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DYNAMIC CONTACT PROBLEMS WITH GIVEN FRICTION FOR VISCOELASTIC BODIES

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1. FORMULATION OF THE PROBLEM AND THE PENALTY METHOD

We assume a bounded domain $\Omega \subset \mathbb{R}^N$ having a $C_{1,1}$-smooth boundary $\partial\Omega$ disjointly divided into the contact part $\partial\Omega_c$ and the remaining parts $\partial\Omega_T$ and $\partial\Omega_u$. All parts are assumed to be measurable. The investigated model is

$$
\begin{gathered}
\ddot{u} - \frac{\partial}{\partial x_j}\sigma_{ij}(u) = f_i, \; i = 1,\dots,N \quad \text{on} \quad Q_{\mathcal{T}} \equiv I_{\mathcal{T}} \times \Omega, \\
u_n \leqslant 0, \quad T_n(u) \leqslant 0, \quad T_n(u)\, u_n = 0 \quad \text{on} \quad S_{c,\mathcal{T}} \equiv I_{\mathcal{T}} \times \partial\Omega_c, \\
|T_t(u)| \leqslant G, \quad T_t(u)\dot{u}_t + G|\dot{u}_t| = 0, \\
|T_t(u)| = G \neq 0 \Longrightarrow \underset{\lambda\leqslant 0 \text{ on } S_{c,\mathcal{T}}}{\exists} \dot{u}_t = \lambda T_t(u) \text{ on } S_{c,\mathcal{T}}, \\
T(u) = T_0 \quad \text{on } S_{T,\mathcal{T}} \equiv I_{\mathcal{T}} \times \partial\Omega_T, \quad u = U \quad \text{on} \quad S_{u,\mathcal{T}} \equiv I_{\mathcal{T}} \times \partial\Omega_u, \\
u(0,\cdot) = u_0, \; \dot{u}(0,\cdot) = u_1 \quad \text{on} \quad \Omega,
\end{gathered} \tag{1}
$$

with the stress tensor $\widetilde{\sigma} \equiv \{\sigma_{ij}\,;\, i,j = 1,\dots,N\}$ given by

$$
\begin{gathered}
\sigma_{ij}(u) = \sigma^I_{ij}(u) + \sigma^V_{ij}(u), \; i,j = 1,\dots,N, \quad \text{where} \\
\sigma^I_{ij}(u) = \frac{\partial W}{\partial e_{ij}}(\cdot, \widetilde{e}(u)), \; i,j = 1,\dots,N, \qquad \text{and} \qquad \widetilde{\sigma}^V \equiv \{\sigma^V_{ij}(u)\} = A\widetilde{e}(\dot{u}).
\end{gathered} \tag{2}
$$

Here and in the sequel the dots denote the appropriate time derivatives, the time interval $I_{\mathcal{T}} \equiv (0, \mathcal{T})$ and

$$
\widetilde{e}(u) = \left\{ \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) ; \, i,j = 1,\dots,N \right\} \tag{3}
$$

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is the small strain tensor. T denotes the boundary stress vector ($T_i(u) \equiv \sigma_{ij}(u)n_j$, $i = 1, \dots, N$, where n is the outer normal vector). For a vector function $w\colon \partial\Omega \to \mathbb{R}^N$ we denote by $w_n \equiv w_i n_i$ its normal component and by $w_t \equiv w - w_n n$ its tangential component. To avoid any confusion with this notation, we shall denote the time variable by τ. The operator A is assumed to be linear in the form $A\colon \widetilde{e}(u) \mapsto a_{ijkl}e_{kl}(u)$ with the coefficients satisfying $|\xi|^{-2}a_{ijkl}(x)\xi_{ij}\xi_{kl} \in (\omega_1, \omega_2)$, $x \in \Omega$, ξ symmetric $\in R^{N^2}$ for some positive constants ω_i, $i = 1, 2$ (independent of x and ξ), and the usual symmetries $a_{ijkl} = a_{jikl} = a_{klij}$, $i, j, k, l = 1, \dots, N$ on Ω. (Here and in the sequel we use the usual summation convention.) The space-dependent stored energy function $W\colon \mathbb{R}^{N+N^2} \to \mathbb{R}$ is assumed to be C_2-smooth on $\overline{\Omega} \times \mathbb{R}^{N^2}$, satisfying $W(\cdot, 0) = 0$, $\frac{\partial W}{\partial \widetilde{e}}(\cdot, 0) = 0$ and having the partial Hess matrix $\frac{\partial^2 W}{\partial \widetilde{e}^2}$ uniformly strongly elliptic with the ellipticity constant β_0 and uniformly bounded (with constant β_1) for almost every $x \in \Omega$. G is assumed to be nonnegative and $-G$ is a given friction force. (We remark that the signs used in the third row in (1) are chosen for the sake of simplicity of notation.) Let us mention that the additional condition $G = -\mathscr{F}T_n(u)$, where nothing else than the coefficient of friction $\mathscr{F}$ is given, formulates the classical contact problem with Coulomb friction.

The mathematical difficulty of the problem, which has a parabolized character, consists in the Signorini boundary condition formulated in displacements. The results of the paper are in a close connection with [6], where the contact problem without friction for nonlinearly elastic material with a singular memory is studied, with [8], where a contact problem for viscoelastic membrane is solved and with [4], where an analogous problem is investigated for linear elasticity but, differently from the above mentioned approach, the Signorini boundary value condition is formulated in velocities.

We remark that the boundedness of Ω can be replaced by the boundedness (and finite measure) of $\operatorname{supp} G$. Then all convergence results used, partially based on imbedding theorems, will be proved for some neighbourhood of $\operatorname{supp} G$ and the results of the paper will remain valid.

To give the variational formulation of the problem and to solve it, we shall use the following notation: For $l \in \mathbb{R}_+$, $p \in (1, +\infty)$ and a domain $M \subset \mathbb{R}^m$ (having a sufficiently smooth boundary) we denote by $W_p^l(M)$ the Sobolev space of $L_p(M)$-functions having the (fractional, if l is non-integer) derivatives in all directions of the order l such that for $i = 1, \dots, m$ these derivatives in the coordinate directions belong to $L_p(M)$—cf. [1], [12], … If $p = 2$, we shall write $W_p^l(M) = H^l(M)$. $\mathring{H}(M)$ denotes the space of functions from $H^l(M)$ having zero traces on ∂M. If $l \in \mathbb{R}_+^2$, then its first coordinate indicates the existence of the appropriate time derivative and the second the existence of the appropriate space derivatives such that all derivatives mentioned belong to $L_p(M)$. $C_k(M)$, $k \geqslant 0$, denotes the space of continously differentiable (or continuous for $k < 1$) functions the highest-order derivatives of which are Hölder continuous with the exponent equal to the fractional part of k. It is equipped with the usual norm. For an interval $I \subset \mathbb{R}$ and a Banach space B, $L_p(I;B)$ denotes the usual Bochner space, and also the introduction of $W_p^l(I;B)$ or $H^l(I;B)$, $l \geqslant 0$ is obvious—cf. [11], [12], [5] etc. $B_0(I;B)$ is the space of bounded functions from I into B with the sup-norm. For a set M, $\operatorname{Int} M$ will denote its interior.

