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COTYPE AND COMPLEMENTED COPIES OF c_0 IN SPACES OF OPERATORS

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1. INTRODUCTION

The main purpose of the paper is to characterize for some Banach spaces F (or E) those Banach spaces E (or F) for which the space of compact operators $K(E,F)$ contains a (complemented) copy of c_0.

First, we consider spaces F of finite cotype p, $2 \leqslant p < \infty$. It turns out that if there is a non-compact map $T\colon l_p \to F$ (as it happens for $F = L_p(\mu)$, C_p—the Schatten class etc.), then we are able to characterize precisely those E for which $K(E,F) \supseteq c_0$. If there is even a non-limited map $T\colon l_p \to F$, then the same condition characterizes even the case when $K(E,F)$ contains a complemented copy of c_0. Since the characterization is given by the conditions of type "there is a non-compact map $T\colon E \to l_p$," it seems that the result has something in common with the following type of results:

(1) If either E or F has some unconditional structure and $L(E,F) \neq K(E,F)$, then $K(E,F)$ contains a copy of c_0 (see [Ka1, Th. 6], [Fe1], [E3, Th. 3], [E4, Cor. 10, 11], [E5, Th. 1, Cor. 3 and Th. 12] and [BDLR, Th. 11]).

Analyzing this connection we prove that if there is a non-limited operator $T\colon E \to F$ factorizing through a space with an unconditional basis, then $K(E,F)$ contains even a complemented copy of c_0.

We also give a version of our results for ε-products and spaces of weak*-weak continuous maps and a sample of applications. There are also some unformulated consequences of our results for those spaces of vector-valued functions or measures which can be interpreted as spaces of operators (see [DzS], [DD2], [DD4], [Dr2], [Dr3], [DrE] and [Me]).

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The results are formulated for E, F being Banach spaces. The proofs are presented in such a form that one can easily observe that the same results are true for E, F being either Fréchet or complete DF-spaces (all four combinations are admitted), where instead of compact operators we consider the space $M(E, F)$ of Montel operators (i.e., operators mapping bounded sets into relatively compact ones). The last section will be devoted to that general setting. In particular, we apply our results to F being non-Montel Köthe sequence spaces $\lambda_p(A)$ or $k_p(V)$, function spaces etc.

We should point out that [Dr3] (and a later note [Dr4]) served as the main source of inspiration. Drewnowski characterized in his paper those Banach spaces E such that $ca(\Sigma, E)$ contains a copy of c_0. Since $ca(\Sigma, E)$ could be interpreted as a space of operators, we analyzed the proof, isolated the main ideas and apply them to our problem.

There is a quite extensive literature on the problem of containment of c_0 or l_∞ in classical spaces of operators. The known results (apart from the results of type (1), see above) could be sorted in three main types:

(2) If F or E' contains a copy of c_0, then a certain space of operators $A(E, F)$ contains a complemented copy of c_0 (see [Fr1, Th. 2.3], [Ry], [Sa], [E2], [E7, Th. 2, Cor. 3]). The results of that type are true for "small" spaces like the space of compact operators $K(E, F)$. They are modelled after the remarkable result of Cembranos [Ce] on containment of complemented copies of c_0 in $C(K, E)$.

(3) If a certain space of operators $A(E, F)$ contains a complemented copy of c_0, then either F or E' contains a copy of c_0 ([Ka1], [E8, Cor. 2]). The results of that type hold mostly for "big" spaces like the space of weakly compact operators $W(E, F)$. The proof is based on results of the following form: if neither F nor E' contains a copy of c_0 (or some similar assumption), then each map $J_0\colon c_0 \to A(E, F)$ extends to a map $J\colon l_\infty \to A(E, F)$ (see [Ka1, Th. 6], [E4, Th. 5, Cor. 9], [BDLR, Th. 10, Cor. 12–18]).

(4) If a certain space of operators $A(E, F)$ contains a copy of l_∞, then either F or E' contains a copy of l_∞ as well (see [Ka1, Th. 6], [Dr2], [BDLR, Th. 9]). These are also "small" space type results.

The results of types (2)–(4) above always exhibit a link between the containment of a Banach space as a (complemented) subspace in $A(E, F)$ and the containment in the spaces E' or F. The aim of this article is to consider cases where such a link does not exist, i.e., the containment in the spaces of operators of a certain Banach space is *not* due to the containment of the same Banach space in the spaces E' or F. Accordingly a new approach and different methods are needed for the problems considered here.

The problem of (complemented) copies of c_0 in spaces of operators is also closely connected with the old question if the space of compact operators can be a non-trivial complemented subspace of the space of all continuous operators (see for example [BDLR], [BL1], [BL2], [DD1], [DD2], [DD3], [DD4], [E1]-[E8], [Fe1], [Fe2], [Ka1] and [Jo1]). In particular, it is known in the Banach setting (see [Fe1, p. 201], [Ka1, proof of Th. 6] and [E3, Th. 2]) that if we consider the following conditions:

(a) $L(E,F) \neq K(E,F)$;
(b) $L(E,F) \supseteq l_\infty$;
(c) $L(E,F) \supseteq c_0$;
(d) $K(E,F) \supseteq c_0$;
(e) $K(E,F)$ is uncomplemented in $L(E,F)$;

then (d) $\Rightarrow$ (c)$\Leftrightarrow$(b) $\Rightarrow$ (a) and (d) $\Rightarrow$ (e) $\Rightarrow$ (a), but, in general, (a) does not imply (d) (see [E9]).

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2. Preliminaries

Our notation and terminology is standard and we refer to the books [J], [LT] and [D1].

The ε-product $E\varepsilon F$ of locally convex spaces (*lcs* in short) E and F is the operator space $L_\varepsilon(E'_{co}, F)$ of all weak*-weakly continuous linear maps from E' into F which transform equicontinuous subsets of E' into relatively compact subsets of F, endowed with the topology of uniform convergence on the equicontinuous sets in E'. The space $L_\varepsilon(E'_{co}, F)$ reduces to the Banach space $K_{w^*}(E', F)$ of compact weak*-weakly continuous linear maps with the norm topology in the Banach space setting. By $L(E'_\mu, F)$ we denote the space of all weak*-weakly continuous linear maps from E' into F. If F and E are complete lcs, then $L_\varepsilon(E'_{co}, F)$ and $L_\varepsilon(E'_\mu, F)$ are also complete lcs. The spaces $L_\varepsilon(E'_{co}, F)$ and $L_\varepsilon(F'_{co}, E)$ (as well as $L_\varepsilon(E'_\mu, F)$ and $L_\varepsilon(F'_\mu, E)$, resp.) are topological isomorphic via the correspondence between h and its adjoint h'. Let $L_w(E,F)$ denote the space $L(E,F)$ endowed with the topology defined by the semi-norms $T \mapsto |y'(T(x))|$, $x \in E, y' \in F'$.

