{\prod\limits_{i=1}^{m}\prod\limits_{j=0}^{k_i-1}(k_i\sqrt[m]{K}+j)}\right]^{\alpha/1-\alpha} & \text{for} \quad -\infty<u\le 0\\[2ex]
\left[\dfrac{x_{x_1}^{k_1}\ldots x_{x_m}^{k_m}}{\prod\limits_{i=1}^{m}\prod\limits_{j=0}^{k_i-1}(k_i\sqrt[m]{K}+j)}\right]^{\alpha/1-\alpha}-\dfrac{\prod\limits_{i=1}^{m}\prod\limits_{j=0}^{k_i-1}(k_i\sqrt[m]{K}+j)}{x_1^{k_1}\ldots x_m^{k_m}}\,u & \\[2ex]
\qquad\qquad \text{for} \quad 0<u\le\left[\dfrac{x_1^{k_1}\ldots x_m^{k_m}}{\prod\limits_{i=1}^{m}\prod\limits_{j=0}^{k_i-1}(k_i\sqrt[m]{K}+j)}\right]^{1/1-\alpha} & \\[2ex]
0 \qquad\qquad \text{for} \quad u>\left[\dfrac{x_1^{k_1}\ldots x_m^{k_m}}{\prod\limits_{i=1}^{m}\prod\limits_{j=0}^{k_i-1}(k_i\sqrt[m]{K}+j)}\right]^{1/1-\alpha} &
\end{cases}
$$

and $K>0$, $0<\alpha<1$. We find out by easy calculations that the function $f_2(X,u)$ is continuous, bounded in the domain E and satisfies the conditions

$$
|f_2(X,u)-f_2(X,v)|\le\frac{\prod\limits_{i=1}^{m}\prod\limits_{j=0}^{k_i-1}(k_i\sqrt[m]{K}+j)}{x_1^{k_1}\ldots x_m^{k_m}}\,|u-v|
$$

$$
|f_2(X,u)-f_2(X,v)|\le|u-v|^{\alpha}
$$

for (X,u), $(X,v)\in E$. Now, let us count the successive approximations (4) for $G(X)=G_0(X)=0$ and $u_0(X)=0$.

a) In the case $k_j\sqrt[m]{K}(1-\alpha)<k_j-(k_j-1)(1-\alpha)$ for $j=1,\ldots,m$ we obtain for each $n=1,2,\ldots$

$$
u_n(X)=\left[\frac{x_1^{k_1}\ldots x_m^{k_m}}{\prod\limits_{i=1}^{m}\prod\limits_{j=0}^{k_i-1}(k_i\sqrt[m]{K}+j)}\right]^{1/1-\alpha}\times
$$

$$
\times\sum_{l=1}^{m}(-1)^{l-1}\left[\prod_{i=1}^{m}\prod_{j=0}^{k_i-1}\frac{(1-\alpha)(k_i\sqrt[m]{K}+j)}{k_i\alpha+(j+1)(1-\alpha)}\right]^{l},
$$

because $\dfrac{(1-\alpha)(k_i \sqrt[m]{K}+j)}{k_i\alpha+(j+1)(1-\alpha)} < 1$ for all $j = 0, 1, \ldots, k_i - 1$ and $i = 1, \ldots, m$. Further, it is obvious that

$$u(X) = \lim_{n\to\infty} u_n(X) = \prod_{i=1}^{m}\prod_{j=0}^{k_i-1} \frac{(1-\alpha)(k_i \sqrt[m]{K}+j)}{k_i\alpha+(j+1)(1-\alpha)} \times$$

$$\times\left[1+\prod_{i=1}^{m}\prod_{j=0}^{k_i-1} \frac{(1-\alpha)(k_i \sqrt[m]{K}+j)}{k_i\alpha+(j+1)(1-\alpha)}\right]^{-1}\left[\frac{x_1^{k_1}\ldots x_m^{k_m}}{\prod_{i=1}^{m}\prod_{j=0}^{k_i-1}(k_i \sqrt[m]{K}+j)}\right]^{1/1-\alpha}.$$

It can be verified immediately, substituting the above function into equation (1') that $u(X)$ is the unique solution of the problem (1'), (2').

b) If $k_j \sqrt[m]{K}(1-\alpha) \geq k_j - (k_j - 1)(1-\alpha)$ for $j = 1, \ldots, m$, then the successive Picard approximations are given as follows:

$$v_n(X) = \frac{1-(-1)^n}{2}\left[\prod_{i=1}^{m}\prod_{j=0}^{k_i-1} \frac{(1-\alpha)(k_i \sqrt[m]{K}+j)}{k_i\alpha+(j+1)(1-\alpha)}\right]\left[\frac{x_1^{k_1}\ldots x_m^{k_m}}{\prod_{i=1}^{m}\prod_{j=0}^{k_i-1}(k_i \sqrt[m]{K}+j)}\right]^{1/1-\alpha}$$

for $n = 0, 1, \ldots$ because of $\dfrac{(1-\alpha)(k_i \sqrt[m]{K}+j)}{k_i\alpha+(j+1)(1-\alpha)} \geq 1$ for each $j = 0, 1, \ldots, k_i - 1$ and $i = 1, \ldots, m$. We can easily see that the sequence of the approximations $\{v_n(X)\}_{n=1}^{\infty}$ does not converge.

Perhaps it is worth mentioning that the results contained in Theorem 2 may be transferred into a system of equations of type (1).