For $w \in H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^N)$ denote $\mathscr{C}_w := \{v \in H^1(\Omega;\mathbb{R}^N);\ v = w \text{ on } \partial\Omega_u,\ v_n \leqslant 0 \text{ a.e. in } \partial\Omega_c\}$. We introduce the variational formulation of the problem: a weak solution to (1) will be a function $u \in B_0\big(I_{\mathscr{T}};H^1(\Omega;\mathbb{R}^N)\big)$ for which $u(\tau,\cdot) \in \mathscr{C}_{U(\tau,\cdot)}$ for a.e. $\tau \in I_{\mathscr{T}}$, $\dot u \in B_0\big(I_{\mathscr{T}};L_2(\Omega;\mathbb{R}^N)\big) \cap L_2\big(I_{\mathscr{T}};H^1(\Omega;\mathbb{R}^N)\big)$, $\dot u(\mathscr{T},\cdot) \in L_2(\Omega;\mathbb{R}^N)$ (therefore $\ddot u \in (H^1(Q_{\mathscr{T}};\ \mathbb{R}^N))^*$) and for all $v \in H^1(Q_{\mathscr{T}};\ \mathbb{R}^N)$ such that $v(\tau,\cdot) \in \mathscr{C}_{U(\tau,\cdot)}$ a.e. in $I_{\mathscr{T}}$ the following inequality holds:

(4)
$$\int_{Q_{\mathscr{T}}} (\sigma_{ij}(u)e_{ij}(v-u) - \dot u_i(\dot v_i - \dot u_i))\,\mathrm{d}x\,\mathrm{d}\tau + \int_{S_{c,\mathscr{T}}} G\left(|v_t + \dot u_t - u_t| - |\dot u_t|\right)\mathrm{d}x\,\mathrm{d}\tau$$
$$+ \int_{\Omega} (\dot u_i(v_i - u_i))\,(\mathscr{T},\cdot)\,\mathrm{d}x \geqslant \int_{\Omega} (u_1)_i(v_i(0,\cdot) - (u_0)_i)\,\mathrm{d}x$$
$$+ \int_{Q_{\mathscr{T}}} f_i(v_i - u_i)\,\mathrm{d}x\,\mathrm{d}\tau + \int_{S_{T,\mathscr{T}}} T_{0,i}(v_i - u_i)\,\mathrm{d}x\,\mathrm{d}\tau.$$

The inequality (4) clearly follows from (1) by multiplying the equilibrium of forces by $v - u$, by integrating the result over $Q_{\mathscr{T}}$, using the Green theorem both in the time and space variables and the boundary value conditions and the initial conditions in (1). For the treatment of the friction term cf. [2], Chapter III, Section 5 and [3].

Now we introduce the penalized problem to (1). We consider the simple penalty function $h\colon z \mapsto \frac{1}{2}(z^+)^2$ with $z^+ \equiv \max(0,z)$, $z \in \mathbb{R}$. Moreover, we introduce smoothing convex functions

(5)
$$K_\eta\colon x \mapsto \begin{cases} |x|, & |x| \geqslant \eta, \\ -\dfrac{1}{8\eta^3}|x|^4 + \dfrac{3}{4\eta}|x|^2 + \dfrac{3}{8}\eta, & |x| \leqslant \eta, \end{cases}$$
$x \in \mathbb{R}^N$, for $\eta > 0$ and
$K_0\colon x \mapsto |x|$, $x \in \mathbb{R}^N$.

For arbitrary $\eta > 0$ we have

(6)
$$K_\eta \in C_2(\mathbb{R}^N),\quad 0 \leqslant K_\eta \text{ are Lipschitz with the constant 1 on } \mathbb{R}^N,$$
$$\operatorname{supp}(K_\eta - K_0) \subset \{x \in \mathbb{R}^N;\ |x| \leqslant \eta\}$$
$$\text{and}\quad \|K_\eta - K_0\|_{C_{1-\beta}(\mathbb{R}^N)} \leqslant \text{const}\ \eta^\beta,\ \beta \in (0,1).$$

Put $\mathscr{H} := \{w \in H^1(\Omega; R^N)); w = 0$ a.e. in $\partial\Omega_u\}$. For $\varepsilon > 0$ and $\eta > 0$ we define that $u_{\varepsilon,\eta}$ is the weak solution of the penalized problem, iff $u_{\varepsilon,\eta} \in U + B_0(I_{\mathscr{T}}; \mathscr{H})$ for which $\dot{u}_{\varepsilon,\eta} \in B_0(I_{\mathscr{T}}; L_2(\Omega; \mathbb{R}^N)) \cap L_2(I_{\mathscr{T}}; H^1(\Omega; \mathbb{R}^N))$ and $\ddot{u}_{\varepsilon,\eta} \in L_2(I_{\mathscr{T}}; \mathscr{H}^*)$, the initial condition in (1) is satisfied and the equation

$$\begin{aligned}(7)\qquad &\int_{Q_{\mathscr{T}}} ((\ddot{u}_{\varepsilon,\eta})_i v_i + \sigma_{ij}(u_{\varepsilon,\eta}) e_{ij}(v)) \,\mathrm{d}x\,\mathrm{d}\tau \\ &+ \int_{S_{c,\mathscr{T}}} \Big(G(\nabla K_\eta)((\dot{u}_{\varepsilon,\eta})_t) v_t + \frac{1}{\varepsilon} h'((u_{\varepsilon,\eta})_n) v_n \Big) \,\mathrm{d}x\,\mathrm{d}\tau \\ &= \int_{Q_{\mathscr{T}}} f_i v_i \,\mathrm{d}x\,\mathrm{d}\tau + \int_{S_{T,\mathscr{T}}} T_{0,i} v_i \,\mathrm{d}x\,\mathrm{d}\tau\end{aligned}$$

holds for any $v \in L_2(I_{\mathscr{T}}; \mathscr{H})$. (The function U is assumed to be defined on $Q_{\mathscr{T}}$ and the prime denotes the derivative of the corresponding function $\mathbb{R} \to \mathbb{R}$.) In fact, the penalized problem consists in replacing the Signorini boundary value condition on $S_{c,\mathscr{T}}$ in (1) by the condition

$$T_n(u_{\varepsilon,\eta}) = -\frac{1}{\varepsilon} h'((u_{\varepsilon,\eta})_n).$$

and in smoothing the Coulomb law condition. This can be proved using (7), where $v = w - u_{\varepsilon,\eta}$ for arbitrary $w \in U + L_2(I_{\mathscr{T}}; \mathscr{H})$ is put and the inequality

(8)

$$\int_{S_{c,\mathscr{T}}} G(\nabla K_\eta)(\dot{u}_{\varepsilon,\eta})_t (w_t - (\dot{u}_{\varepsilon,\eta})_t) \,\mathrm{d}x\,\mathrm{d}\tau \leqslant \int_{S_{c,\mathscr{T}}} G(K_\eta(w_t) - K_\eta((\dot{u}_{\varepsilon,\eta})_t)) \,\mathrm{d}x\,\mathrm{d}\tau,$$

which holds due to the convexity of K_η and to the non-negativity of G for any $\eta, \varepsilon > 0$ and each $w \in L_2(I_{\mathscr{T}}; H^{\frac{1}{2}}(\partial\Omega; \mathbb{R}^N))$, provided $G \in L_2(I_{\mathscr{T}}; (H^{\frac{1}{2}}(\partial\Omega_c)^*))$, is applied. The resulting variational inequality will be denoted by (7′).

