Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 46 (1996), No. 3, 513–522

Persistent URL: `http://dml.cz/dmlcz/127312`

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Czechoslovak Mathematical Journal, 46 (121) 1996, Praha

INVERSE SEMIRINGS AND THEIR LATTICE OF CONGRUENCES

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(Received August 16, 1994)

To the memory of Otakar Borůvka

Universal Algebra Theory, algebras whose congruences form a modular (distributive, boolean) lattice with respect to inclusion have attracted great attention. For example, in semigroup theory see [1]. The aim of this paper is to describe all inverse semirings having a modular (distributive, boolean) congruence lattice. The notion of an associative inverse semiring can be found in [2]. The congruence lattices of these semirings have also been studied in [3].

1. Introduction

We shall fix the type $\tau = (t, \mathrm{ar})$ with $t = (+, \cdot, -)$, $\mathrm{ar}(+) = \mathrm{ar}(\cdot) = 2$ and $\mathrm{ar}(-) = 1$. *An inverse semiring* is a τ-algebra $\mathscr{S} = (S, \tau)$ satisfying the axioms

(1.1) $(S, +)$ is a commutative semigroup,

(1.2) $x(y+z) = xy + xz, (y+z)x = yx + zx$,

(1.3) $x = (x-x) + x, -x = -x + (x-x)$ where $xy = x \cdot y$, $xy + z = (xy) + z$ and $x - y = x + (-y)$,

(1.4) $(x-x) + (y-y) = (x-x)(y-y)$.

Note that we need not use and suppose the associativity of the multiplication.

By $S(\mathscr{S})$ we denote the set of all elements of an inverse semiring $\mathscr{S}$. We put $E(\mathscr{S}) = \{x \in S(\mathscr{S}), x = x + x\}$. It follows from (1.3) that $x - x \in E(\mathscr{S})$ for every $x \in S(\mathscr{S})$. Let $0\colon S(\mathscr{S}) \to E(\mathscr{S})$ be a mapping such that $0x = x - x$ for all $x \in S(\mathscr{S})$. According to (1.1) and (1.3) we have

$$x + 0x = x = 0x + x.$$

In some proofs the following implication will be used:

(1.5) $x = x + y + x, y = y + x + y \Rightarrow y = -x$ for every $x, y \in S(\mathscr{S})$.

Proof. Suppose that $x = x+y+x$ and $y = y+x+y$. Then according to (1.1) and (1.3) we have $y = y+x+y = y+x+(-x)+x+y = (x+y+x)+y+(-x) = x+y+(-x) = x + y + (-x) + x + (-x) = (-x) + (x + y + x) + (-x) = (-x) + x + (-x) = -x$. $\square$

From (1.1), (1.2) and (1.5) it is easy to show the following:

(1.6) $-(-x) = x$,

(1.7) $-(x + y) = (-x) + (-y)$,

(1.8) $-(xy) = (-x)y = x(-y)$.

It follows from (1.5) and (1.4) that

(1.9) $e = -e = 0e = e^2$ for every $e \in E(\mathscr{S})$ and

(1.10) $e + f = ef \in E(\mathscr{S})$ for every $e, f \in E(\mathscr{S})$.

This implies that $\mathscr{E}(\mathscr{S}) = (E(\mathscr{S}), \tau)$ is an inverse subsemiring of $\mathscr{S}$ which is *a semilattice* and so for $e, f \in E(\mathscr{S})$ we can put

(1.11) $e \leqslant f$ if and only if $ef = e$,

(1.12) $e < f$ if and only if $e \leqslant f$ and $e \neq f$,

(1.13) $e \parallel f$ if and only if $e \neq ef \neq f$.

According to (1.2), (1.9), (1.8) amd (1.10) for every $e \in E(\mathscr{S})$ and every $x \in S(\mathscr{S})$ we have

(1.14) $xe = (0x)e = e(0x) = ex \in E(\mathscr{S})$.

Indeed, we have $xe = x(e + e) = x(e - e) = xe - xe = (x - x)e = (0x)e = (0x) = \ldots = ex$. It follows from (1.1), (1.6), (1.7), (1.9) and (1.14) that

(1.15) 0 is the homomorphism projection of $\mathscr{S}$ onto $\mathscr{E}(\mathscr{S})$.

2. Property (M)

Let $\mathscr{S}$ be an inverse semiring. By $(\mathrm{Con}(\mathscr{S}), \wedge, \vee)$ (or briefly $\mathrm{Con}(\mathscr{S})$) we denote the lattice of all congruences on $\mathscr{S}$ with respect to set inclusion. For $x, y \in S(\mathscr{S})$ we denote by $\Theta_{\mathscr{S}}(x, y)$ (or briefly $\Theta(x, y)$) the least congruence on $\mathscr{S}$ containing (x, y).

