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THE LARGEST PROPER REGULAR IDEAL OF $S(X)$

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1. Introduction

An ideal of a semigroup S is regular if it is a regular subsemigroup of S. It is immediate that if a semigroup S with identity contains a proper regular ideal, then it has a largest proper regular ideal. It is simply the union of all proper regular ideals. Furthermore, suppose J is an ideal of S and every element of J is a regular element of S. Then every element of J is also regular in J. Specifically, if $a \in J$ is regular in S, then $axa = a$ for some $x \in S$. But then $a(xax)a = a$ and $xax \in J$. If a topological space X has more than one point, then $S(X)$, the semigroup of all continuous selfmaps of X, has a largest proper regular ideal. Our task here is to describe the elements in that ideal. This problem was suggested to us by F. Pastijn who, together with D. Hardy, solved it for the semigroup of all binary relations on a set in [2].

Throughout this paper, $LPR(X)$ will denote the largest proper regular ideal of $S(X)$ and $K(X)$ will denote the kernel of $S(X)$ which consists of all the constant functions. It is evident that $K(X)$ is a regular ideal. Consequently, $LPR(X)$ exists and contains $K(X)$ if X has more than one point. In all that follows, $\operatorname{Ran}(f)$ will denote the range of a function f. Section 2 is devoted to an explicit description of $LPR(X)$ for a great many spaces and we investigate some of the properties of the semigroup $LPR(X)$ in Section 3.

2. A DESCRIPTION OF THE ELEMENTS IN $LPR(X)$

Theorem 2.1. *Let X be a completely regular Hausdorff space which contains an arc and whose components are open. Then $LPR(X) = K(X)$.*

Proof. We must show that each $f \in LPR(X)$ is constant. We do this in stages and we first show that

$$f \text{ is constant on components.} \tag{2.1.1}$$

Suppose f is not constant on some component C of X. Choose $a, b \in C$ such that $f(a) \neq f(b)$. Then there exists a continuous function g from X into $I = [0, 1]$ such that $g(a) = 0$ and $g(b) = 1$. Let h be any continuous surjective selfmap of I which is not injective on any subarc of I. For example, any surjective, continuous, nowhere differentiable selfmap will serve the purpose. Then let k be a homeomorphism from I onto a subarc A of X and note that $k \circ h \circ g \in S(X)$. According to Theorem 3.1 of [6], an element $t \in S(X)$ is regular if and only if $\mathrm{Ran}(t)$ is a retract of X and t maps some subspace homeomorphically onto $\mathrm{Ran}(t)$. Now, $\mathrm{Ran}(k \circ h \circ g \circ f) = A$ is certainly a retract of X. However, $k \circ h \circ g \circ f$ cannot map anything homeomorphically onto A since h is not injective on any subarc of I. Hence, $k \circ h \circ g \circ f \notin LPR(X)$ which means $f \notin LPR(X)$. This contradiction means that (2.1.1) must hold.

We next prove

$$|\,\mathrm{Ran}(f) \cap C| \leqslant 1 \quad \text{for each component } C. \tag{2.1.2}$$

Suppose $\mathrm{Ran}(f) \cap C$ contains two distinct points a and b for some component C and suppose f maps Y bijectively onto $\mathrm{Ran}(f)$. Since f is constant on components, it follows that distinct points in Y must lie in distinct components and since the components are open, this means that Y is a discrete subspace of X. However, $\mathrm{Ran}(f)$ is not discrete. To see this, note that Theorem 3.1 of [6] tells us that $\mathrm{Ran}(f)$ must be a retract and this means that there is an idempotent continuous selfmap v of X such that $\mathrm{Ran}(v) = \mathrm{Ran}(f)$. Since $v(a) = a$ and $v(b) = b$, we see that $v[C]$ is a nondegenerate connected subspace of X which is contained in $R(f)$. Evidently, $\mathrm{Ran}(f)$ is not discrete so that f cannot possibly map any subspace Y homeomorphically onto $\mathrm{Ran}(f)$. According to Theorem 3.1 of [6], f is not regular and we again have a contradiction. Thus, we have verified (2.1.2). Now we show, by way of contradiction, that f is constant. We know that f must satisfy both (2.1.1) and (2.1.2). Suppose, however, that f is not constant. Then $f(a) \neq f(b)$ for some $a, b \in X$. This means that $f(a)$ and $f(b)$ must lie in different components since f satisfies (2.1.2). Let