A subset B of a lcs E is called *limited*, if every equicontinuous, $\sigma(E', E)$-null sequence in E' converges uniformly to zero on B. A continuous linear map from a lcs E into a lcs F is called *limited* if it transforms bounded sets into limited sets. By $\mathrm{Li}(E,F)$, $W(E,F)$ respectively, we denote the space of all limited, weakly compact maps from E into F. A lcs E has the *Gelfand-Phillips property*, if every limited set in E is relatively compact.

We say that a Banach space E has the *p-Orlicz property* if for each unconditionally convergent series $\sum x_n$ in E we have: $\sum\limits_{n\in\mathbb{N}} \|x_n\|^p < \infty$. The space E has *cotype p*, if there is a constant C such that for each finite family of vectors $x_1, \dots, x_n$ the following condition holds: $\left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p} \leqslant C\left(\int_0^1 \|\sum_{i=1}^{n} r_i(t)x_i\|^p \, dt\right)^{1/p}$, where (r_n) are Rademacher functions. It is well known that cotype p implies p-Orlicz property, the converse is not true for $p = 2$ even for Banach lattices (see [T1] or [DF, p. 103]). For all Banach spaces cotype p and p-Orlicz property are equivalent for $2 < p < \infty$ [T2].

It is well known that for $1 \leqslant p < \infty$ and a Banach space E the space of sequences

$$l^p_w(E) := \left\{(x_n)_{n\in\mathbb{N}} \subseteq E \colon \forall\, x' \in E' \ \sum |x'(x_n)|^p < \infty\right\}$$

is a Banach space if equipped with the seminorm (see [J, 16.5]):

$$\varepsilon^p((x_n)_{n\in\mathbb{N}}) := \sup_{\|x'\|\leqslant 1} \left(\sum |x'(x_n)|^p\right)^{1/p}.$$

This class is closely connected with operators $T\colon l_q \to E$ as it is explained in the following known lemma (see [AD, Prop. B], [Dr3, Lemma 1] or [A, Prop. 2.2]):

Lemma 1. *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then the following assertions are equivalent for each Banach space E:*

(a) $(x_n) \in l^p_w(E)$.

(b) *The series $\sum a_n x_n$ converges unconditionally for every sequence $(a_n) \in l_q$.*

(c) *The map $(a_n) \to \sum_{n=1}^{\infty} a_n x_n$, $(a_n) \in l_q$, defines an operator $T\colon l_q \to E$.*

We need two other classes of sequences: $c_0(E)$—the class of null sequences; $l(E)$—the class of limited sequences. At least the first part of the following lemma is also known (see [Dr3, p. 748] or [Ct]).

Lemma 2. *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and let E be a Banach space.*

(1) *The following assertions are equivalent:*

(a) $K(l_p, E) \neq L(l_p, E)$.

(b) $l^q_w(E) \setminus c_0(E) \neq \emptyset$.

(c) *There is an operator $T\colon l_p \to E$ such that $T(e_n) \nrightarrow 0$ as $n \to \infty$.*

(2) *The following assertions are equivalent:*

(a) $\mathrm{Li}(l_p, E) \neq L(l_p, E)$.

(b) $l^q_w(E) \setminus l(E) \neq \emptyset$.

(c) *There is an equicontinuous weak* null sequence $(x'_n)_{n\in\mathbb{N}} \subseteq E'$ and an operator $T\colon l_p \to E$ such that $x'_n(T(e_n)) = 1$ for all n.*

P r o o f. Both proofs are very similar so we restrict ourselves only to the part (2).

(a) $\Rightarrow$ (b): Let $T\colon l_p \to E$ be a non-limited operator. Since l_p is reflexive, we will find a normalized, weakly null sequence (u_n) in l_p such that (Tu_n) is non-limited. Thus we find an equicontinuous weak* null sequence $(x'_n) \subseteq E'$ such that $x'_n(Tu_n) > 1$. By passing to a subsequence if necessary, we may assume that (u_n) is a basic sequence in l_p equivalent to the standard basis in l_p (see the proof of [AD, Prop. D]). Since the standard basis in l_p belongs to $l^q_w(l_p)$, the sequence (x_n), $x_n = Tu_n$, belongs to $l^q_w(E) \setminus l(E)$.

(b) $\Rightarrow$ (c): We define T according to Lemma 1 (c).

(c) $\Rightarrow$ (a): Obviously, such T is non-limited. □

Since $C(K)$ contains a copy of l_1 if and only if K is a non-scattered compact Hausdorff space [PS], the following result is an improvement of Theorem 3.2 in [A] (comp. also [OP, Prop. 3]).

Corollary 3. *Let E be a Banach space containing a copy of l_1 and let F be a Banach space. If $W(E, F) = K(E, F)$, then $L(l_2, F) = K(l_2, F)$.*

P r o o f. Let $T\colon l_1 \to l_2$ be the inclusion mapping. Since T is 2-summing, the Pietsch Domination Theorem says that there exists a regular Borel probability measure μ defined on some compact Hausdorff space such that T factors through $L^\infty(\mu)$. Then, by the injectivity of $L^\infty(\mu)$, T extends to a non-compact continuous linear map S from E into l_2. Suppose that $L(l_2, F) \neq K(l_2, F)$, by Lemma 2, there is $U\colon l_2 \to F$, $U(e_n) \nrightarrow 0$. Obviously, $U \circ S \in W(E, F)$, if $U \circ S$ were compact, then $U \circ S(f_n) = U(e_n) \to 0$, where (f_n) and (e_n) are unit vectors in l_1 and l_2, respectively; a contradiction. □

Corollary 4. *If E is a Banach space, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then $L(E, l_p) \neq K(E, l_p)$ iff $L(l_q, E') \neq K(l_q, E')$.*