Recall that $\mathrm{Ker}\, 0 = \{(x, y);\ x, y \in S(\mathscr{S}) \text{ and } 0x = 0y\} \in \mathrm{Con}(\mathscr{S})$. By $[\mathrm{Ker}\, 0)$ we denote the principal filter of $\mathrm{Con}(\mathscr{S})$ generated by $\mathrm{Ker}\, 0$, i.e. $[\mathrm{Ker}\, 0) = \{X \in \mathrm{Con}(\mathscr{S});\ \mathrm{Ker}\, 0 \subseteq X\}$. For every $X \in [\mathrm{Ker}\, 0)$ we put $\varphi(X) = X \cap (E(\mathscr{S}) \times E(\mathscr{S}))$. It is clear that $\varphi\colon [\mathrm{Ker}\, 0) \to \mathrm{Con}(\mathscr{E}(\mathscr{S}))$.

Lemma 2.1. *The mapping φ is an isomorphism of the lattice $[\mathrm{Ker}\, 0)$ onto the lattice $\mathrm{Con}(\mathscr{E}(\mathscr{S}))$.*

P r o o f. First we shall show that $X_1 \subseteq X_2$ if and only if $\varphi(X_1) \subseteq \varphi(X_2)$ for all $X_1, X_2 \in [\operatorname{Ker} 0)$. It is clear that $X_1 \subseteq X_2$ implies $\varphi(X_1) \subseteq \varphi(X_2)$. Suppose that $\varphi(X_1) \subseteq \varphi(X_2)$. Let $(x, y) \in X_1$. Then $(0x, 0y) \in X_1$ and so $(0x, 0y) \in \varphi(X_1) \subseteq \varphi(X_2) \subseteq X_2$. It is easy to show that $(x, 0x), (0y, y) \in \operatorname{Ker} 0 \subseteq X_2$ and so $(x, y) \in X_2$. We have $X_1 \subseteq X_2$.

Now, we shall prove that φ is a surjective mapping. Let $Y \in \operatorname{Con}(\mathscr{E}(\mathscr{S}))$. Put $X = \{(x, y);\, x, y \in S(\mathscr{S}) \text{ and } (0x, 0y) \in Y\}$. According to (1.15), we have $X \in \operatorname{Con}(\mathscr{S})$. If $(x, y) \in \operatorname{Ker} 0$, then $0x = 0y$ and so $(x, y) \in X$. Therefore we have $\operatorname{Ker} 0 \subseteq X$ and so $X \in [\operatorname{Ker} 0)$. Finally, we obtain $\varphi(X) = X \cap (E(\mathscr{S}) \times E(\mathscr{S})) \subseteq Y \subseteq \varphi(X)$. Hence we have $\varphi(X) = Y$. $\square$

Recall that a semilattice $\mathscr{E}$ is *a tree* if for any pair of elements $e, f \in S(\mathscr{E})$ with $e \parallel f$ there is no element $g \in S(\mathscr{E})$ such that $e \leqslant g$ and $f \leqslant g$.

Lemma 2.2. *Let $\mathscr{S}$ be an inverse semiring.*
If $\operatorname{Con}(\mathscr{S})$ is modular, then $\mathscr{E}(\mathscr{S})$ is a tree.
If $\mathscr{E}(\mathscr{S})$ is a tree, then $[\operatorname{Ker} 0)$ is distributive.

The proof follows from Lemma 2.1 and Theorem 4.4 of [1]. $\square$

Lemma 2.3. *If the lattice $\operatorname{Con}(\mathscr{S})$ is modular, then $a + f = f$ for all elements a, f of an inverse semiring $\mathscr{S}$, where $f = 0f < 0a$.*

P r o o f. Suppose that $\operatorname{Con}(\mathscr{S})$ is modular and $a+f \neq f$, where $f = 0f < 0a = e$. It follows from (1.4) that $a \neq e$. Put $A = \Theta(a + f, f), B = \Theta(e, f)$ and $C = \Theta(a, e)$. We have $(a + f, f) = (a, e) + (f, f) \in C$ and so $A \subseteq C$.