C_a, C_b denote the components containing a and b respectively. Choose two distinct points c and d in the arc $A \subseteq X$ and define a function g by

$$g(x) = \begin{cases} c & \text{for } x \in C_a, \\ d & \text{for } x \in C_b, \\ x & \text{for } x \notin C_a \cup C_b. \end{cases}$$

The restriction of g to each component is continuous and since components are open, this means that g is a continuous selfmap of X. However, $g \circ f$ does not satisfy condition (2.1.2) since the component which contains the arc A contains two points of $\operatorname{Ran}(g \circ f)$. This means $g \circ f \notin LPR(X)$ and hence, $f \notin LPR(X)$ since $LPR(X)$ is an ideal of $S(X)$. This is, again, a contradiction and we conclude that f, is indeed, constant. This completes the proof. □

Corollary 2.2. *Let X be a completely regular, locally connected Hausdorff space which contains an arc. Then $LPR(X) = K(X)$.*

Proof. The components are open in a locally connected space. □

One easily verifies that $K(X)$ is the smallest ideal of $S(X)$ for any space X whatsoever. So we have just shown that the largest proper regular ideal of $S(X)$ coincides with the smallest ideal of $S(X)$ for a great many spaces X. The situation is different, however, for 0-dimensional spaces. We need to recall some terminology. A *clopen* set is one which is both closed and open. A space X is *totally separated* if for each pair of distinct points $a, b \in X$, there exists a clopen set containing a but not b and it is *0-dimensional* if it has a basis of clopen sets. A space is *realcompact* if it is homeomorphic to a closed subspace of a product of real lines. One may consult [1] for further information about realcompact spaces, which turn out to be quite abundant. For example, Theorem 15.24 of [1] tells us that every metric space of nonmeasurable cardinal is realcompact. Finally, we denote by $FR(X)$, the family of all functions in $S(X)$ with finite range. Our next result gives us some information about $FR(X)$.

Theorem 2.3. *$FR(X)$ is an ideal of $S(X)$ and for disconnected X, it is a regular ideal if and only if X is totally separated.*

Proof. It is immediate that $FR(X)$ is an ideal of $S(X)$ for any space X. Suppose X is totally separated and $f \in FR(X)$. We must show that f is regular. Let $A = \{a_n\}_{n=1}^{N}$ be any finite subset of X. Choose n such that $1 \leqslant n \leqslant N$. For each $j \neq n$, there exists a clopen set H_j containing a_n but not a_j. Let $G_n = \bigcap\{H_j\}_{j \neq n}$. For each n, G_n is a clopen set containing a_n but not a_j for $j \neq n$. Then let $V_n = (X \setminus \bigcup\{G_j\}_{j \neq n}) \cap G_n$. The sets V_n are clopen, mutually disjoint and $a_n \in V_n$

for each n. Define a selfmap w of X by

$$w(x) = \begin{cases} a_n & \text{for } x \in V_n,\ 1 \leqslant n \leqslant N, \\ a_1 & \text{for } x \in X \setminus \bigcup\{V_n\}_{n=1}^N. \end{cases}$$

It is immediate that w is continuous and hence, is an idempotent element of FR(X) with the property that Ran$(w) = A$. We have shown that every finite subset of X is a retract of X. Thus, Ran(f) is a retract of X. For each $y \in$ Ran(f), choose $x_y \in f^{-1}(y)$ and let $B = \{x_y : y \in \text{Ran}(f)\}$. Since both B and Ran(f) are discrete, it follows that f maps B homeomorphically onto Ran(f) and it follows from Theorem 3.1 of [6] that f is regular.