P r o o f. Let $T\colon E \to l_p$ be a non-compact operator, then its dual map $T'\colon l_q \to E'_b$ is non-compact. On the other hand, if $L(l_q, E') \neq K(l_q, E')$, then there is $(x'_n) \in l^p_w(E') \setminus c_0(E')$. It is easily seen that the map $T\colon E \to l_p$, $T(x) = (x'_n(x))_{n \in \mathbb{N}}$ is not compact. □

In order to describe operators into l_p, $1 \leqslant p < \infty$, we need the class of weak* l_p-sequences:

$$l^p_{w^*}(E') = \Big\{ (x'_n) \subseteq E'_b \colon \forall x \in E \quad \sum |x'_n(x)|^p < \infty \Big\}.$$

Proposition 5. *If E is a Banach space, then:*

(a) *if $T\colon E \to l_p$ is an operator, then $(T'(e_n)) \in l^p_{w^*}(E')$;*

(b) *if* $(x'_n) \in l^p_{w^*}(E')$ *then* $T\colon E \to l_p$, $T(x) = (x'_n(x))$ *is a continuous operator.*

3. The main results

We start with the following:

Theorem 6. *Let E, F be Banach spaces and let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.*

If $L(l_q, E) \neq K(l_q, E)$ and $L(l_p, F) \neq K(l_p, F)$, then there exists a topological embedding $J\colon l_\infty \to L_\varepsilon(E'_\mu, F)$ such that $J(c_0) = J(l_\infty) \cap L_\varepsilon(E'_{co}, F)$. In particular, $L_\varepsilon(E'_{co}, F) \neq L_\varepsilon(E'_\mu, F)$.

If, additionally, $L(l_p, F) \neq \mathrm{Li}(l_p, F)$ or $L(l_q, E) \neq \mathrm{Li}(l_q, E)$, then we can even obtain $J(c_0)$ complemented in $L_\varepsilon(E'_{co}, F)$.

Corollary 7. *Let E, F be Banach spaces and let $1 < p < \infty$.*

If $L(E, l_p) \neq K(E, l_p)$ and $L(l_p, F) \neq K(l_p, F)$, then there exists a topological embedding $J\colon l_\infty \to W(E, F)$ such that $J(c_0) = J(l_\infty) \cap K(E, F)$. In particular, $W(E, F) \neq K(E, F)$.

If, additionally, $L(l_p, F) \neq \mathrm{Li}(l_p, F)$ or $L(l_q, E'_b) \neq \mathrm{Li}(l_q, E'_b)$, then we can even obtain $J(c_0)$ complemented in $K(E, F)$.

Remark. Under the assumptions of Th. 6 and Cor. 7 we can prove that $K(E, F)$ ($L_\varepsilon(E'_{co}, F)$) is uncomplemented in $L(E, F)$ ($L_\varepsilon(E'_\mu, F)$, resp.); see [E3, Th. 2], [E4], comp. [BDLR, Th. 32 and Cor. 33].

Proof of Corollary 7. By [CoRu, Ex. 0.2] (comp. [BDLR, Cor. 7]), we can identify $W(E, F)$ with $L_\varepsilon((E')'_\mu, F)$ and $K(E, F)$ with $L_\varepsilon((E')'_{co}, F)$. Moreover, by Cor. 4, $L(E, l_p) \neq K(E, l_p)$ implies that $L(l_q, E') \neq K(l_q, E')$ for q conjugate to p. This completes the proof by Theorem 6. □

Proof of Theorem 6. By the assumptions (see Lemma 2), there are sequences $(x_n) \in l^p_w(E) \setminus c_0(E)$, $(z_n) \in l^q_w(F) \setminus c_0(F)$. Of course, (x_n) is weakly null but we may assume that $\|x_n\| \geqslant 1$. By the Bessaga-Pełczyński Selection Principle (see [D1, p. 42]), taking a subsequence if necessary, we can find a sequence of functionals (x'_n) biorthogonal to (x_n).

Now we can define a linear map $T\colon E' \to l_p$ by $T(x') = (x'(x_n))_n$. Then $T \in L(E'_\mu, l_p)$ or equivalently T is weak*-weak continuous from E' into l_p. Indeed, by Lemma 1, the series $\sum_{n=1}^{\infty} \xi_n x_n$ converges unconditionally in E for all $(\xi_n) \in l_q$. Since for each $(\xi_n) \in l_q = (l_p)'$

$$\left|\sum_{n=1}^{\infty} \xi_n x'(x_n)\right| = \left|x'\left(\sum_{n=1}^{\infty} \xi_n x_n\right)\right| \quad \text{for all } x' \in E',$$

we conclude that $T \in L(E'_\mu, l_p)$.

Similarly, there is a continuous linear map $S\colon l_p \to F$ such that $S(e_n) = z_n$ (see Lemma 2 (c)) hence $\|S(e_n)\| \geqslant 1$ for all n. Thus we have a continuous linear map $S \circ T\colon E'_\mu \to F$ with $\|S(T(x'_n))\| = \|S(e_n)\| \geqslant 1$ for all n. Let $\xi = (\xi_n) \in l^\infty$ and let $D_\xi\colon l_p \to l_p$ be the diagonal map $(\lambda_n) \mapsto (\xi_n \lambda_n)$. Then we can define a continuous linear map

$$J\colon l^\infty \to L_\varepsilon(E'_\mu, F) \quad \text{by} \quad J(\xi)x' = (S \circ D_\xi \circ T)(x').$$

Now, $J\colon l^\infty \to L_\omega(E'_\mu, F)$ is continuous. In fact, for $y' \in F'$ and $x' \in E'$ we have that

$$|y'(J(\xi)x')| \leqslant M \cdot \left(\sum_{n=1}^{\infty} |x'(x_n)|^p\right)^{1/p} \cdot \sup_n |\xi_n| \quad \text{for all } \xi = (\xi_n) \in l^\infty,$$

where $M > 0$ is some constant. Then J has closed graph in $l^\infty \times L_\varepsilon(E'_\mu, F)$ and it follows that $J\colon l^\infty \to L_\varepsilon(E'_\mu, F)$ is continuous.