Let $g \in E(\mathscr{S})$ and put $G_g = \{(x, y);\, x, y \in S(\mathscr{S}), \text{ where } x = y \text{ or } 0x = 0y \leqslant g\}$. We shall show that $G_g \in \operatorname{Con}(\mathscr{S})$. Evidently G_g is an equivalence on $S(\mathscr{S})$. Suppose that $(x, y) \in G_g$ and $x \neq y$. By (1.15) we have $(-x, -y) \in G_g$. If $z \in S(\mathscr{S})$, then by virtue of (1.4) we obtain $0(x + z) = 0(xz) = (0x)(0z) = (0y)(0z) = 0(yz) = 0(y + z) \leqslant g$ and $0(xz) = 0(yz) = 0(zx) = 0(zy) \leqslant g$. Therefore $G_g \in \operatorname{Con}(\mathscr{S})$.

We have $(a + f, f) \in G_f$ and so $A \subseteq G_f$. From $(a, e) \in G_e$ it follows that $C \subseteq G_e$.

Let $D = \{(x, y);\, x, y \in S(\mathscr{S}) \text{ and } x + f = y + f\}$. We shall prove that $D \in \operatorname{Con}(\mathscr{S})$. It is clear that D is an equivalence on $S(\mathscr{S})$. Assume that $(x, y) \in D$. By (1.7) and (1.9) we have $(-x, -y) \in D$. If $z \in S(\mathscr{S})$, then (1.1) implies $(x + z, y + z) \in D$. It remains to show that $(xz, yz) \in D$ (and dually $(zx, zy) \in D$) for every $z \in S(\mathscr{S})$. We have $x + f = y + f$ and so, by (1.15) amd (1.2), we obtain

$$0x + f = 0y + f \quad \text{and} \quad xz + fz = yz + fz. \tag{2.1}$$

Using (1.14) and (1.10) we have

$$xz + f + 0z = yz + f + 0z. \tag{2.2}$$

From (2.1) and (2.2) it follows that $xz + f = xz + 0(xz) + f = xz + 0x + 0z + f = yz + 0x + 0z + f = yz + 0y + 0z + f = yz + 0(yz) + f = yz + f$. Thus we have $D \in \mathrm{Con}(\mathscr{S})$.

Since $(e, f) \in D$, we have $B \subseteq D$ and so $B \cap C \subseteq D \cap G_e = D \wedge G_e$. We have $(a, a + f) = (a, a) + (e, f) \in B$, $(a + f, f) \in A$ and $(f, e) \in B$. This implies $(a, e) \in (A \vee B) \wedge C = A \vee (B \wedge C) \subseteq G_f \vee (D \wedge G_e)$. Then there exists a finite sequence $a = x_0, x_1, \ldots, x_n = e$ of elements from $S(\mathscr{S})$ such that $(x_{i-1}, x_i) \in G_f$ or $(x_{i-1}, x_i) \in D \cap G_e$. We can suppose that the length $n \geqslant 1$ is minimal.

If $(x_0, x_1) \in G_f$, then $x_0 = x_1$, which is a contradiction. We have $x_0 \neq x_1$ and $(x_0, x_1) \in D \cap G_e$. Then $0x_1 = 0x_0 = e$, $a + f = x_0 + f = x_1 + f$ and so $x_1 \neq e$.Therefore $n \geqslant 2$ and $(x_1, x_2) \in G_f$. This implies that $x_1 = x_2$, a contradiction.

Consequently, we have $a + f = f$. □

Definition 2.1. We shall say that *an inverse semiring* $\mathscr{S}$ has property (M) if

$$a + f = f$$

for all $a, f \in S(\mathscr{S})$, where $f = 0f < 0a$.

Lemma 2.4. *If an inverse semiring $\mathscr{S}$ has property* (M), *then for $a, b \in S(\mathscr{S})$ we have*

(i) $a + b = b$ *for* $0b < 0a$,

(ii) $ab = ba = 0b$ *for* $0b < 0a$,

(iii) $a + b = ab = 0(ab)$ *for* $0a \parallel 0b$.

Proof. (i) and (ii). If $0b < 0a$, then by (1.3) and Definition 2.1 we obtain $a + b = a + 0b + b = 0b + b = b$. Further we have $ab + b^2 = b^2$ (see (1.2)) and so $ab + 0b^2 = 0b^2$. It is easy to show that $0(ab) = 0b = 0b^2$. Hence we have $ab = 0b$. Analogously we can show that $ba = 0b$.

(iii). Suppose that $0a \parallel 0b$. According to (1.3), (1.4) and Definition 2.1, we have $a+b = a+b+0(a+b) = a+b+0(ab) = a+0(ab) = 0(ab)$. This implies that $a^2+ab = a0(ab)$ and so, by (1.14), (1.15) and (i), we have $ab = a^2 + ab = (0a)0(ab) = 0(ab)$. □

3. Property (D)

Let $\mathscr{S}$ be an inverse semiring and let $e \in E(\mathscr{S})$. By $\mathscr{S}_e$ we denote the inverse subsemiring of $\mathscr{S}$ satisfying $S(\mathscr{S}_e) = \{x \in S(\mathscr{S});\, 0x = e\}$. By virtue of (1.15) it is easy to show that $E(\mathscr{S}_e) = \{e\}$ and so $\mathscr{S}_e$ is a subring of $\mathscr{S}$.