Suppose, conversely, that $FR(X)$ is a regular ideal of $S(X)$ and let a and b be any two distinct points of X. Since X is not connected, it contains a proper, nonempty clopen subset H. Define

$$f(x) = \begin{cases} a & \text{for } x \in H, \\ b & \text{for } x \in X \setminus H. \end{cases}$$

Since f is regular, Ran(f) is a retract of X by Theorem 3.1 of [6]. That is, there exists a continuous idempotent selfmap, w, of X such that Ran$(w) =$ Ran$(f) = \{a, b\}$. Then $w^{-1}(a)$ is a clopen set containing a but not b. □

To help with the proof of our next theorem, we will first establish a lemma.

Lemma 2.4. *Suppose X is a realcompact, noncompact, 0-dimensional metric space. Then X can be decomposed into an infinite number of clopen sets.*

Proof. Since X is both realcompact and noncompact, there exists a positive unbounded continuous function f from X to the reals R. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of distinct points in X such that $f(x_n) = r_n$ where $r_n + 2 < r_{n+1}$ for each n. Let $G_n = (r_n - 1, r_n + 1)$ and choose a clopen set H_n such that $x_n \in H_n \subseteq f^{-1}[G_n]$. Evidently, $W = \bigcup\{H_n\}_{n=1}^{\infty}$ is open. We assert that it is also closed. Suppose $x \notin W$. Then $x \notin H_n$ for all n.

Case 1: $x \in f^{-1}[G_m]$ for some m.

Then $V = (f^{-1}[G_m]) \setminus H_m$ is a neighborhood of x such that $V \cap H_n = \emptyset$ for all n which means $V \cap W = \emptyset$.

Case 2: $x \notin f^{-1}[G_n]$ for all n.

Suppose $r_m + 1 < f(x) < r_{m+1} - 1$ for some m. Then choose a, b such that $r_m + 1 < a < f(x) < b < r_{m+1} - 1$ and $f^{-1}(a, b)$ is a neighborhood of x which doesn't intersect W. On the other hand, if $f(x) = r_m + 1$ for some m, then $V = (f^{-1}(r_m, r_m + 2)) \setminus H_m$ is a neighborhood of x such that $V \cap H_n = \emptyset$ for all n and we conclude that $V \cap W = \emptyset$. The same sort of argument applies if $f(x) = r_m - 1$ for some m and we have now shown that W is clopen. Consequently, $X \backslash W$ together with $\{H_n\}_{n=1}^{\infty}$ is an infinite family of clopen subsets of X which forms a decomposition of X. □

Theorem 2.5. *Let X be a realcompact 0-dimensional metric space which is not discrete. Then $LPR(X) = FR(X)$.*

Proof. In view of Theorem 2.3, we have $FR(X) \subseteq LPR(X)$ so we must show that $LPR(X) \subseteq FR(X)$.

Case 1: X is not compact.

Suppose $f \notin FR(X)$. We must show that $f \notin LPR(X)$.

Case 1.1: $\mathrm{Ran}(f)$ is not discrete.

Choose a countably infinite subset $Y \subseteq \mathrm{Ran}(f)$ which is not closed and then choose $x_y \in f^{-1}(y)$ for each $y \in \mathrm{Ran}(f)$. By Lemma 2.4, there exists a decomposition $\{H_y\}_{y \in Y}$ of X into mutally disjoint nonempty clopen subsets. Define $g(x) = x_y$ for each $x \in H_y$. Then $g \in S(X)$. However, $\mathrm{Ran}(f \circ g) = Y$ cannot be a retract of X since it is not closed and Theorem 3.1 of [6] assures us that $f \circ g$ is not regular. Since $f \circ g \notin LPR(X)$, we must also have $f \notin LPR(X)$.

Case 1.2: $\mathrm{Ran}(f)$ is discrete.