Moreover, we have that $\|J(e_n)x'_n\| = \|S(e_n)\| \geqslant 1$ for all n and $(x'_n) \subset E'$ is equicontinuous, so $J(e_n) \not\to 0$ in $L_\varepsilon(E'_\mu, F)$. Thus there exists an infinite subset $M \subset \mathbb{N}$ such that $J\colon l^\infty(M) \to L_\varepsilon(E'_\mu, F)$ is an isomorphism (see [Dr1]). Let us assume that $M = \mathbb{N}$. Since $J(\xi)x'_n = \xi_n S(e_n)$ and $S(e_n) \to 0$ weakly in F, it is clear that $J(\xi) \in L(E'_{co}, F)$ implies that $\xi = (\xi_n) \in c_0$. Conversely, suppose that $\xi = (\xi_n) \in c_0$. Then the diagonal map $D_\xi\colon l_p \to l_p, (\lambda_n) \mapsto (\xi_n \lambda_n)$, is compact and therefore $J(\xi) \in L(E'_{co}, F)$. Thus we have

$$J(c_0) = J(l^\infty) \cap L_\varepsilon(E'_{co}, F).$$

Finally, if we assume that $L(l_p, F) \neq \mathrm{Li}(l_p, F)$, then, by Lemma 2, there are a weak*-null sequence $(y'_n) \subset F'$ such that $y'_n(S(e_n)) = 1$ for all n. We define P: $L_\varepsilon(E'_{co}, F) \to c_0$ by $P(T) = (y'_n(T(x'_n)))_n$. Then P is well-defined and continuous, since $(x'_n) \subset E'$ is equicontinuous. For every $\xi = (\xi_n) \in c_0$ we have that

$$P(J(\xi)) = (y'_n(S(\xi_n e_n)))_n = (\xi_n)_n = \xi.$$

This means that $J(c_0)$ is complemented in $L_\varepsilon(E'_{co}, F)$, and the proof is complete.

The case $L(l_q, E) \neq \mathrm{Li}(l_q, E)$ is very similar, we take as (y'_n) a sequence biorthogonal to (z_n) and as (x'_n) a weak*-null sequence in E' such that $x'_n(x_n) = 1$ for all $n \in \mathbb{N}$. $\square$

Remark. The map J above could be explicitly defined (see the proof of Lemma 2):

$$J(\xi)x' = \sum_{n \in \mathbb{N}} \xi_n x'(x_n) z_n.$$

Theorem 8. *Let E ,F be Banach spaces and let $2 \leqslant p < \infty$, $\frac{1}{q} + \frac{1}{p} = 1$.*

(1) If F has the p-Orlicz property, then

(a) if $L_\varepsilon(E'_\mu, F)$ contains a copy of c_0, then $K(l_q, E) \neq L(l_q, E)$;

(b) if $L(E, F)$ contains a copy of c_0, then $K(E, l_p) \neq L(E, l_p)$.

(2) If E has the p-Orlicz property and $L_\varepsilon(E'_\mu, F)$ contains a copy of c_0, then $K(l_q, F) \neq L(l_q, F)$.

(3) If E' has the p-Orlicz property and $L_b(E, F)$ contains a copy of c_0, then $K(l_q, F) \neq L(l_q, F)$.

Proof. (1)(a): We can assume that F does not contain a copy of c_0, since c_0 has no Orlicz property and p-Orlicz property is inherited by subspaces. Suppose that $J\colon c_0 \to L_\varepsilon(E'_\mu, F)$ is an embedding. Then we can define a continuous linear map $J_{x'}\colon c_0 \to F$ by $J_{x'}(\xi) = J(\xi)x'$, $x' \in E'$. Since $c_0 \not\subset F$ we get by Lemma 1 and Lemma 2(b) in [BDLR] (see also [Ka2, Th. 2.4] and [Da, Lemma 2.2]) that $\sum\limits_{n=1}^{\infty} \xi_n J(e_n)x'$ is subseries convergent in F for all $x' \in E'$ and all $\xi = (\xi_n) \in l^\infty$. For $x' \in E'$ the series $\sum\limits_{n=1}^{\infty} J(e_n)x'$ is unconditionally convergent in F. Since F has the p-Orlicz property, it follows that $\sum\limits_{n=1}^{\infty} \|J(e_n)x'\|^p < \infty$ for all. Since $J(e_n) \not\to 0$ in $L_\varepsilon(E'_\mu, F)$, there are an equicontinuous sequence (x'_n) in E' and an equicontinuous sequence (y'_n), $\|y'_n\| \leqslant 1$, such that $|y'_n(J(e_n)x'_n)| \geqslant 1$ for all n. Let $x_n = y'_n \circ J(e_n) \in E = (E'_\mu)'$. Then $x_n \not\to 0$ in E. We have also $\|J(e_n)x'\| \geqslant |y'_n(J(e_n)x')|$ for all n, and consequently $\sum\limits_{n=1}^{\infty} |y'_n(J(e_n)x')|^p < \infty$. Hence $\sum\limits_{n=1}^{\infty} |x'(x_n)|^p < \infty$ for all $x' \in E'$, and the proof is complete by Lemma 2.

(b): We can prove similarly as above, that there is $(x'_n) \in l^p_{w^*}(E') \setminus c_0(E'_b)$. As in Prop. 5, we define an operator $T\colon E \to l_p$, $T(x) = (x'_n(x))_{n\in\mathbb{N}}$. By the description of relatively compact sets in l_p, T is not compact.

(2) and (3): The proof is quite similar — it suffices to use the map $J^{y'}$, $J^{y'}(\xi) = y' \circ J(\xi)$ for $y' \in F'$ instead of $J_{x'}$. $\square$

Now, we give a sample of immediate consequences.

Corollary 9. *Let E, F be Banach spaces. If F has the p-Orlicz property $(2 \leqslant p < \infty)$ and $L(l_p, F) \neq K(l_p, F)$, then the following assertions are equivalent:*

(a) $L(E, F)$ contains a copy of l^∞.

(b) $L(E, F)$ contains a copy of c_0.

(c) $K(E, F)$ contains a copy of c_0.

(d) There is an embedding $J\colon l^\infty \to W(E, F)$ such that $J(c_0) = J(l^\infty) \cap K(E, F)$.

(e) $L(E, l_p) \neq K(E, l_p)$.