The introduced problems will be solved under the following set of assumptions:

$$\begin{aligned}(9)\qquad & u_0 \in \mathscr{C}_{U(0,\cdot)},\ u_1 \in H^1(\Omega; \mathbb{R}^N), \\ & U \in H^2(Q_{\mathscr{T}}; \mathbb{R}^N) \text{ such that } U(0,\cdot)\big|_{\partial\Omega_u} = u_0\big|_{\partial\Omega_u}, \\ & \frac{\partial U}{\partial\tau}(0,\cdot)\Big|_{\partial\Omega_u} = u_1\big|_{\partial\Omega_u} \text{ and } U = 0 \text{ a.e. in } S_{c,\mathscr{T}}, \\ & T_0 \in L_2(I_{\mathscr{T}}; (H^{\frac{1}{2}}(\partial\Omega_T; \mathbb{R}^N))'), \\ & f \in L_2(I_{\mathscr{T}}; (H^1(\Omega; \mathbb{R}^N))^*) \text{ and } 0 \leqslant G \in L_2(I_{\mathscr{T}}; (H^{\frac{1}{2}}(\partial\Omega_c))^*).\end{aligned}$$

The sign of G is understood in the usual dual sense. We remark that the assumptions can be a little weakened (some of such possibilities will be mentioned in the sequel).

To solve problem (7), we use the usual Galerkin approximation. We denote $Q_\tau \equiv I_\tau \times \Omega$, $\tau \in I_{\mathcal{T}}$, and use the same notation for S_τ, $S_{T,\tau}$, $S_{c,\tau}$. Putting

$$v = \begin{cases} \dot U - \dot u_{\varepsilon,\eta} & \text{on } Q_\tau, \\ 0 & \text{on } Q_{\mathcal{T}} \setminus Q_\tau \end{cases}$$

and exploiting (8) we obtain

$$\begin{aligned}(10)\qquad & \int_\Omega \Big(\frac{1}{2}|\dot u_{\varepsilon,\eta}|^2 + W(\cdot, \widetilde{e}(u_{\varepsilon,\eta}))\Big)(\tau,\cdot)\,\mathrm{d}x \\ & + \int_{Q_\tau} A(\widetilde{e}(\dot u_{\varepsilon,\eta}))\widetilde{e}(\dot u_{\varepsilon,\eta})\,\mathrm{d}x\,\mathrm{d}s + \int_{\partial\Omega_c} \frac{1}{2\varepsilon}|(u_{\varepsilon,\eta})_n^+|^2(\tau,\cdot)\,\mathrm{d}x \\ & \leqslant \int_{S_{c,\tau}} GK_\eta(\dot U_t)\,\mathrm{d}x\,\mathrm{d}s + \int_{S_{T,\tau}} T_{0,i}(\dot u_{\varepsilon,\eta} - \dot U)_i\,\mathrm{d}x\,\mathrm{d}s \\ & + \int_\Omega \Big(\frac{1}{2}|u_1|^2 + W(\cdot, \widetilde{e}(u_0)) - u_1\dot U(0,\cdot) + (\dot u_{\varepsilon,\eta}\dot U)(\tau,\cdot)\Big)\,\mathrm{d}x \\ & + \int_{Q_\tau} \Big(A(\widetilde{e}(\dot u_{\varepsilon,\eta}))\widetilde{e}(\dot U) + \sigma^I_{ij}(u)e_{ij}(\dot U) - \dot u_{\varepsilon,\eta}\ddot U + f_i(\dot u_{\varepsilon,\eta} - \dot U)_i\Big)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Here we used the relations $\int_{\partial\Omega_c} \frac{1}{\varepsilon}(u_0)_n^-(u_0)_n^+\,\mathrm{d}x = 0$, $\int_{S_{c,\tau}} G\,K_\eta((\dot u_{\varepsilon,\eta})_t)\,\mathrm{d}x\,\mathrm{d}s \geqslant 0$ and $\frac{\partial}{\partial\tau}W(\cdot,\widetilde{e}(u_{\varepsilon,\eta})) = \frac{\partial W}{\partial e_{ij}}(\cdot,\widetilde{e}(u_{\varepsilon,\eta}))\widetilde{e}(\dot u_{\varepsilon,\eta})$.

Let us denote by $\mathscr{R}$ the space of all $u \in H^1(\Omega;\mathbb{R}^N)$ for which $\int_\Omega e_{ij}(u)e_{ij}(u)\,\mathrm{d}x = 0$ and by Y its orthogonal complement in $H^1(\Omega;\mathbb{R}^N)$. It is well-known that $\mathscr{R}$ is the space of all shifts and rotations of Ω as a rigid and undeformable body and that $\dim\mathscr{R} = \frac{N(N+1)}{2}$. Let π_Y denote the orthogonal projection $H^1(\Omega;\mathbb{R}^N) \to Y$. From the assumption on W we obtain

$$\begin{aligned}(11)\qquad W(\cdot,\widetilde{e}(w)) &= \int_0^1 (1-\theta)\frac{\partial^2 W}{\partial e_{ij}\partial e_{kl}}(\cdot,\widetilde{e}(\theta w))\,e_{ij}(w)\,e_{kl}(w)\,\mathrm{d}\theta \\ &\in \Big(\frac{1}{2}\beta_0 e_{ij}(w)\,e_{ij}(w), \frac{1}{2}\beta_1 e_{ij}(w)\,e_{ij}(w)\Big), \\ & \Big(\frac{\partial W}{\partial e_{ij}}(\cdot,\widetilde{e}(w)) - \frac{\partial W}{\partial e_{ij}}(\cdot,\widetilde{e}(v))\Big)\,e_{ij}(w-v) \\ &= \int_0^1 \frac{\partial^2 W}{\partial e_{ij}\partial e_{kl}}(\cdot,\widetilde{e}(w+\theta(v-w)))e_{ij}(v-w)\,e_{kl}(v-w)\,\mathrm{d}\theta \\ &\in (\beta_0 e_{ij}(v-w)\,e_{ij}(v-w), \beta_1 e_{ij}(v-w)\,e_{ij}(v-w))\end{aligned}$$