Put $H_e = \{(x, y);\, x, y \in S(\mathscr{S})$, where $x = y$ or $0x = e = 0y\}$.

Lemma 3.1. *Let $\mathscr{S}$ be an inverse semiring having property* (M) *and let $e \in E(\mathscr{S})$. Then $H_e \in \mathrm{Con}(\mathscr{S})$ and the principal ideal $(H_e]$ of $\mathrm{Con}(\mathscr{S})$ generated by H_e is isomorphic to $\mathrm{Con}(\mathscr{S}_e)$.*

Proof. Suppose that an inverse semiring $\mathscr{S}$ has property (M) and $e \in E(\mathscr{S})$. Let $X \in \mathrm{Con}(\mathscr{S}_e)$ and put $\psi(X) = X \cup \mathrm{id}_{S(\mathscr{S})}$.

First we shall show that $\psi(X) \in \mathrm{Con}(\mathscr{S})$. Evidently, $\psi(X)$ is an equivalence on $S(\mathscr{S})$. Suppose that $(x, y) \in \psi(X)$ and $x \neq y$. Then $(x, y) \in X$ and so $0x = e = 0y$. It is clear that $(-x, -y) \in X \subseteq \psi(X)$, Let $z \in S(\mathscr{S})$. If $0z = e$, then evidently $(x+z, y+z), (xz, yz), (zx, zy) \in X \subseteq \psi(X)$. From Lemma 2.4 we obtain the following implications. If $0z < e$, then $(x + z, y + z) = (z, z)$ and $(xz, yz) = (zx, zy) = (0z, 0z)$ belong to $\psi(X)$. If $e < 0z$, then $(x+z, y+z) = (x, y)$ and $(xz, yz) = (zx, zy) = (e, e)$ belong to $X \subseteq \psi(X)$. If $e \parallel 0z$, then $(x + z, y + z) = (xz, yz) = (zx, zy) = (ez, ez) \in \psi(X)$. Hence $\psi(X) \in \mathrm{Con}(\mathscr{S})$ and so $\psi\colon \mathrm{Con}(\mathscr{S}_e) \to \mathrm{Con}(\mathscr{S})$.

It is easy to see that $X \subseteq Y$ if and only if $\psi(X) \subseteq \psi(Y)$ for all $X, Y \in \mathrm{Con}(\mathscr{S}_e)$. Clearly $\psi(S(\mathscr{S}_e) \times S(\mathscr{S}_e)) = H_e$ and so $H_e \in \mathrm{Con}(\mathscr{S})$ and $\psi(\mathrm{Con}(\mathscr{S}_e)) \subseteq (H_e] = \{X \in \mathrm{Con}(\mathscr{S});\, X \subseteq H_e\}$. It remains to prove that $\psi(\mathrm{Con}(\mathscr{S}_e)) = (H_e]$. Let $Y \in (H_e]$. Then $Y \subseteq H_e$. Put $X = Y \cap (S(\mathscr{S}_e) \times S(\mathscr{S}_e))$. Clearly $X \in \mathrm{Con}(\mathscr{S}_e)$. Suppose that $(x, y) \in \psi(X)$. Then $(x, y) \in X$ or $x = y$ and so $(x, y) \in Y$. Hence we have $\psi(X) \subseteq Y$. Assume that $(x, y) \in Y$. Then $(x, y) \in H_e$ and this implies $x, y \in S(\mathscr{S}_e)$ or $x = y$. Consequently, we obtain that $(x, y) \in X \cup \mathrm{id}_{S(\mathscr{S})} = \psi(X)$. Thus we have $Y \subseteq \psi(X)$ and so $Y = \psi(X)$. □

Definition 3.1. We shall say that *an inverse semiring $\mathscr{S}$ has property* (D) if for each $e \in E(\mathscr{S})$ the lattice $\mathrm{Con}(\mathscr{S}_e)$ is distributive.

Lemma 3.2. *If the lattice $\mathrm{Con}(\mathscr{S})$ is distributive, then an inverse semiring $\mathscr{S}$ has property* (D).