If $L(l_p, F) \neq \mathrm{Li}(l_p, F)$, then the above conditions are also equivalent to:

(f) *The condition (d) holds with $J(c_0)$ complemented in $K(E, F)$.*

(g) *$K(E, F)$ contains a complemented copy of c_0.*

Remark. The implication (b) $\Rightarrow$ (a) holds for all Banach spaces E, F, see [Ka1, proof of Th. 6 (iii) $\Rightarrow$ (ii)].

Corollary 10. *Let E, F be Banach spaces. If E' has the p-Orlicz property ($2 \leqslant p < \infty$) and $L(l_p, E') \neq K(l_p, E')$, then the following assertions are equivalent:*

(a) *$L(E, F)$ contains a copy of l^∞.*

(b) *$L(E, F)$ contains a copy of c_0.*

(c) *$K(E, F)$ contains a copy of c_0.*

(d) *There is an embedding $J\colon l^\infty \to W(E, F)$ such that $J(c_0) = J(l^\infty) \cap K(E, F)$.*

(e) *$L(l_q, F) \neq K(l_q, F)$.*

If $L(l_p, E') \neq \mathrm{Li}(l_p, E')$, then the above conditions are also equivalent to:

(f) *The condition* (d) *holds with $J(c_0)$ complemented in $K(E, F)$.*

(g) *$K(E, F)$ contains a complemented copy of c_0.*

P r o o f o f C o r o l l a r i e s 9 a n d 1 0. Obviously, (d) $\Rightarrow$ (c) $\Rightarrow$ (b), (d) $\Rightarrow$ (a) $\Rightarrow$ (b). By Theorem 6, Cor. 7 and 4, (e) $\Rightarrow$ (d) and, by Theorem 8, (b) $\Rightarrow$ (e). Moreover, obviously (f) $\Rightarrow$ (g) $\Rightarrow$ (b) and, by Theorem 6 and Cor. 7, (e) $\Rightarrow$ (f). $\square$

Corollary 11. *Let E', F be Banach spaces with the 2-Orlicz property. Then the following assertions are equivalent:*

(a) *$L(E, F)$ contains a copy of l^∞.*

(b) *$L(E, F)$ contains a copy of c_0.*

(c) *$K(E, F)$ contains a copy of c_0.*

(d) *There is an embedding $J\colon l^\infty \to L(E, F)$ such that $J(c_0) = J(l^\infty) \cap L(E, F)$.*

(e) *$L(E, l_2) \neq K(E, l_2)$ and $L(l_2, F) \neq K(l_2, F)$.*

P r o o f. (b) $\Rightarrow$ (e): Follows from Theorem 8, on the other hand, (e) $\Rightarrow$ (d) follows from Theorem 6. The rest is obvious (see the proof of Cor. 9). $\square$

Remarks. (1) Similar results also hold for $L_\varepsilon(E'_\mu, F)$ and $L_\varepsilon(E'_{co}, F)$ spaces.

(2) If F has the Gelfand-Phillips property, in particular, if F is separable or reflexive, then $L(l_p, F) \neq K(l_p, F)$ iff $L(l_p, F) \neq \mathrm{Li}(l_p, F)$ and we can improve the results above.

(3) A particular case of the result for $E = P$ and $F = P'$, where P is the so-called Pisier space was proved in [Jo2, Prop. 1].

The dual of each C^*-algebra has cotype 2 [TJ]. By Cor. 11, we get (comp. [E3, Th. 4]):

Corollary 12. *Let E be a C^*-algebra, F the dual of any C^*-algebra, then the following assertions are equivalent:*

(a) $L(E,F) \supseteq l_\infty$;
(b) $L(E,F) \supseteq c_0$;
(c) $K(E,F) \supseteq c_0$;
(d) $L(E,l_2) \neq K(E,l_2)$ *and* $L(l_2,F) \neq K(l_2,F)$.

The Schatten class C_p is of cotype $\max(2,p)$ [TJ]. Thus, by Cor. 9 and recalling that each reflexive Banach space has the Gelfand-Phillips property, we get:

Corollary 13. *Let $2 \leqslant p < \infty$, then for each Banach space E the following assertions are equivalent:*

(a) $L(E,C_p) \supseteq l_\infty$;
(b) $L(E,C_p) \supseteq c_0$;
(c) $K(E,C_p) \supseteq c_0$;
(d) $K(E,C_p)$ *contains a complemented copy of* c_0;
(e) $L(E,l_p) \neq K(E,l_p)$.

The next result is a particular case of Th. 16 from the next section. Thus, we present it without proof. The result strengthens [Ka1, Th. 6 (ii) $\Rightarrow$ (i)] (comp. [Fe1, Th. 4]).

Corollary 14. *If E and F are Banach spaces such that there exists a non-limited map $T\colon E \to F$ factorizing through a Banach space with an unconditional basis, then $K(E,F)$ contains a complemented copy of c_0.*

Remarks. (1) The result has a very nice form if $\mathrm{Li}(E,F) = K(E,F)$, i.e., when F has the Gelfand-Phillips property (comp. a similar result [E4, Th. 19]).

(2) Let us note that in the proof of Cor. 7 we show in fact that there is a non-compact map (or a non-limited map) $J(\alpha)\colon E \to F$, $\alpha = (1,1,\ldots)$, which factorizes through l_p. Thus Cor. 7 is a consequence of Cor. 14.

By Cor. 9 and Cor. 14 we get:

Corollary 15. *Let $1 \leqslant p < \infty$, μ a purely non-atomic measure, then for each Banach space E the folowing assertions are equivalent:*

(a) $L(E,L_p(\mu)) \supseteq l_\infty$;

(b) $L(E, L_p(\mu)) \supseteq c_0$;
(c) $K(E, L_p(\mu)) \supseteq c_0$;
(d) $K(E, L_p(\mu))$ *contains a complemented copy of* c_0;
(e) $L(E, l_{\max(2,p)}) \neq K(E, l_{\max(2,p)})$.

If $p > 1$, *then the above conditions are equivalent to:* (f) $L(E, L_p(\mu)) \neq K(E, L_p(\mu))$.