for arbitrary displacements w and v on $Q_{\mathcal{T}}$. By virtue of the strong ellipticity of A and the nonnegativity and strong convexity (11) of W we derive from (10) in the standard way (with the help of the Hölder inequality and the trace theorem) the a priori estimate

$$\tag{12} \sup_{\tau\in I_{\mathscr T}}\Big(\|\dot u_{\varepsilon,\eta}(\tau,\cdot)\|^2_{L_2(\Omega;\mathbb R^N)}+\frac{1}{\varepsilon}\|(u_{\varepsilon,\eta})^+_n(\tau,\cdot)\|^2_{L_2(\partial\Omega_c)}\Big) +\|\nabla\dot u_{\varepsilon,\eta}\|^2_{L_2(Q_{\mathscr T};\mathbb R^{N^2})}\leqslant c_0,\ \text{where } c_0\equiv c_0(\mathscr J)\quad\text{with}$$

$$\mathscr J\equiv\Big[\beta_0,\beta_1,\omega_1,\omega_2,\|u_0\|_{H^1(\Omega;\mathbb R^N)},\|u_1\|_{L_2(\Omega;\mathbb R^N)},\|G\|_{L_2\left(I_{\mathscr T};(H^{\frac12}(\partial\Omega_c))^*\right)},\ \|f\|_{L_2(I_{\mathscr T};(H^1(\Omega;\mathbb R^N))^*)},\|T_0\|_{L_2\left(I_{\mathscr T};(H^{\frac12}(\partial\Omega;\mathbb R^N))^*\right)},\|U\|_{H^2(Q_{\mathscr T};\mathbb R^N)}\Big].$$

In fact, such an estimate is nearly obvious if $\operatorname{mes}_{N-1}\partial\Omega_u>0$. If $\operatorname{mes}_{N-1}\partial\Omega=0$, it holds evidently for $\pi_Y\nabla\dot u_{\varepsilon,\eta}$. However, due to the finite dimension of $\mathscr R$ the following inequalities with c_1 and c_2 independent of η and ε hold:

$$\tag{13} \int_0^{\mathscr T}\|\nabla\pi_{\mathscr R}\dot u_{\varepsilon,\eta}(\tau,\cdot)\|^2_{L_2(\Omega;\mathbb R^{N^2})}\,\mathrm d\tau\leqslant c_1\int_0^{\mathscr T}\|\pi_{\mathscr R}\dot u_{\varepsilon,\eta}(\tau,\cdot)\|^2_{L_2(\Omega;\mathbb R^N)}\,\mathrm d\tau \leqslant c_2\|\dot u_{\varepsilon,\eta}\|^2_{L_2(Q_{\mathscr T};\mathbb R^N)},$$

therefore the estimate (12) is valid in this case, too.

An arbitrary function $w\in L_2\big(I_{\mathscr T};\mathring H^1(\Omega,\mathbb R^N)\big)$ can be put into (7). The estimates (12) together with (11) and the Gronwall-lemma-type arguments yield that

$$\tag{14} \|\ddot u_{\varepsilon,\eta}\|^2_{L_2(I_{\mathscr T};H^{-1}(\Omega;\mathbb R^N))}\leqslant c_3\|\nabla\dot u_{\varepsilon,\eta}\|^2_{L_2(Q_{\mathscr T};\mathbb R^{N^2})}+c_4\|f\|^2_{L_2(I_{\mathscr T};H^{-1}(\Omega;\mathbb R^N))}$$

with c_3, c_4 independent both of ε, η and of any boundary data. Here and in the sequel, $H^{-1}(\Omega;\mathbb R^N)\equiv\big(\mathring H^1(\Omega;\mathbb R^N)\big)^*$. Now, we apply the interpolation theory for Sobolev spaces of the Hilbert type (cf. [9], Chapter 1—the technique of the local straightenning of the boundary studied e.g. in [3] shows that the requirement of the high smoothness of the boundary is redundant) for the spaces $H^1(\Omega;\mathbb R^N)$ and $H^{-1}(\Omega;\mathbb R^N)$. This and the estimate (14) lead to the estimate

$$\tag{15} \|\dot u_{\varepsilon,\eta}\|^2_{H^{\frac12}(I_{\mathscr T};L_2(\Omega;\mathbb R^N))}\leqslant c_5\|\nabla\dot u_{\varepsilon,\eta}\|^2_{L_2(Q_{\mathscr T};\mathbb R^{N^2})}+c_6\|f\|^2_{L_2(I_{\mathscr T};H^{-1}(\Omega;\mathbb R^N))},$$

where for c_5, c_6 the same assertion as for c_3 and c_4 holds. To prove it, we extend the solutions $\mathring u_{\varepsilon,\eta}\equiv\dot u_{\varepsilon,\eta}-u_1$ in time in such a way that $\mathring u_{\varepsilon,\eta}\equiv0$ for $\tau\in(-\infty,0)$, ε, $\eta>0$. Moreover, we extend f, T_0 and G by 0 onto $(\mathscr T,\infty)\times\Omega$ and U onto the same set in such a way that the appropriate conditions in (9) still hold (for such an extension see [9], Chapter 1). We employ a nonincreasing cut-off function $\varrho_0\in C_2(\mathbb R)$ such that $\varrho_0\equiv1$ on $(-\infty,\mathscr T\rangle$ and $\varrho_0\equiv0$ on $\langle2\mathscr T,+\infty)$. For $\|\nabla(\varrho_0\mathring u_{\varepsilon,\eta})\|_{L_2(\mathbb R\times\Omega;\mathbb R^{N^2})}$ the estimate of the type (12) which is uniform in ε and η remains valid. The uniform estimate of the type (14) for $\frac{\partial}{\partial\tau}(\varrho_0 \mathring{u}_{\varepsilon,\eta})$ in $L_2\left(\mathbb{R}; H^{-1}(\Omega;\mathbb{R}^N)\right)$ remains valid as well. From this and with the help of the partial Fourier transformation in time, the estimate (15) follows immediately by the use of the Hölder inequality.