Since $f \notin FR(X)$, $\mathrm{Ran}(f)$ is also (countably) infinite. We may assume that f is regular since otherwise, it is immediate that $f \notin LPR(X)$. Then $\mathrm{Ran}(f)$ is closed since it is a retract of X. Choose a closed countable nondiscrete subspace Y of X. Let g map $\mathrm{Ran}(f)$ onto Y and extend to a continuous map of X onto Y by Corollary 3, page 281 of [3]. Let $h = g \circ f$. Suppose h maps some subset B of X bijectively onto $\mathrm{Ran}(h) = Y$. Now $\{f^{-1}(y) \colon y \in \mathrm{Ran}(f)\}$ is a decomposition of X into mutually disjoint nonempty clopen subsets since $\mathrm{Ran}(f)$ is discrete. Let any $b \in B$ be given. Then $b \in f^{-1}(y)$ for some $y \in \mathrm{Ran}(f)$ and thus, $\{b\} = B \cap f^{-1}(y)$. That is, b is an isolated point of B. Consequently, all points of B are isolated and B is discrete. This means that h cannot possibly map B homeomorphically onto its range Y since Y is not discrete. Then h is not regular by Theorem 3.1 of [6] and since $h \notin LPR(X)$, we must have $f \notin LPR(X)$.

Case 2: X is compact.

We also divide Case 2 into two subcases.

Case 2.1: X has exactly one cluster point.

Then $S(X)$ is regular according to the theorem in [4]. Thus, $LPR(X)$ coincides with the largest proper ideal of $S(X)$ and according to Theorem 4.10 of [5], this consists of all those functions $f \in S(X)$ such that if A and B are two retracts of X, both homeomorphic to X, then f does not map A homeomorphically onto B. But one readily verifies that this is the case if and only if $\mathrm{Ran}(f)$ is finite.

Case 2.2: X has more than one cluster point.

Suppose $f \notin FR(X)$. Again, we must show that $f \notin LPR(X)$. Choose a compact subset $A \subseteq \mathrm{Ran}(f)$ with exactly one cluster point p. Then $A = \{p\} \cup \{a_n\}_{n=1}^{\infty}$ where $\lim_{n\to\infty} a_n = p$. Now choose $b_n \in f^{-1}(a_n)$ for each n. Then $\{b_n\}_{n=1}^{\infty}$ has a convergent subsequence which converges to some point q and there is no loss of generality if we assume that subsequence is $\{b_n\}_{n=1}^{\infty}$ itself. Let $B = \{q\} \cup \{b_n\}_{n=1}^{\infty}$. Now choose a compact subset C, disjoint from B with exactly one cluster point r. Then $C = \{r\} \cup \{c_n\}_{n=1}^{\infty}$ and $\lim_{n\to\infty} c_n = r$. Choose mutually disjoint clopen subsets $\{G_n\}_{n=1}^{\infty}$ and $\{H_n\}_{n=1}^{\infty}$ such that $G_n \cap B = \{b_n\}$, $H_n \cap C = \{c_n\}$, $\lim_{n\to\infty} \mathrm{Diam}\, G_n = 0$ and $\lim_{n\to\infty} \mathrm{Diam}\, H_n = 0$ where *Diam* means diameter. Now define a selfmap g of X by

$$g(x) = \begin{cases} b_{2n-1} & \text{for } x \in G_n, \\ b_{2n} & \text{for } x \in H_n, \\ q & \text{for } x \in X \setminus \bigcup\{G_n \cup H_n\}_{n=1}^{\infty}. \end{cases}$$

We need to show that g is continuous. First of all, g is continuous at all points of G_n and H_n since these sets are clopen. Let a be any point of $X \setminus \bigcup\{G_n \cup H_n\}_{n=1}^{\infty}$ and let V be any neighborhood of $g(a) = q$. Then there is a positive integer N such that $b_n \in V$ for $n > N$. Let $x \in V \setminus \bigcup\{G_n \cup H_n\}_{n=1}^{N}$. If $x \in G_n$ for some $n > N$, we have $g(x) = b_{2n-1} \in V$ and if $x \in H_n$ for some $n > N$, we have $g(x) = b_{2n} \in V$. The remaining case is where $x \notin G_n \cup H_n$ for all $n > N$ and hence $x \notin G_n \cup H_n$ for all n. For this case, we have $g(x) = q \in V$. Thus, $V \setminus \bigcup\{G_n \cup H_n\}_{n=1}^{N}$ is a neighborhood of a which g maps into V and we conclude that g is continuous at all points of X. That is, $g \in S(X)$. We then see that

$$f \circ g(x) = \begin{cases} a_{2n-1} & \text{for } x \in G_n, \\ a_{2n} & \text{for } x \in H_n, \\ p & \text{for } x \in X \setminus \bigcup\{G_n \cup H_n\}_{n=1}^{\infty}. \end{cases}$$