The proof follows from Lemma 2.3 and Lemma 3.1. □

Lemma 4.1. *Let $X, Y \in \mathrm{Con}(\mathscr{S})$, where $\mathscr{S}$ is an inverse semiring. Then $(e, f) \in X \vee Y$ for $e, f \in E(\mathscr{S})$ if and only if there is a finite sequence $e = x_0, x_1, \ldots, x_n = f$ of elements from $E(\mathscr{S})$ such that $(x_{i-1}, x_i) \in X \cup Y$ for $i = 1, 2, \ldots, n$.*

Proof. It is well known that $(e, f) \in X \vee Y$ if and only if there is a finite sequence $e = y_0, y_1, \ldots, y_n = f$ of elements from $S(\mathscr{S})$ such that $(y_{i-1}, y_i) \in X \cup Y$ for $i = 1, 2, \ldots, n$. Suppose that $e, f \in E(\mathscr{S})$ and put $x_i = 0y_i$ for $i = 0, 1, \ldots, n$. Then we have $(x_{i-1}, x_i) \in X \cup Y$ for $i = 1, 2, \ldots, n$ and $x_0 = e, x_n = f$ and $x_i \in E(\mathscr{S})$. □

Lemma 4.2. *Let $X \in \mathrm{Con}(\mathscr{S})$, where S is an inverse semiring. If for $e, f \in E(\mathscr{S})$ we have $(e, f) \in X \vee \mathrm{Ker}\, 0$, then $(e, f) \in X$.*

Proof. According to Lemma 4.1, there is a finite sequence $e = x_0, x_1, \ldots, x_n = f$ of elements from $E(\mathscr{S})$ such that $(x_{i-1}, x_i) \in X \cup \mathrm{Ker}\, 0$ for $i = 1, 2, \ldots, n$. If $(x_{i-1}, x_i) \in \mathrm{Ker}\, 0$, then $x_{i-1} = 0x_{i-1} = 0x_i = x_i$ and so $(x_{i-1}, x_i) \in X$. Therefore we have $(e, f) \in X$. □

Lemma 4.3. *Let $A, B, C \in \mathrm{Con}(\mathscr{S})$, where $\mathscr{S}$ is an inverse semiring in which $\mathscr{E}(\mathscr{S})$ is a tree. If for $e, f \in E(\mathscr{S})$ we have $(e, f) \in (A \vee B) \wedge C$, then $(e, f) \in (A \wedge C) \vee (B \wedge C)$.*

Proof. Suppose that $(e, f) \in (A \vee B) \wedge C$, where $e, f \in E(\mathscr{S})$. Put $A' = A \vee \mathrm{Ker}\, 0$, $B' = B \vee \mathrm{Ker}\, 0$ and $C'' = C \vee \mathrm{Ker}\, 0$. Clearly we have $(e, f) \in (A' \vee B') \wedge C''$. It follows from Lemma 2.2 that $(e, f) \in (A' \wedge C'') \vee (B' \wedge C'')$. By Lemma 4.1 there is a finite sequence $e = x_0, x_1, \ldots, x_n = f$ of elements from $E(\mathscr{S})$ such that $(x_{i-1}, x_i) \in (A' \wedge C') \cup (B' \wedge C'')$ for $i = 1, 2, \ldots, n$. According to Lemma 4.2, we have $(x_{i-1}, x_i) \in (A \wedge C) \cup (B \wedge C)$. Consequently, $(e, f) \in (A \wedge C) \vee (B \wedge C)$. □

Lemma 4.4. *Let $X, Y \in \mathrm{Con}(\mathscr{S})$, where $\mathscr{S}$ is an inverse semiring. If $(x, z) \in X$, $(z, y) \in Y$ and $0x = 0z = 0y$, then there exists w such that $(x, w) \in Y$, $(w, y) \in X$ and $0w = 0z$.*

Proof. Put $w = x - z + y$. It follows from (1.4) that $0w = 0z$ and $(x, w) = (x, x) - (z, z) + (z, y) \in Y$, $(w, z) = (x, z) - (z, z) + (y, y) \in X$. □

Lemma 4.5. *Let $X \in \mathrm{Con}(\mathscr{S})$, where $\mathscr{S}$ is an inverse semiring having property (M). Let $(x, y) \in X$. If $0x < 0y$ or $0x \parallel 0y$, then $(u, v) \in X$ for all $u, v \in S(\mathscr{S})$ with $0u = 0y = 0v$.*

P r o o f. Suppose that $0x < 0y$. Then according to Lemma 2.4, we have $(u,x) = (u-y, u-y) + (y,x) \in X$ and $(x,v) = (v-y, v-y) + (x,y) \in X$. Therefore $(u,v) \in X$.