The precise trace theorem and (15) yields that

$$\|\dot{u}_{\varepsilon,\eta}\|_{H^{\frac{1}{4},\frac{1}{2}}(S_{\mathcal{T}};\mathbb{R}^N)} \leqslant c_7 \|\dot{u}_{\varepsilon,\eta}\|_{H^{\frac{1}{2},1}(Q_{\mathcal{T}};\mathbb{R}^N)}, \tag{16}$$

where for c_7 the same assertion as for $c_3,\ldots,c_6$ holds. In fact, the localization technique (as in [3]) and the just defined extension in time yields that we can restrict ourselves to the case of the functions defined on $\mathcal{Q} \equiv \mathbb{R}\times\Omega$, where $\Omega \equiv \mathbb{R}^{N-1}\times\mathbb{R}^+$, and having uniformly bounded supports there. Then the extension to $\mathbb{R}\times\mathbb{R}^N$ is possible like in [9] and similarly to [10] we introduce two Fourier transformations: the first one in all variables—the transforms will be denoted by hats—and the other one with respect to the time and the tangential space variable only—the transforms will be denoted by checks. The dual time variable will be denoted by υ, the dual space variable by ξ. Then we have

(17)

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\check{\dot{u}}_{\varepsilon,\eta}(\upsilon,\xi_1,\ldots,\xi_{N-1},0)|^2 \left(1+|\upsilon|+|\xi_1|^2+\ldots+|\xi_{N-1}|^2\right)^{\frac{1}{2}} \,\mathrm{d}\upsilon\,\mathrm{d}\xi_1\ldots\mathrm{d}\xi_{N-1}\\
&= \int_{\mathbb{R}^N} \frac{1}{2\pi}\left|\int_{\mathbb{R}} \widehat{\dot{u}}_{\varepsilon,\eta}(\upsilon,\xi)\,\mathrm{d}\xi_N\right|^2 \left(1+|\upsilon|+|\xi_1|^2+\ldots+|\xi_{N-1}|^2\right)^{\frac{1}{2}} \,\mathrm{d}\upsilon\,\mathrm{d}\xi_1\ldots\mathrm{d}\xi_{N-1}\\
&\qquad\leqslant \frac{1}{2\pi}\int_{\mathbb{R}^{N+1}} |\widehat{\dot{u}}_{\varepsilon,\eta}(\upsilon,\xi)|^2\left(1+|\upsilon|+|\xi|^2\right)\,\mathrm{d}\upsilon\,\mathrm{d}\xi\\
&\qquad\times\int_{\mathbb{R}} \frac{\left(1+|\upsilon|+|\xi_1|^2+\ldots+|\xi_{N-1}|^2\right)^{\frac{1}{2}}}{1+|\upsilon|+|\xi|^2}\,\mathrm{d}\xi_N.
\end{aligned}$$

The last integral is equal to π. The appropriate expression of the Sobolev-Slobodeckii norms (cf. [5], Lemma 1) and (17) yield (16).

Let $\mathcal{Z} \equiv \{z^r;\, r\in\mathbb{N}\}$ be a basis of the space $\mathcal{H}$ which is $L_2(\Omega;\mathbb{R}^N)$-orthogonal and such that the first $\frac{N(N+1)}{2}$ elements of $\mathcal{Z}$ create a basis of $\mathcal{R}$ if $\mathrm{mes}_{N-1}\partial\Omega = 0$. The existence of such a basis is a consequence of the spectral theory. Let

$$X_m \equiv \left\{\sum_{r=1}^{m} q_r z^r;\; q_r \in L_2(I_{\mathcal{T}};\mathbb{R}^N)\right\}.$$

An element $u_{\varepsilon,\eta,m} \in U + X_m$ will be an approximate solution to (7) if it satisfies the approximate version of the initial condition and for every $v \in X_m$ the variational equation

$$
(18)\qquad \int_{Q_{\mathcal{T}}} ((\ddot u_{\varepsilon,\eta,m})_i v_i + \sigma_{ij}(u_{\varepsilon,\eta,m}) e_{ij}(v))\, \mathrm{d}x\, \mathrm{d}\tau
$$

$$
+ \int_{S_{c,\mathcal{T}}} \Big(\frac{1}{\varepsilon} h'((u_{\varepsilon,\eta,m})_n) v_n + G\, \nabla K_\eta((\dot u_{\varepsilon,\eta,m})_t) v_t \Big)\, \mathrm{d}x\, \mathrm{d}\tau
$$

$$
= \int_{Q_{\mathcal{T}}} f_i v_i\, \mathrm{d}x\, \mathrm{d}\tau + \int_{S_{T,\mathcal{T}}} T_{0,i} v_i\, \mathrm{d}x\, \mathrm{d}\tau \quad \forall v \in X_m
$$

holds, where all the terms have a good sense. The existence and unicity of such $u_{\varepsilon,\eta,m}$ for $\eta, \varepsilon > 0$ and $m \in \mathbb{N}$ is obvious as usual from the theory of ordinary differential equations. The estimate (12) for $u_{\varepsilon,\eta,m}$, being uniform with respect to $\varepsilon, \eta > 0$ and $m \in \mathbb{N}$, can be verified in the same way as the original estimate (12).

Let us denote by $(\cdot,\cdot)$ and $[\cdot,\cdot]$ the $L_2(Q_{\mathcal{T}};\mathbb{R}^N)$- and $L_2(\Omega;\mathbb{R}^N)$-scalar product, respectively. The L_2-orthogonality of $\mathscr{Z}$ yields that for $\frac{\partial^2}{\partial\tau^2} u_{\varepsilon,\eta,m} \equiv \sum_{i=1}^{m} [\ddot u_{\varepsilon,\eta,m}, z_i]\, z_i + \ddot U$ and for an arbitrary $v \in L_2(I_{\mathcal{T}};\mathscr{H})$ it holds $(\ddot u_{\varepsilon,\eta,m}, v) = (\ddot u_{\varepsilon,\eta,m}, \pi_{X_m} v) + (\ddot U, v - \pi_{X_m} v)$. From (18), (9), (12) and from the uniform boundedness of the projections π_{X_m} in $H^1(\Omega;\mathbb{R}^N)$ we prove the essential boundedness of $\{\ddot u_{\varepsilon,\eta,m};\, \varepsilon, \eta > 0, m \in \mathbb{N}\}$ in $L_2(I_{\mathcal{T}};\mathscr{H}^*)$.

Now we prove the convergence of the Galerkin approximate solutions for fixed ε and η. Due to (12), (15) and (16) which validity is now verified for $\{u_{\varepsilon,\eta,m};\, \varepsilon, \eta > 0, m \in \mathbb{N}\}$ by the same arguments as for $\{u_{\varepsilon,\eta};\, \varepsilon, \eta > 0\}$, there is a subsequence $m_k \to +\infty$ such that for every fixed $\varepsilon > 0$, $\eta > 0$ and for $k \to +\infty$ the following convergences are valid:

$$
(19)\quad \dot u_{\varepsilon,\eta,m_k} \to \dot u_{\varepsilon,\eta} \text{ in } L_2(Q_{\mathcal{T}};\mathbb{R}^N), \quad \dot u_{\varepsilon,\eta,m_k}(\mathcal{T},\cdot) \rightharpoonup \dot u_{\varepsilon,\eta}(\mathcal{T},\cdot) \text{ in } L_2(\Omega;\mathbb{R}^N),
$$