Thus, $\mathrm{Ran}(f \circ g) = A$. Suppose $f \circ g$ maps some subset Y bijectively onto A. Then $Y \cap G_n = \{d_n\}$ and $Y \cap H_n = \{s_n\}$ for each n. Let any ε be given and let $N(q, \varepsilon) = \{x \in X : \delta(q, x) < \varepsilon\}$ where $\delta(q, x)$ denotes the distance between q and x. There exists a positive integer N_1 such that $b_n \in N(q, \varepsilon/2)$ for $n > N_1$ and since $\lim_{n\to\infty} \mathrm{Diam}\, G_n = 0$, there exists a positive integer N_2 such that $\mathrm{Diam}\, G_n < \varepsilon/2$ for $n > N_2$. Let $N_3 = \max\{N_1, N_2\}$. For $n > N_3$, we have $\delta(q, d_n) \leqslant \delta(q, b_n) + \delta(b_n, d_n) < \varepsilon$. Thus $d_n \in N(q, \varepsilon)$ for $n > N_3$ and we see that $\lim_{n\to\infty} d_n = q$. In the same manner, one shows that $\lim_{n\to\infty} s_n = r$ and we see that both q and r are cluster points of Y. But $Y = \{d_n\}_{n=1}^{\infty} \cup \{s_n\}_{n=1}^{\infty} \cup \{y\}$ where $y \in X \setminus \bigcup\{G_n \cup H_n\}_{n=1}^{\infty}$. Evidently, Y cannot contain both q and r and is, therefore, not closed. Consequently, $f \circ g$ cannot map Y homeomorphically onto $\mathrm{Ran}(f \circ g)$ and according to Theorem (3.1) of [6], $f \circ g$ is not regular. Since $f \circ g \notin LPR(X)$, we must have $f \notin LPR(X)$ and the proof is finally complete. □

Theorem 2.6. *Let X be discrete. Then $LPR(X)$ consists of all those selfmaps f of X such that $|\mathrm{Ran}(f)| < |X|$.*

Proof. $LPR(X)$ coincides with the largest proper ideal, $M(S(X))$, of $S(X)$ since $S(X)$ is regular. According to Theorem 4.10 of [5], $M(S(X))$ consists of all those functions f such that if A and B are any two retracts of X, both homeomorphic to X, then f does not map A homeomorphically onto B. Since X is discrete, it is immediate that $f \in LPR(X)$ if and only if $|\mathrm{Ran}(f)| < |X|$. □

3. Some properties of the semigroup $FR(X)$

The semigroup $FR(X)$ is not very sensitive for distinguishing between spaces when the spaces are not 0-dimensional and Hausdorff. For example, let X and Y be any two connected spaces with the same cardinality. Then $FR(X) = K(X)$ and $FR(Y) = K(Y)$. But $K(X)$ and $K(Y)$ are left zero semigroups with identities whose cardinalities are equal, and hence, they are isomorphic. For another example, suppose $X = A \cup B$ and $Y = C \cup D$ where A and B are components of X, C and D are components of Y and $|A| = |C|$ and $|B| = |D|$. Then both $FR(X)$ and $FR(Y)$ contain many functions whose ranges consist of two elements. Let h map A bijectively onto C and B bijectively onto D. One can verify that the mapping φ which is defined by $\varphi(f) = h \circ f \circ h^{-1}$ is an isomorphism from $FR(X)$ onto $FR(Y)$. However the situation is quite different for 0-dimensional Hausdorff spaces as our next theorem shows.