Assume that $0x \parallel 0y$. Then we have $0(xy) < 0y$. Using (1.3), (1.4) and (1.14) we obtain $(0(xy), y) = (0y, 0y) + (x,y) \in X$. The rest of the proof follows from its first part. $\square$

Lemma 4.6. *Let $A, B, C \in \mathrm{Con}(\mathscr{S})$, where $\mathscr{S}$ is an inverse semiring having property* (M), *and $\mathscr{E}(\mathscr{S})$ is a tree.*

(i) *If $A \subseteq C$, then $(A \vee B) \wedge C \subseteq A \vee (B \wedge C)$.*

(ii) *If $\mathscr{S}$ has property (D), then $(A \vee B) \wedge C \subseteq (A \vee C) \vee (B \wedge C)$.*

P r o o f. Let $(u,v) \in (A \vee B) \wedge C$. Then there exists a finite sequence $u = x_0, x_1, \ldots, x_n = v$ of elements from $S(\mathscr{S})$ such that $(x_{i-1}, x_i) \in A \cup B$ for $i = 1, 2, \ldots, n$. Further, we have $(u,v) \in C$. Put $e = 0u$ and $f = 0v$. Clearly $(e,f) \in (A \vee B) \wedge C$. We have the following possibilities:

Case 1. $e = f$. Put $y_i = x_i + e$. It follows from (1.4) that $0y_i \leqslant e, u = y_0, v = y_n$ and $(y_{i-1}, y_i) \in A \cup B$ for $i = 1, 2, \ldots, n$.

Subcase 1a. $0y_i = e$ for all $i = 1, 2, \ldots, n$. It follows from Lemma 4.4 that there is an element w of $S(\mathscr{S})$ such that $(u,w) \in A, (w,v) \in B$ and $0w = e$.

If $A \subseteq C$ then $(w,v) \in C$ and so $(u,v) \in A \vee (B \wedge C)$.

If $\mathscr{S}$ has property (D), then according to Lemma 3.1, the lattice $(H_e]$ is distributive. Clearly $(u,w), (w,v), (u,v) \in H_e$ and so $(u,w) \in A'$, $(w,v) \in B'$, $(u,v) \in C'$, where $A' = A \wedge H_e$, $B' = B \wedge H_e$ and $C' = C \wedge H_e$. It is clear that $A', B', C' \in (H_e]$ and so we have $(u,v) \in (A' \vee B') \wedge C' = (A' \wedge C') \vee (B' \wedge C') \subseteq (A \wedge C) \vee (B \wedge C)$.

Subcase 1b. $0y_i < e$ for some i. It follows from Lemma 4.5 that $(u,v) \in A$ or $(u,v) \in B$. Thus we have $(u,v) \in (A \wedge C) \vee (B \wedge C)$.

Case 2. $f < e$. It follows from Lemma 4.3 that $(f,e) \in (A \wedge C) \vee (B \wedge C)$. According to Lemma 4.5, we have $(u,e) \in (A \wedge C) \vee (B \wedge C)$ and so $(u,f) \in (A \wedge C) \vee (B \wedge C) \subseteq (A \vee B) \wedge C$. This implies that $(f,v) \in (A \vee B) \wedge C$. Using Case 1 we can continue our proof.

If $A \subseteq C$, then $(f,v) \in A \vee (B \wedge C)$ and so $(u,v) \in A \vee (B \wedge C)$. If $\mathscr{S}$ has property (D), then $(f,v) \in (A \wedge C) \vee (B \wedge C)$ and so $(u,v) \in (A \wedge C) \vee (B \wedge C)$.

Case 3. $e < f$. This is dual to Case 2.

Case 4. $e \parallel f$. According to Lemma 4.3, we have $(e,f) \in (A \wedge C) \vee (B \wedge C)$. It follows from Lemma 4.5 that $(u,e), (f,v) \in (A \wedge C) \vee (B \wedge C)$. Therefore $(u,v) \in (A \wedge C) \vee (B \wedge C)$. $\square$

Theorem 4.1. *Let $\mathscr{S}$ be an inverse semiring. Then*
(i) $\mathrm{Con}(\mathscr{S})$ *is modular if and only if* $\mathscr{E}(\mathscr{S})$ *is a tree and* $\mathscr{S}$ *has property* (M);
(ii) $\mathrm{Con}(\mathscr{S})$ *is distributive if and only if* $\mathscr{E}(\mathscr{S})$ *is a tree and* $\mathscr{S}$ *has properties* (M) *and* (D).

The proof follows from Lemmas 2.2, 2.3, 3.2 and 4.6. □

5. Property (B)

Definition 5.1. We shall say that *an inverse semiring has property* (B) if $\operatorname{card} S(\mathscr{S}_e) > 1$ implies that e is the zero of $\mathscr{E}(\mathscr{S})$ and $\mathrm{Con}(\mathscr{S}_e)$ is boolean.