$$
\dot u_{\varepsilon,\eta,m_k} \rightharpoonup \dot u_{\varepsilon,\eta} \text{ in } H^{\frac14,\frac12}(S_{c,\mathcal{T}};\mathbb{R}^N) \implies \dot u_{\varepsilon,\eta,m_k} \to \dot u_{\varepsilon,\eta} \quad \&
$$

$$
K_\eta(\dot u_{\varepsilon,\eta,m_k}) \to K_\eta(\dot u_{\varepsilon,\eta}) \text{ both in } H^{\frac14-\alpha,\frac12-\alpha}(S_{c,\mathcal{T}};\mathbb{R}^N),\ \alpha \in \Big(0,\frac14\Big),
$$

$$
\nabla u_{\varepsilon,\eta,m_k} \to \nabla u_{\varepsilon,\eta} \text{ in } L_2(Q_{\mathcal{T}};\mathbb{R}^{N^2}), \quad \nabla \dot u_{\varepsilon,\eta,m_k} \rightharpoonup \nabla \dot u_{\varepsilon,\eta} \text{ in } L_2(Q_{\mathcal{T}};\mathbb{R}^{N^2}),
$$

$$
\ddot u_{\varepsilon,\eta,m_k} \rightharpoonup \ddot u_{\varepsilon,\eta} \text{ in } L_2(I_{\mathcal{T}};\mathscr{H}^*) \text{ and } \tilde\sigma(u_{\varepsilon,\eta,m_k}) \rightharpoonup \tilde\sigma(u_{\varepsilon,\eta}) \text{ in } L_2(Q_{\mathcal{T}};\mathbb{R}^{N^2}).
$$

The strong convergence of velocities holds by virtue of the compact imbedding $H^{\frac12,1}(Q_{\mathcal{T}};\mathbb{R}^N) \hookrightarrow L_2(Q_{\mathcal{T}};\mathbb{R}^N)$. The weak convergence of their traces is a consequence of (16), the strong convergence follows from a certain more general compact imbedding theorem (see e.g. [9]). The strong convergence of their K_η-images holds due to the same reasons and to (6). For the gradients the weak convergence is obvious. For the projections $\pi_{m_k} \equiv \pi_{U+X_{m_k}}$ in $H^1(Q_{\mathcal{T}};\mathbb{R}^N)$ we put $v = \pi_{m_k} u_{\varepsilon,\eta} - u_{\varepsilon,\eta,m_k}$ into (18), change (18) with the help of an appropriate version of (8) and add $\int_{Q_{\mathcal{T}}} \sigma_{ij}(u_{\varepsilon,\eta})e_{ij}(u_{\varepsilon,\eta,m_k} - u_{\varepsilon,\eta}) + \sigma_{ij}(u_{\varepsilon,\eta,m_k})e_{ij}(u_{\varepsilon,\eta} - \pi_{m_k} u_{\varepsilon,\eta})\,dx\,d\tau$ to both sides of the inequality. On the left hand side of the resulting inequality we keep $\int_{Q_{\mathcal{T}}} (\sigma_{ij}(u_{\varepsilon,\eta}) - \sigma_{ij}(u_{\varepsilon,\eta,m_k}))\, e_{ij}(u_{\varepsilon,\eta} - u_{\varepsilon,\eta,m_k})\,dx\,d\tau$ and we can check that its right hand side tends to 0. (In particular, we exploit the weak convergence of $K_\eta((\pi_{m_k} u_{\varepsilon,\eta})_t - (u_{\varepsilon,\eta,m_k})_t + (\dot{u}_{\varepsilon,\eta,m_k})_t)$ and of $K_\eta((\dot{u}_{\varepsilon,\eta,m_k})_t)$ to $K_\eta(\dot{u}_{\varepsilon,\eta})_t)$ in $H^{\frac{1}{4},\frac{1}{2}}(S_{c,\mathcal{T}})$ which follows from their boundedness in that space (due to (6), (12) and (16)) and their convergence almost everywhere in $S_{c,\mathcal{T}}$.) The strong convergence of gradients then follows from the strong monotonicity of the employed operators A and $\frac{\partial W}{\partial \tilde{e}}$ (cf. (11)) provided $\mathrm{mes}_{N-1}\partial\Omega_u > 0$. The last convergence in (19) holds again due to the strong monotonicity of $\frac{\partial W}{\partial \tilde{e}}$ which yields its maximal monotonicity, due to the almost-everywhere pointwise convergence of (a subsequence of) gradients and due to the linearity of A. From this, it is easy to see that $u_{\varepsilon,\eta}$ is a solution of (7′).

If $\mathrm{mes}_{N-1}\partial\Omega_u = 0$, we use for the strong convergence of $\nabla\pi_{\mathcal{R}} u_{\varepsilon,\eta,m_k}$, which remains to be proved, the inequality (13) and the fact that all space derivatives of such elements of orders higher than one are zero (therefore $\{\nabla\pi_{\mathcal{R}} u_{\varepsilon,\eta,m_k}\}$ is bounded in $H^1\Big(I_{\mathcal{T}}; L_2(\Omega;\mathbb{R}^{N^2})\Big) \cap L_2\Big(I_{\mathcal{T}}; H^r(\Omega;\mathbb{R}^{N^2})\Big)$ for any $r \in \mathbb{N}$ and the compact imbedding theorem can be used). The rest of the proof is the same as above.

To complete the proof, we take a subnet of $(\nabla K_\eta)(\dot{u}_{\varepsilon,\eta,m_k})$ tending to some $\tilde{K}$ in the w^*-topology of $L_\infty(S_{c,\mathcal{T}};\mathbb{R}^N)$. Let the sequence $\{G_p\} \subset L_2(S_{c,\mathcal{T}})$ tend to G in $L_2(I_{\mathcal{T}}; (H^{\frac{1}{2}}(\partial\Omega_c))^*)$. Then for arbitrary $w \in L_2\Big(I_{\mathcal{T}}; H^{\frac{1}{2}}(\partial\Omega_c;\mathbb{R}^N)\Big)$ we have $\int_{S_{c,\mathcal{T}}} G((\nabla K_\eta)(\dot{u}_{\varepsilon,\eta,m_k}) - \tilde{K})w\,dx\,d\tau = \int_{S_{c,\mathcal{T}}} (G - G_p)((\nabla K_\eta)(\dot{u}_{\varepsilon,\eta,m_k}) - \tilde{K})w\,dx\,d\tau + \int_{S_{c,\mathcal{T}}} G_p((\nabla K_\eta)(\dot{u}_{\varepsilon,\eta,m_k}) - \tilde{K})w\,dx\,d\tau$. The first term tends to zero for that net also due to the boundedness of $\Big\{(\nabla K_\eta)(\dot{u}_{\varepsilon,\eta,m_k}) - \tilde{K}\Big\}$ in $L_\infty(S_{c,\mathcal{T}};\mathbb{R}^N)$ and the second also due to the boundedness of $\{G_p w\}$ in $L_1(S_{c,\mathcal{T}};\mathbb{R}^N)$. The strong L_2-convergence of the traces of velocities yields the identity $\tilde{K} = (\nabla K_\eta)(\dot{u}_{\varepsilon,\eta})$ and therefore for some subsequence (denoted again by m_k) the convergence $G\,(\nabla K_\eta)(\dot{u}_{\varepsilon,\eta,m_k}) \rightharpoonup G\,(\nabla K_\eta)(\dot{u}_{\varepsilon,\eta})$ holds in $L_2\Big(I_{\mathcal{T}}; (H^{\frac{1}{2}}(\partial\Omega_c))^*\Big)$. We have proved