Theorem 3.1. *Let X and Y be 0-dimensional Hausdorff spaces and let φ be an isomorphism from $FR(X)$ onto $FR(Y)$. Then there exists a homeomorphism h from X onto Y such that $\varphi(f) = h \circ f \circ h^{-1}$.*

Proof. The constant function which maps everything into the point x will be denoted by $\langle x\rangle$. It will be clear from context what the domain of $\langle x\rangle$ is. The left zeros of $FR(X)$ are precisely the constant functions so that φ must map $K(X)$ bijectively onto $K(Y)$. Define a bijection h from X onto Y by $h(x) = y$ if and only if $\varphi\langle x\rangle = \langle y\rangle$. It follows that $\varphi\langle x\rangle = \langle h(x)\rangle$ for all $x \in X$. For any $f \in FR(X)$, we have

$$\langle h(f(x))\rangle = \varphi\langle f(x)\rangle = \varphi(f \circ \langle x\rangle) = \varphi(f) \circ \varphi\langle x\rangle = \varphi(f) \circ \langle h(x)\rangle = \langle \varphi(f)(h(x))\rangle$$

which implies

$$\varphi(f) \circ h = h \circ f \text{ for all } f \in FR(X). \tag{3.1.1}$$

It follows immediately from (3.1.1) that $\varphi(f) = h \circ f \circ h^{-1}$ for each $f \in FR(X)$. It remains to show that h is a homeomorphism. Let $\mathscr{B}$ be a basis for Y which consists of clopen sets. Choose two distinct points p and q in Y and for each $B \in \mathscr{B}$ define a function $g_B \in FR(Y)$ by

$$g_B(x) = \begin{cases} p & \text{for } x \in B, \\ q & \text{for } x \in X \setminus B. \end{cases}$$

Let $B \in \mathscr{B}$ be given and let $\varphi(f) = g_B$. We then use (3.1.1) to get

$$\begin{aligned} h^{-1}[B] &= h^{-1}[g_B^{-1}(p)] = (g_B \circ h)^{-1}(p) \\ &= (\varphi(f) \circ h)^{-1}(p) = (h \circ f)^{-1}(p) = f^{-1}(h^{-1}(p)). \end{aligned}$$

Since h is bijective and $f \in FR(X)$, it follows that $f^{-1}(h^{-1}(p))$ is a clopen subset of X and we conclude that h is continuous. It follows that h^{-1} must also be continuous since φ^{-1} is an isomorphism from $FR(Y)$ onto $FR(X)$. Thus, h is a homeomorphism and the proof is complete. □

Corollary 3.2. *Let X and Y be 0-dimensional Hausdorff spaces. Then $FR(X)$ and $FR(Y)$ are isomorphic if and only if X and Y are homeomorphic.*

Proof. If $S(X)$ and $S(Y)$ are isomorphic, then X and Y are homeomorphic by the previous theorem. If h is a homeomorphism from X onto Y, then φ is an isomorphism from $S(X)$ onto $S(Y)$ where $\varphi(f) = h \circ f \circ h^{-1}$. □

In view of Corollary 3.2, there is a one-to-one correspondence between the class of all 0-dimensional spaces and their semigroups of continuous selfmaps with finite ranges. In other words, we have here a rather extensive class of mutually nonisomorphic semigroups. And yet, they all share a number of similar properties as the next several results indicate. We wish to determine Green's relations for these semigroups and this can be done directly without appeal to other results. But it can also be done quite easily by appealing to some general results already in the literature after a few definitions are introduced and this is the direction we choose to take. In what follows, $T(X)$ is any semigroup of selfmaps of a set X and $T(X)^1 = T(X) \cup \{\delta\}$ where δ is the identity map.

Definition 3.3. Let $A, B \in X$. An element $f \in T(X)^1$ is said to map A *T-isomorphically* onto B if $f[A] \subseteq B$ and there exists a $g \in T(X)^1$ such that $g[B] \subseteq A$, $f \circ g|B = \delta|B$ and $g \circ f|A = \delta|A$. In this case, we say that A and B are *T-isomorphic* and that $f|A$ is a T-isomorphism from A onto B.