Lemma 5.1. *If the lattice* $\mathrm{Con}(\mathscr{S})$ *is boolean, the the inverse semiring* $\mathscr{S}$ *has property* (B).

Proof. Assume that $\mathrm{Con}(\mathscr{S})$ is boolean and $e \in E(\mathscr{S})$. It follows from Lemma 2.3 that $\mathscr{S}$ has property (M). According to Lemma 3.1, the lattice $\mathrm{Con}(\mathscr{S}_e)$ is isomorphic to the principal ideal $(H_e]$ of $\mathrm{Con}(\mathscr{S})$. It is well known that $(H_e]$ is boolean and so $\mathrm{Con}(\mathscr{S}_e)$ is boolean. Suppose by way of contradiction that $\operatorname{card} S(\mathscr{S}_e) > 1$ and e is no zero of $\mathscr{E}(\mathscr{S})$. Then there exists some f of $E(\mathscr{S})$ such that $f < e$. Since $\mathrm{Con}(\mathscr{S})$ is boolean, there is $Y \in \mathrm{Con}(\mathscr{S})$ such that $H_e \wedge Y = \mathrm{id}_{S(\mathscr{S})}$ and $H_e \vee Y = S(\mathscr{S}) \times S(\mathscr{S})$. We have $(e,f) \in H_e \vee Y$ and so, by Lemma 4.1, there is a finite sequence $e = x_0, x_1, \dots, x_n = f$ of elements from $E(\mathscr{S})$ such that $(x_{i-1}, x_i) \in H_e \cup Y$ for $i = 1, 2, \dots, n$. If $(x_{i-1}, x_i) \in H_e$, then $x_{i-1} = x_i$ and so $(x_{i-1}, x_i) \in Y$. Thus we have $(e,f) \in Y$. Let $a \in S(\mathscr{S}_e)$. By (M), we obtain $(a,f) = (e,f) + (a,a) \in Y$. Hence $(a,e) \in Y \cap H_e = \mathrm{id}_{S(\mathscr{S})}$. This implies that $a = e$ and so $\operatorname{card} S(\mathscr{S}_e) = 1$, a contradiction.

Consequently, $\mathscr{S}$ has property (B). □

Recall that a tree $\mathscr{E}$ is said to be *locally finite* if every interval of $\mathscr{E}$ is a finite chain.

Lemma 5.2. *Let E be a semilattice. Then* $\mathrm{Con}(\mathscr{E})$ *is boolean if and only if* $\mathscr{E}$ *is a locally finite tree.*

Proof. See Theorem 4.5 of [1]. □

Theorem 5.1. *Let $\mathscr{S}$ be an inverse semiring. Then the lattice* $\mathrm{Con}(\mathscr{S})$ *is boolean if and only if* $\mathscr{E}(\mathscr{S})$ *is a locally finite tree and* $\mathscr{S}$ *has property* (B).

Proof. 1. Suppose that $\mathrm{Con}(\mathscr{S})$ is boolean. According to Lemma 5.1, $\mathscr{S}$ has property (B). It follows from Lemma 2.1 that the lattice $\mathrm{Con}(\mathscr{E}(\mathscr{S}))$ is isomorphic to the principal filter $[\mathrm{Ker}\,0)$ of $\mathrm{Con}(\mathscr{S})$, which is boolean. Thus $\mathrm{Con}(\mathscr{E}(\mathscr{S}))$ is boolean. It follows from Lemma 5.2 that $\mathscr{E}(\mathscr{S})$ is a locally finite tree.

2. Now we assume that $\mathscr{E}(\mathscr{S})$ is a locally finite tree and $\mathscr{S}$ has property (B). Lemma 2.1 and Lemma 5.2 imply that the principal filter $[\mathrm{Ker}\,0)$ of $\mathrm{Con}(\mathscr{S})$ is boolean.

If $E(\mathscr{S}) = S(\mathscr{S})$, then $\mathrm{Ker}\,0 = \mathrm{id}_{S(\mathscr{S})}$ and so $[\mathrm{Ker}\,0) = \mathrm{Con}(\mathscr{S})$, which is boolean.

Assume that $E(\mathscr{S}) \neq S(\mathscr{S})$. Property (B) implies that $\mathscr{E}(\mathscr{S})$ has the zero $e, \mathrm{Con}(\mathscr{S}_e)$ is boolean and according to (1.10), $\mathscr{S}$ has property (M). Lemma 3.1 implies that the principal ideal $(\mathrm{Ker}\,0]$ of $\mathrm{Con}(\mathscr{S})$ is boolean, because $H_e = \mathrm{Ker}\,0$. Hence $(\mathrm{Ker}\,0] \times [\mathrm{Ker}\,0)$ is a boolean lattice.