Proof. For $p > 1$ the space $L_p(\mu)$ has an unconditional basis and for $p \geqslant 1$ there is a non-limited map $T\colon l_{\max(2,p)} \to L_p(\mu)$. Indeed, it is clear for $p > 1$, for $p = 1$ the Rademacher functions are non-limited and they span a copy of l_2 in $L_1(\mu)$. □

4. The case of Fréchet and DF-spaces

We will be interested in the so-called admissible pairs of of lcs (E, F), i.e., one of the following cases holds:

(1) Both E and F are Fréchet spaces.
(2) Both E and F are complete DF-spaces.
(3) E is a Fréchet space and F is a complete DF-space.
(4) E is a complete DF-space and F is a Fréchet space.

We use further on (without reference) the following facts on admissible pairs. If $G = E$ or E'_b or $(E'_b)'_b$ etc. and $H = F$ or F'_b or $(F'_b)'_b$ etc. the following conditions hold (see the beginning of Section 2 in [BDLR]): (i) both G and H are complete and weakly angelic [Fl]; (ii) G is $\aleph_0$-barreled (i.e., the Banach-Steinhaus Theorem for sequences holds); (iii) the space $L_b(G, H)$ admits a strict web in the sense of De Wilde (i.e., the Closed Graph Theorem holds for maps from any Banach space into $L_b(G, H)$, see [J]).

The notions of cotype p and of the p-Orlicz property can be extended in an obvious way to general lcs. If a sequentially complete lcs E has the p-Orlicz property or cotype p for $p < 2$, then it has 1-Orlicz property and cotype 1, equivalently, it is nuclear (comp. [KRT, Cor. 5, p. 107]). It is easily seen that any projective limit of Banach spaces with cotype p (or p-Orlicz property) has cotype p (p-Orlicz property, respectively).

Let us recall that $M(E, F)$ denotes the space of Montel operators (i.e., those mapping bounded sets into relatively compact ones). Now, we give a promised general form of Cor. 14.

Theorem 16. *Let* (E, F) *be an admissible pair of non-Montel lcs. Assume that there is a map* $T \in L(E, F) \setminus \mathrm{Li}(E, F)$ *which factorizes through a* $\aleph_0$*-barrelled space* *with an unconditional basis (for example, through a Fréchet or complete DF-space with an unconditional basis). Then $M_b(E,F)$ contains a complemented copy of c_0.*

First we need an auxiliary lemma due to Drewnowski [Dr5]. It generalizes a Banach space result due to Emmanuele [E2, Th. 1] and Schlumprecht [S]. The proof is similar.

Lemma 17. *Let (x_k) be a sequence in a lcs E such that there is an embedding $J\colon c_0 \to E$ with $J(e_k) = x_k$ for all $k \in \mathbb{N}$. Assume that there is an equicontinuous, weak*-null sequence (x'_n) in E' which is not uniformly convergent to zero on (x_k). Then a subsequence of (x_k) spans a complemented copy of c_0 in E.*

Lemma 18. *Let E be a Banach space and F a lcs. Suppose that the following properties are satisfied for $S \in L(E,F)$:*

(i) *There is an equicontinuous weak*-null sequence $(y'_n) \subset F'$ such that for all $n \in \mathbb{N}$*

$$\sup_{x\in B_E} |y'_n(S(x))| \geqslant \varepsilon_0 \qquad \text{for some } \varepsilon_0 > 0.$$

(ii) *There is a sequence (S_m) in $K(E,F)$ such that $S_m(x) \to S(x)$ in F for every $x \in E$, when $m \to \infty$.*

Then there exist an $\varepsilon > 0$ and strictly increasing subsequences (m_j) and (k_j) in $\mathbb{N}$ such that for all $j \in \mathbb{N}$

$$\sup_{x\in B_E} |y'_{k_j}((S_{m_{2j}} - S_{m_{2j-1}})(x))| \geqslant \varepsilon.$$

Proof. We first show that there exists an $\varepsilon > 0$ such that for all $m_0 \in \mathbb{N}$ there are $m > n \geqslant m_0$ with

$$\sup_{x\in B_E} \sup_{k\geqslant m_0} |y'_k((S_m - S_n)(x))| \geqslant \varepsilon. \tag{1}$$

On the contrary, let us assume that for all $\varepsilon > 0$ there is $m_0 \in \mathbb{N}$ such that for all $m > n \geqslant m_0$ it follows that

$$|y'_k((S_m - S_n)(x))| < \varepsilon \quad \text{for all } k \geqslant m_0 \quad \text{and all } x \in B_E.$$

Let $\varepsilon = \varepsilon_0/3$. By assumption there exists $m_0 \in \mathbb{N}$ such that for every $m > m_0$ we have that $|y'_k((S_m - S_{m_0})(x))| < \varepsilon_0/3$ for all $k \geqslant m_0$ and all $x \in B_E$. Now we fix $k \geqslant m_0$ and $x \in B_E$ and let $m \to \infty$. Hence $|y'_k((S_{m_0} - S)(x))| \leqslant \varepsilon_0/3$. Thus we obtain that $|y'_k(S(x))| \leqslant \varepsilon_0/3 + |y'_k(S_{m_0}(x))|$ for all $k \geqslant m_0$ and all $x \in B_E$. Since

$S_{m_0} \in K(E,F)$, there is a $k_0 \in \mathbb{N}$ such that $|y_k'(S_{m_0}(x))| < \varepsilon_0/3$ for all $x \in B_E$ and all $k \geqslant k_0$. Now it follows for $k > \max(k_0, m_0)$ and $x \in B_E$ that $|y_k'(S(x))| \leqslant 2\varepsilon_0/3$, which is a contradiction to (i). Thus we have shown (1).

The result follows easily by induction. □

Proof of Theorem 16. First we observe that *if E, F are complete non-Schwartz DF or non-Montel Fréchet spaces, $F \supseteq c_0$, then $L_\varepsilon(E'_{co}, F)$ contains a complemented copy of c_0.* In the setting of Banach spaces this result has been obtained by Ryan [Ry]. For the sake of completeness we briefly recall the main ingredients of the proof.