Let $\mathbf{U}(X) = (u_1(X), \ldots, u_p(X))$ denote a vector function defined in the domain R and let $\mathbf{F}(X, \mathbf{U}) = (f_1(X, \mathbf{U}), \ldots, f_p(X, \mathbf{U}))$ be also a vector function of $m+p$ variables defined on $H = R \times \prod_{i=1}^{p}\{-\infty < u_i < +\infty\}$. Suppose that the vector function $\Phi_r^{(i_r)}(X_r) = ({}^1\varphi_r^{(i_r)}(X_r), \ldots, {}^p\varphi_r^{(i_r)}(X_r))$ is sufficiently regular on R_r for each $i_r = 0, 1, \ldots, k_r - 1$ and $r = 1, \ldots, m$.

Then we define the initial problem

$$(15)\qquad D_{x_1}^{k_1}\ldots D_{x_m}^{k_m}\mathbf{U} = \mathbf{F}(X, \mathbf{U}),\ X \in R^0,\ k_i \geq 1,\ i = 1, \ldots, m$$

$$(16)\qquad [D_{x_r}^{i_r}\mathbf{U}(X)]_{x_r=0} = \Phi_r^{(i_r)}(X_r),\ X_r \in R_r,\ i_r = 0, 1, \ldots, k_r - 1;\ r = 1, \ldots, m$$

$$[D_{x_s}^{j_s}\Phi_r^{(i_r)}(X_r)]_{x_s=0} = [D_{x_r}^{i_r}\Phi_s^{(j_s)}(X_s)]_{x_r=0},\ X_{rs} \in R_{rs}$$

$$r \neq s,\ i_r = 0, 1, \ldots, k_r - 1;\ j_s = 0, 1, \ldots k_s - 1;\ r, s = 1, \ldots, m$$

and its solution as a regular vector function in the domain R satisfying the conditions (15), (16).

The initial problem thus defined is equivalent to a system of integral equations of type (3).

If $||\mathbf{B}||$ denotes the norm of the vector $\mathbf{B} = (b_1, \ldots, b_p)$ given as the sum of the absolute values of its coordinates or an arbitrary norm equivalent (in the sense of the convergence) to the first one, then:

Theorem 3. *Let the vector function* $\mathbf{F}(X, \mathbf{U})$ *be defined, continuous, bounded on* H *and let the following conditions be satisfied*:

$$\text{(17)} \qquad ||\mathbf{F}(X, \mathbf{U}) - \mathbf{F}(X, \mathbf{V})|| \leq \frac{K}{x_1^{k_1} \ldots x_m^{k_m}} ||\mathbf{U} - \mathbf{V}||,\ K > 0$$

in the domain $H^0 = R^0 \times \prod_{i=1}^{p} \{-\infty < u_i < +\infty\}$

$$\text{(18)} \qquad ||\mathbf{F}(X, \mathbf{U}) - \mathbf{F}(X, \mathbf{V})|| \leq C||\mathbf{U} - \mathbf{V}||^{\alpha},\ C > 0,\ 0 < \alpha < 1$$

for $(X, \mathbf{U}), (X, \mathbf{V}) \in H$, *where the constants* k_j, K, α *and* m *are connected by the relation* (11) *from Theorem* 2. *Then the initial problem* (15), (16) *has one and only one regular solution which may be expressed as the limit of the uniformly convergent sequence* $\{\mathbf{U}_n(X)\}_{n=1}^{\infty}$ *of the Picard approximations*

$$\mathbf{U}_{n+1}(X) = \bar{\mathbf{G}}_0(X) + \int_{\Sigma_{x1}^{k1}} \mathrm{d}\mu_1 \ldots \int_{\Sigma_{xm}^{km}} \mathbf{F}[\Xi, \mathbf{U}_n(\Xi)]\mathrm{d}\mu_m,$$

where $\mathbf{U}_0(X)$ *is an arbitrary continuous vector function on* R *and the vector function* $\bar{\mathbf{G}}_0(X)$ *is given by the equation*

$$\bar{\mathbf{G}}_0(X) = \sum_{j=1}^{m} \sum_{i_1, \ldots, i_j} \sum_{l_1, \ldots, l_j} (-1)^{j-1} \frac{x_{i_1}^{l_1} \ldots x_{i_j}^{l_j}}{l_1! \ldots l_j!} [D_{x_{i_1}}^{l_1} \ldots D_{x_{i_j}}^{l_j} \mathbf{U}(X)]_{\substack{x_{i_1}=0 \\ \vdots \\ x_{i_j}=0}}$$

for $0 \leq l_1 \leq k_{i_1} - 1, \ldots, 0 \leq l_j \leq k_{i_j} - 1$ *and any combination* $(i_1, \ldots, i_j)$ *of the* m *natural numbers* $(1, \ldots, m)$ j *at a time* $(i_1 < \ldots < i_j)$.

The proof of this theorem coincides formally with that, of Theorem 2 and will be omitted.

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ERRATA

E. Jucovič, *Corrections of my paper „On a problem in map colouring"*, Mat. časop. *19* (1969), 225—227.

(1) In the 12th line on p. 225 there should be $\geqq 3$ instead of $\leqq 3$.

(2) The map P in Theorem 3 on p. 226 is supposed to be on the surface of the sphere.

(3) H. Izbicki lectured on simultaneous colouring of elements of a map also at the colloquium on graph theory in Manebach 1967. See his paper *Verallgemeinerte Farbenzahlen* in *Beiträge zur Graphentheorie*, Leipzig 1968, pp. 81—83.

(3) Recently much work was done with simultaneous colouring of edges and vertices of graphs (total colouring). For detailed references see M. Behzad, *The Total Chromatic Number of a Graph*: A Survey, Proceedings of the 1969 Conference on Combinatorial Mathematics Held in Oxford, England, Academic Press 1970.