Lemma. *Let the assumptions concerning Ω, its boundary, the operator A, the function W and the assumptions (9) be fulfilled. Then there exists a solution to the problem (7) for every $\varepsilon > 0$ and $\eta > 0$.*

Remark. The assumption $u_1 \in H^1(\Omega;\mathbb{R}^N)$ in (9) was imposed in order to make possible to consider more general G (see Theorem below). If we restrict ourselves to G from (9), the usual assumption $u_1 \in L_2(\Omega;\mathbb{R}^N)$ is sufficient both for Lemma and for Theorem below.

2. Existence theorem and regularity result

The aim of this section is to prove

Theorem. *Let all the assumptions of Lemma be fulfilled with the exception of G for which we assume $G \in \left(H^{\frac14,\frac12}(S_{c,\mathcal{T}})\right)^*$. Then there exists a weak solution to the contact problem* (1).

To prove Theorem, two limit procedures must be carried out. First we make the limit procedure for η. Due to (12) (15) and (16) it is easy to see that there is a sequence $\{\eta_k\}$ such that $\eta_k \to 0$ for $k \to +\infty$ and all the weak convergences of the derivatives similarly as in (19) are valid, where the limit elements will be denoted by u_ε. Moreover, the strong convergence of velocities $\dot{u}_{\varepsilon,\eta} \to \dot{u}_\varepsilon$ in $L_2(Q_{\mathcal{T}})$ and of their traces like in (19) holds. It is easy to see that for any $\alpha \in (0, \frac12)$, each $\beta, \eta \in (0, 1)$ and any $w \in H^{\frac12}(\partial\Omega_c)$ the following inequality holds:

$$
\begin{aligned}
(20) \qquad \|(K_\eta - K_0)(w)\|^2_{H^{\frac12-\alpha}(\partial\Omega_c;\mathbb{R}^N)} &\leqslant \|K_\eta - K_0\|^2_{C_0(\mathbb{R})}(\mathrm{mes}_{N-1}\partial\Omega_c) \\
&+ \|K_\eta - K_0\|^2_{C_{1-\beta}(\mathbb{R})} \int_{\partial\Omega_c}\int_{\partial\Omega_c} \frac{|w(x) - w(y)|^{2-2\beta}}{|x-y|^{N+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

The continuous imbedding $H^{\frac12}(\partial\Omega_c) \hookrightarrow W_{2-2\beta}^{\frac{1-2\alpha}{2-2\beta}}(\partial\Omega_c)$, $\beta \in (0, 2\alpha)$, based on the results of [12], Chapter 2, Sec. 4 (cf. [1], too), and the relations (6) yield that the left hand side in (20) tends to 0 uniformly with respect to bounded sets in $H^{\frac12}(\partial\Omega_c;\mathbb{R}^N)$. Using this and the compactness of the operator $w \mapsto |w|$ from $H^{\frac14,\frac12}(S_{c,\mathcal{T}})$ to $L_2(I_{\mathcal{T}}; H^{\frac12-\alpha}(\partial\Omega_c))$, $\alpha > 0$, we obtain

$$
\begin{aligned}
(21) \qquad K_{\eta_k}((\dot{u}_{\varepsilon,\eta_k})_t) &= (K_{\eta_k}((\dot{u}_{\varepsilon,\eta_k})_t) - K_0((\dot{u}_{\varepsilon,\eta_k})_t)) + K_0((\dot{u}_{\varepsilon,\eta_k})_t) \\
&\xrightarrow{L_2(I_{\mathcal{T}};H^{\frac12-\alpha}(\partial\Omega_c))} K_0((\dot{u}_\varepsilon)_t).
\end{aligned}
$$

On the other hand, $\{K_{\eta_k}((\dot{u}_{\varepsilon,\eta_k})_t)\}$ is bounded in $H^{\frac14,\frac12}(S_{c,\mathcal{T}})$ due to (6) (12) and (16), therefore there is its subsequence having a weak limit there. From (21) we can derive in the standard way that this limit is $K_0((\dot{u}_\varepsilon)_t)$ and it is the limit of the whole sequence. Analogously, $K_{\eta_k}((u_\varepsilon)_t - (u_{\varepsilon,\eta_k})_t + (\dot{u}_{\varepsilon,\eta_k})_t) \xrightarrow{H^{\frac14,\frac12}(S_{c,\mathcal{T}})} K_0((\dot{u}_\varepsilon)_t)$. Let us put $w = u_\varepsilon$ in (7′). Similarly to the proof of Lemma, we add $\int_{Q_{\mathcal{T}}} \sigma_{ij}(u_\varepsilon)e_{ij}(u_{\varepsilon,\eta} - u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}\tau$ to both sides of (7′). With the same arguments as in the proof of Lemma we prove the strong convergence of gradients like in (19) and then the weak convergence of stresses. Thus we prove that u_ε is the solution of the variational inequality

$$(22)\ \int_{Q_{\mathcal{T}}} (\ddot{u}_\varepsilon)_i(w-u_\varepsilon)_i + \sigma_{ij}(u_\varepsilon)e_{ij}(w-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}\tau + \int_{S_{c,\mathcal{T}}} \frac{1}{\varepsilon}h'((u_\varepsilon)_n)(w-u_\varepsilon)_n$$
$$+G\left(|w_t+(\dot{u}_\varepsilon)_t-(u_\varepsilon)_t|-|(\dot{u}_\varepsilon)_t|\right)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\geqslant \int_{Q_{\mathcal{T}}} f_i(w-u_\varepsilon)_i\,\mathrm{d}x\,\mathrm{d}\tau + \int_{S_{T,\mathcal{T}}} T_{0,i}(w-u_\varepsilon)_i\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\forall w\in U+L_2(I_{\mathcal{T}};\mathcal{H}).$$