Since E is non-Montel (or non-Schwartz, resp.) there is an equicontinuos, weak*-null sequence (x_n') in E' and a bounded sequence (x_n) in E such that $x_n'(x_n) = 1$ for all $n \in \mathbb{N}$ [BL2, Th. 9 and Cor. 14] (comp. [L] and [LS]). Let (y_n) be a copy in F of the unit basis (e_n) in c_0. Then there is an equicontinuous sequence (y_n') in F' such that $y_m(y_n) = \delta_{mn}$. Put $z_n = x_n \otimes y_n \in E \otimes F$. Now, Freniche [Fr1, Th. 2.3] showed that there is an embedding $J\colon c_0 \to E\hat{\otimes}_\varepsilon F \subset L_\varepsilon(E'_{co}, F)$ given by $J(\xi) = \sum_{n=1}^{\infty} \xi_n z_n$. For each $n \in \mathbb{N}$ define $u_n \in L_\varepsilon(E'_{co}, F)'$ by $u_n(T) = \langle y_n', T(x_n')\rangle$. Notice that $F = (F'_{co})'$ and that the dual map $T'\colon F'_{co} \to E$ is continuous. The sequence $(u_n) \subset L_\varepsilon(E'_{co}, F)'$ is equicontinuous, and for each $T \in L(E'_{co}, F)$,

$$|u_n(T)| = |\langle T'(y_n'), x_n'\rangle| \to 0, \quad \text{as } n \to \infty.$$

Thus $u_n \to 0$ weak* in $L_\varepsilon(E'_{co}, F)'$. Since $J(e_n) = z_n$ and

$$|u_n(z_n)| = |x_n'(x_n) y_n'(y_n)| = 1$$

for all $n \in \mathbb{N}$, it follows from Lemma 17 that $L_\varepsilon(E'_{co}, F)$ contains a complemented copy of c_0.

Since the Fréchet space E is Montel if and only if E_b' is Schwartz [J, 11.6.1], we obtain from the above result that *if $F \supseteq c_0$, then $M_b(E,F) = L_\varepsilon((E_b')'_{co}, F)$ contains a complemented copy of c_0.*

Now, let us assume that F does not contain a copy of c_0. Let $T = f \circ g$ be the factorization through a $\aleph_0$-barrelled space G with unconditional basis (u_n) and let (u_n') denote the associated sequence of coefficient functionals. Hence there is a sequence of continuous linear projections $P_m\colon G \to G$ of finite rank, defined by $P_m(u) = \sum_{n=1}^{m} u_n'(u) u_n$. For every $u \in G$ we have $u = \lim_m P_m(u)$ and for each 0-neighbourhood U there is a 0-neighbourhood V such that for all $M \subset \mathbb{N}$ and all appropriate scalar sequences (α_n) we have $\| \sum_{M \subset \mathbb{N}} \alpha_n u_n \|_U \leqslant \| \sum_{n=1}^{\infty} \alpha_n u_n \|_V$ (cf. [Wl], Var. 1.21). Clearly $T_m = f \circ P_m \circ g \in M(E,F)$ for every $m \in \mathbb{N}$. Now $T_m(x) \to T(x)$ in F for all $x \in E$. Since T is not limited, there are a weak*-null sequence $(y'_n) \subset F'$, an $\varepsilon_0 > 0$ and a Banach disk B_0 in E such that $\sup_{x \in B_0} |y'_n(T(x))| \geqslant \varepsilon_0$ for all $n \in \mathbb{N}$. Let $S = T \circ j \in L(H,F)$ and $S_m = T_m \circ j \in K(H,F)$, where $H = E_{B_0}$ is a Banach space and $j\colon H \to E$ is the continuous injection. Since $\sup_{x \in B_0} |y'_n(S(x))| \geqslant \varepsilon_0$ for all $n \in \mathbb{N}$ and $S_m(x) \to S(x)$ in F for every $x \in H$, when $m \to \infty$ we can apply Lemma 18 to obtain $\varepsilon > 0$ and strictly increasing subsequences (m_k) and (n_k) in $\mathbb{N}$ such that for all $k \in \mathbb{N}$

$$\sup_{x \in B_0} |y'_{n_k}((S_{m_{2k}} - S_{m_{2k-1}})(x))| \geqslant \varepsilon. \tag{2}$$

Put $Q_k = T_{m_{2k}} - T_{m_{2k-1}} \in M(E,F)$. Exactly as in [BL1, proof of Prop. 14], we can prove that for $x \in E$ and $\xi = (\xi_k) \in l_\infty$ the set $\{\sum_{k \in \triangle} \xi_k Q_k x\colon \triangle \subset \mathbb{N} \text{ finite}\}$ is bounded in F, which, by [MR, Th. 5], is equivalent to

$$\sum_{k=1}^{\infty} |y'(\xi_k Q_k x)| < \infty$$

for each $y' \in F'$. Now, since F does not contain a copy of c_0, Theorem 4.5 in [Ka2] gives that $\sum_{k=1}^{\infty} \xi_k Q_k x$ is convergent in F for all $x \in E$ and all $\xi = (\xi_k) \in l^\infty$. Thus we can define a linear map

$$R\colon l^\infty \to L(E,F), \xi \mapsto \left(x \mapsto \sum_{k=1}^{\infty} \xi_k Q_k x\right).$$

Since E is $\aleph_0$-barrelled, $R(\xi) \in L(E,F)$. Since $R\colon l^\infty \to L_w(E,F)$ is continuous, the graph of R is closed in $l^\infty \times L_b(E,F)$. The space $L_b(E,F)$ is a webbed space and we conclude that $R\colon l^\infty \to L_b(E,F)$ is continuous (cf. [J, Theorem 5.4.1]).

Since $(y'_{n_k}) \subset F'$ is equicontinuous, $Q_k \not\to 0$ in $M_b(E,F)$ by (2). Thus $R(e_k) = Q_k \not\to 0$, when $k \to \infty$. Hence there exists an infinite subset $M \subset \mathbb{N}$ such that R: $l^\infty(M) \to L_b(E,F)$ is an isomorphism [Dr1]. Assume that $M = \mathbb{N}$. It is easily seen that $R\colon c_0(M) \to M_b(E,F)$. Moreover, by (2), there is a sequence $(x_k) \subset B_0$ such that $|y'_{n_k}(Q_k(x_k))| \geqslant \varepsilon/2$ for all $k \in \mathbb{N}$. For $T \in M(E,F)$,

$$|(y'_{n_k} \otimes x_k)(T)| = |\langle T(x_k), y'_{n_k}\rangle| \to 0, \quad \text{when } k \to \infty.$$

Thus the sequence $(y'_{n_k} \otimes x_k)$ converges weak* to zero in $M_b(E,F)'$. Now we can apply Lemma 17, and it follows that $c_0(L)$ is complemented in $M_b(E,F)$ for some infinite subset L of M. This completes the proof. ☐

The authors have checked that if we replace the spaces of compact operators and of weakly compact operators by spaces of Montel operators (i.e., those mapping bounded sets into relatively compact sets) and of reflexive operators (i.e., those mapping bounded sets into relatively weakly compact sets) respectively, then the whole set of results from Section 3 (1–15) remain true for admissible pairs (E, F). Since the proofs require slight modifications, we omit details and we give only some applications.