Definition 3.4. A subset A of X is a *T-retract* of X if it is the range of an idempotent of $T(X)^1$.

Since we are dealing in this section with 0-dimensional Hausdorff spaces, the FR-retracts here are precisely the finite subsets of X and f maps a subset A FR-isomorphically onto a subset B if and only if both A and B are finite and f maps A bijectively onto B. Consequently, two subsets of X are FR-isomorphic if and only if they are finite and have the same number of elements. For any function f, let $\pi(f) = \{f^{-1}(y) : y \in \operatorname{Ran}(f)\}$. The symbols $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$ and $\mathscr{J}$ denote the usual Green's relations.

Theorem 3.5. *Let X be a 0-dimensional Hausdorff space and let $f, g \in FR(X)$. Then $f\mathscr{L}g$ if and only if $\pi(f) = \pi(g)$, $f\mathscr{R}g$ if and only if $\operatorname{Ran}(f) = \operatorname{Ran}(g)$ and, consequently, $f\mathscr{H}g$ if and only if $\pi(f) = \pi(g)$ and $\operatorname{Ran}(f) = \operatorname{Ran}(g)$.*

Proof. This is an immediate consequence of Theorems 2.6 and 2.7 of [5]. □

Theorem 3.6. *Let X be a 0-dimensional Hausdorff space. Then the following statements about $f, g \in FR(X)$ are equivalent.*

(3.6.1) *f and g are $\mathscr{D}$-equivalent.*

(3.6.2) *f and g are $\mathscr{J}$-equivalent.*

(3.6.3) *$Ran(f)$ and $Ran(g)$ have the same number of elements.*

Proof. Theorem 2.11 of [5] tells us that $f\mathscr{D}g$ if and only if $\operatorname{Ran}(f)$ and $\operatorname{Ran}(g)$ are FR-isomorphic and Theorem 2.12 of [5] tells us that $f\mathscr{J}g$ if and only if the range of each contains an FR-retract which is FR-isomorphic to the range of the other. The FR-retracts are precisely the finite subsets and two subsets of X are FR-isomorphic if and only if they are finite and have the same number of elements. It is now apparent that (3.6.1), (3.6.2) and (3.6.3) are all equivalent. □

Theorem 3.7. *Let X be an infinite 0-dimensional Hausdorff space and let J be a proper ideal of $FR(X)$. Then there exists a positive integer N such that $J = \{f \in FR(X) : |\operatorname{Ran}(f)| \leqslant N\}$. Moreover, J is a principal ideal.*

Proof. Theorem 2.12 of [5] tells us that $g \in FR(X)^1 \circ f \circ FR(X)^1$ if and only if $\operatorname{Ran}(g)$ is FR-isomorphic to an FR-retract contained in $\operatorname{Ran}(f)$, or alternatively, if and only if $|\operatorname{Ran}(g)| \leqslant |\operatorname{Ran}(f)|$. It follows that $\{|\operatorname{Ran}(f)| : f \in J\}$ has a greatest element N and $J = \{f \in FR(X) : |\operatorname{Ran}(f)| \leqslant N\}$. Furthermore, J is generated by any f such that $|\operatorname{Ran}(f)| = N$. □

In our concluding result, we describe the maximal subgroups of $FR(X)$.

Theorem 3.8. *Let X be an infinite 0-dimensional Hausdorff space. Then each maximal subgroup of $FR(X)$ is isomorphic to the full symmetric group on a finite set. Conversely, each full symmetric group on a finite set is isomorphic to a maximal subgroup of $FR(X)$.*

Proof. Corollary 2.10 of [5] tells us that every maximal subgroup of $FR(X)$ is isomorphic to FR-automorphism group of an FR-retract of X and conversely, every FR-automorphism group of an FR-retract of X is isomorphic to a maximal subgroup of $FR(X)$. But the FR-automorphism groups of FR-retracts of X are precisely the full symmetric groups on finite subsets of X. □

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