Now, the a priori estimate (12) can be recalculated using the ideas leading to (14), (15) and (16). For the test functions v such that $v_t\in H^{\frac14,\frac12}(S_{c,\mathcal{T}};\mathbb{R}^N)$ we use the estimate of the friction term

$$(23)\qquad \int_{S_{c,\mathcal{T}}} G\left(|v_t+(\dot{u}_\varepsilon)_t-(u_\varepsilon)_t|-|(\dot{u}_\varepsilon)_t|\right)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\leqslant \check{c}\|G\|_{\left(H^{\frac14,\frac12}(S_{c,\mathcal{T}})\right)^*}\left(\|\dot{u}_\varepsilon\|_{H^{\frac14,\frac12}(S_{c,\mathcal{T}};\mathbb{R}^N)}+\|v_t\|_{H^{\frac14,\frac12}(S_{c,\mathcal{T}};\mathbb{R}^N)}\right)$$

with $\check{c}$ independent of $\varepsilon>0$. On the other hand for $\widetilde{\mathcal{I}}$ summing all suitable norms of the input data with the exception of G, the test function $v=\dot{U}$ put into (22) yield

$$(24)\qquad \|\dot{u}_\varepsilon\|^2_{H^{\frac12,1}(Q_{\mathcal{T}};\mathbb{R}^N))}\leqslant c_5\|\dot{u}_\varepsilon\|^2_{L_2(I_{\mathcal{T}};H^1(\Omega;\mathbb{R}^N))}+c_6\|f\|^2_{L_2(I_{\mathcal{T}};H^{-1}(\Omega;\mathbb{R}^N))}$$
$$\leqslant c_8(\widetilde{\mathcal{I}})+c_9\|G\|_{\left(H^{\frac14,\frac12}(S_{c,\mathcal{T}})\right)^*}\|\dot{u}_\varepsilon\|_{H^{\frac12,1}(Q_{\mathcal{T}};\mathbb{R}^N)}$$

with the constants c_5, c_6 from (15), and c_8, c_9 independent of $\varepsilon>0$. Therefore the solutions u_ε satisfy (12) with $\mathcal{I}$, where $\|G\|_{(H^{\frac14,\frac12}(S_{c,\mathcal{T}}))^*}$ replaces $\|G\|_{L_2(I_{\mathcal{T}};(H^{\frac12}(\partial\Omega_c))^*)}$. Then the used technique gives easily that the penalized problem has a solution for any $G\in\left(H^{\frac14,\frac12}(S_{c,\mathcal{T}})\right)^*$.

For the second limit procedure for $\varepsilon\to0$ we verify again the validity of the convergences like in (19)† to a certain limit u for some sequence $\varepsilon_k\to0$. In particular, we can prove that $\dot{u}_{\varepsilon_k}\to\dot{u}$ in $L_2(Q_{\mathcal{T}};\mathbb{R}^N)$ which is important due to the sign at $\|\dot{u}\|^2_{L_2(Q_{\mathcal{T}};\mathbb{R}^N)}$ in (4) excluding the use of the weak lower semicontinuity arguments. The proofs of the remaining strong convergences are based on the same ideas as in the preceding limit procedures and then the weak convergence of stresses is clear. It is obvious that the limit u satisfies (4). Redefining the set of the test functions for (4) in such a way that the appropriate anologue of estimates (23) and (24) can be performed, we prove the existence of a solution for each $G\in\left(H^{\frac14,\frac12}(S_{c,\mathcal{T}})\right)^*$. Theorem is proved.

† The accelerations converge in $L_2(I_{\mathcal{T}};H^{-1}(\Omega;\mathbb{R}^N))$.

Corollary. *Under the assumptions of Theorem let, moreover, the coefficients of* A *be* C_1*-smooth on* $\overline{\Omega}$, *let* W *be* C_2*-smooth on* $\overline{\Omega} \times \mathbb{R}^{N^2}$, *let* $f \in L_2(Q_{\mathcal{T}}; \mathbb{R}^N)$, *and let* $G \in \left(H^{\frac{1}{4},0}(S_{c,\mathcal{T}})\right)^*$. *Then the solution found belongs to* $B_0\left(I_T; H^{\frac{3}{2},1}(\Omega'; \mathbb{R}^N)\right)$, *where* $\Omega' \subset \Omega$ *is a domain along the contact part of the boundary, the first index denotes the tangential and the second the normal regularity of* u. *Moreover, the solution belongs to any space* $H^{\frac{3}{2},\frac{3}{2},1}(Q'_{\mathcal{T}}; \mathbb{R}^N)$ *for any* $Q'_{\mathcal{T}} \equiv I_{\mathcal{T}} \times \Omega'$, Ω' *as above. (Here the first component of the vector-index of the space corresponds to the time variable, the second to the tangential space variables and the last to the normal variable.)*

The proof of the time regularity (the second term) was in fact done without any additional assumption. Due to the strong monotonicity of A and of $\frac{\partial W}{\partial \bar{e}}$, the space regularity in the tangential direction will be proved, after the local straightening of the boundary, by the usual shift method. By this method, a difference of displacements (at the original points and at the points shifted in a certain tangential direction) multiplied by a suitable smooth localization function is put as a test function $(v - u)$ into (4) and into its shifted version. The result is multiplied by an appropriate power of the Euclidean norm of the difference of points and integrated. In the estimates of the fractional derivative seminorm, the velocity is treated as a part of the right hand side of the problem and its space regularity (cf. (12)) is exploited. For details see Remark 3.2 of [8], where an analogous proof for the case of a membrane is done, and [3], where the use of the shift method is described in all detail. The use of the method requires the smoothness of both the "coefficients" and the boundary. Of course, the strong monotonicity of A and of $\frac{\partial W}{\partial \bar{e}}$ is here essentially employed.

Remark. The $B_0(I_{\mathcal{T}}; H^{2-\varepsilon}(\tilde{\Omega}))$-regularity of the solution for any $\varepsilon > 0$ on any $\tilde{\Omega} \subset \Omega$ such that $\operatorname{dist}(\tilde{\Omega}, \partial\Omega) > 0$ can be proved via the shift method as above. Here naturally no constraint to directions of shifts occurs and the smooth localization function multiplying the difference of displacements vanishes outside $\tilde{\tilde{\Omega}}$ such that $\overline{\tilde{\Omega}} \subset \tilde{\tilde{\Omega}}$ and $\overline{\tilde{\tilde{\Omega}}} \subset \Omega$. This result can give the strong convergence of the gradients in the limit procedures and the pointwise convergence of a subsequence almost everywhere on $Q_{\mathcal{T}}$ which yields the weak convergence of stresses. Inside Ω, naturally, some better time regularity of u can be proved, too, particularly in the case of the linear viscoelasticity. The regularity along the contact part of the boundary, however, is particularly important for the possibility to solve the original contact problem with friction (where only the coefficient of friction is given) which can be solved by means of the fixed point approach (cf. [3], [10]). The impossibility to use velocities in the shift technique bounds its result to that mentioned in Corollary which is far from the possibility to use such a procedure. Differently from [7], where the contact condition is formulated in velocities, the classical shift technique does not seem to be sufficient to prove the existence of a solution to the original contact problem with Coulomb friction.

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