In particular, the proof of Th. 8 also works for $p < 2$. As we mentioned already, every lcs with the p-Orlicz property, $p < 2$, is nuclear. Accordingly, the proof of Th. 8 shows the following result:

Corollary 19. *If the pair (E, F) is admissible and F is nuclear then the space $L_\varepsilon(E'_\mu, F)$ or $L(E, F)$ contains c_0 if and only if $E \supseteq c_0$ or $E \supseteq l_1$ complemented, respectively.*

Let us note that, by (1) from the introduction, if F has an unconditional basis, then either $M(E, F) \neq L(E, F)$ and, then $M_b(E, F) \supseteq c_0$, or $M(E, F) = L(E, F)$. Moreover, if $M(E, F) = L(E, F)$ and F contains no copy of c_0, then by [Ka1, proof of Th. 6 (iii) $\Rightarrow$ (ii)] (comp. [BDLR, Cor. 12]), $M_b(E, F) \supseteq l_\infty$ and, by [BDLR, Th. 9] (see also [Ka1, Th. 4]), either E' or F contains a copy of l_∞. Thus in our case $M_b(E, F) \supseteq c_0$ iff either E' or F contains c_0 or $M(E, F) \neq L(E, F)$.

If we try to repeat the above considerations for complemented copies of c_0 we are in trouble. Namely, if E' or F contains a copy of c_0 and the other space is non-Montel, then, by [Ry] (see also the first part of the proof of Theorem 16), $M_b(E, F)$ contains a complemented copy of c_0. The remaining case is unclear:

Problem. [Dr5] *Let E' and F contains no complemented copy of c_0 and let E' or F be a Montel space. Is it possible that $M_b(E, F)$ contains a complemented copy of c_0?*

Remark. Since, by [Ra, p. 98], $C(K)$ contains a complemented copy of c_0 iff it has not the Grothendieck property, the problem is solved for E Montel and $F = C(K)$ by [Fr2]. More general solutions has been found very recently in [DL].

We denote by $\lambda_p(A)$ the Köthe echelon space of order p, $1 \leqslant p < \infty$ (see [B]). We get:

Corollary 20. *Let E and F be non-Montel Fréchet spaces. Under one of the following conditions:*

(a) $E = \lambda_1(A)$,
(b) $F = \lambda_0(A)$,

(c) *E and F are hilbertizable,*

(d) $E = \lambda_p(A)$, $F = \lambda_q(B)$, $1 \leqslant p \leqslant q < \infty$,

we have that $M_b(E, F)$ contains a complemented copy of c_0.

Proof. First we prove (a). The space E contains a complemented copy of l_1. It follows that if every $T\colon E \to F$ belongs to $\mathrm{Li}(E, F)$, then every bounded sequence (x_n) in F is limited in F. But since F is non-Montel, there is a sequence (x'_n) in F' which converges to zero in $\sigma(F', F)$ but is not $\beta(F', F)$-null [BL2]. This is a contradiction, and consequently there exists a $T \in L(E, F) \setminus \mathrm{Li}(E, F)$ which factors through l_1. We apply Theorem 16.

Now, we prove (b), (c) and (d). Since every reflexive Banach space is a Gelfand-Phillips space, it is clear that all hilbertizable Fréchet spaces are Gelfand-Phillips. Further, it is also well-known that separable Fréchet spaces are Gelfand-Phillips. Hence $M(E, F) = \mathrm{Li}(E, F)$. From the proof of Theorem 3 (a) in [BL1] we have in all three cases that there exists $T \in L(E, F) \setminus M(E, F)$ which factors through a Banach space with an unconditional bases. This completes the proof by Theorem 16. □

In the paper [Rh] Reiher considered Fréchet weighted Köthe function spaces (see also [Dz]). It is proved that such a space $L_\varrho(A)$ over purely non-atomic space always contains a Banach subspace of the form L_ϱ. Now, if we take $L_\varrho = L_p(0, 1)$ we get easily Cor. 15 for weighted L_p spaces instead of L_p.

Corollary 21. *Let $2 \leqslant p < \infty$ and E be Fréchet or complete DF-space. The following conditions are equivalent:*

(a) $L(E, \lambda_p(A)) \supseteq l_\infty$;

(b) $L(E, \lambda_p(A)) \supseteq c_0$;

(c) $M(E, \lambda_p(A)) \supseteq c_0$;

(d) *$M(E, \lambda_p(A))$ contains a complemented copy of c_0;*

(e) *either $E'_b \supseteq l_\infty$ or there is a non-Montel map $T\colon E \to l_p$ and $\lambda_p(A)$ contains a copy of l_p.*

Proof. (e) $\Rightarrow$ (d) and (a) follows from the Fréchet or DF version of Cor. 9.

(b) $\Rightarrow$ (e): If λ_p is non-Montel, then $\lambda_p \supseteq l_p$ and, by Cor. 9, $M(E, l_p) \neq L(E, l_p)$. If $\lambda_p(A)$ is a Montel space, then $L(E, \lambda_p(A)) = M(E, \lambda_p(A)) \supseteq c_0$. Now, either $E'_b \supseteq c_0$ or $L(E, \lambda_p(A)) \supseteq l_\infty$ (see [BDLR, Cor. 12]). By [BDLR, Th. 9], if $M(E, \lambda_p(A)) \supseteq l_\infty$, then $E'_b \supseteq l_\infty$. Similarly, by [BDLR, Th. 8], if $E'_b \supseteq c_0$, then $E'_b \supseteq l_\infty$. □

The results similar to Cor. 20 and 21 also hold for coechelon Köthe sequence DF-spaces.

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