Now, we shall prove that $\mathrm{Ker}\,0$ has a complement in the lattice $\mathrm{Con}(\mathscr{S})$. Put $P\{(x,y)\,;\, x = y \in S(\mathscr{S})$ or $x, y \in E(\mathscr{S})\}$. We shall show that $P \in \mathrm{Con}(\mathscr{S})$. Clearly P is an equivalence on $S(\mathscr{S})$. Assume that $(x,y) \in P, x \neq y$, and $z \in S(\mathscr{S})$. Then $x, y \in E(\mathscr{S})$ and so, by (1.14), we have $xz, yz, zx, zy \in E(\mathscr{S})$. This means that $(xz, yz), (zx, zy) \in P$. If $z \in E(\mathscr{S})$, then it follows from (1.15) that $x + z, y + z \in E(\mathscr{S})$. Thus we have $(x + z, y + z) \in P$. Suppose that $z \notin E(\mathscr{S})$. Then $z \in S(\mathscr{S}_e)$ and so, by (1.3), (1.4), we have $x + z = x + z + e = z = y + z + e = y + z$. This implies that $(x + z, y + z) \in P$. Consequently, $P \in \mathrm{Con}(\mathscr{S})$. It is easy to show that $P \wedge \mathrm{Ker}\,0 = \mathrm{id}_{S(\mathscr{S})}$ and $P \vee \mathrm{Ker}\,0 = S(\mathscr{S}) \times S(\mathscr{S})$.

Finally, if $\mathscr{S}$ has property (B), then it has properties (M) and (D) and so according to Theorem 4.1, $\mathrm{Con}(\mathscr{S})$ is distributive. It follows from Theorem 6 (Section 7) of [4] that $\mathrm{Con}(\mathscr{S})$ is isomorphic to the boolean lattice $(\mathrm{Ker}\,0] \times [\mathrm{Ker}\,0)$. Hence $\mathrm{Con}(\mathscr{S})$ is boolean. □

6. Inverse Δ-semirings

Following the semigroup theory, an inverse semiring $\mathscr{S}$ is called an inverse Δ-semiring if the lattice $\mathrm{Con}(\mathscr{S})$ is a chain.

Lemma 6.1. *If $\mathscr{S}$ is an inverse Δ-semiring, then* $\mathrm{card}\, E(\mathscr{S}) \leqslant 2$.

Proof. It follows from Lemma 2.1 that $\mathrm{Con}(E(\mathscr{S}))$ is a chain and so, by Lemma 3 of [5], we obtain $\mathrm{card}\, E(\mathscr{S}) \leqslant 2$. □

Lemma 6.2. *Let $\mathscr{S}$ be an inverse Δ-semiring. If $E(\mathscr{S}) = \{e, f\}, e < f$, then* $\mathrm{card}\, S(\mathscr{S}_e) = 1$.

Proof. Put $Q = \{(x,y)\,;\, x, y \in S(\mathscr{S})$ and $x + e = y + e\}$. It is easy to show that $Q \in \mathrm{Con}(\mathscr{S})$. Since $\mathrm{Con}(\mathscr{S})$ is a chain we have the following two possibilities:

Case 1. $Q \subseteq \operatorname{Ker} 0$. We have $(e,f) \in Q$ and so $(e,f) \in \operatorname{Ker} 0$. This means that $e = f$, a contradiction.

Case 2. $\operatorname{Ker} 0 \subseteq Q$. For $x, y \in S(\mathscr{S}_e)$ we have $(x,y) \in \operatorname{Ker} 0$ and so $(x,y) \in Q$. Thus $x = x + e = y + e = y$. Consequently, $\operatorname{card} S(\mathscr{S}_e) = 1$. □

Let $\mathscr{R}$ be a ring. By $\mathscr{R}^0$ we denote the τ-algebra, where $S(\mathscr{R}^0) = S(\mathscr{R}) \cup \{h\}, h \notin S(\mathscr{R})$, and

$$-x = \begin{cases} x & \text{for} \quad x = h, \\ -x & \text{for} \quad x \in S(\mathscr{R}). \end{cases}$$

The addition and the multiplication on $S(\mathscr{R}^0)$ are defined as follows:

If $x, y \in S(\mathscr{R})$, then $x + y$ (xy, respectively) is the same as in $\mathscr{R}$.

If $x \in S(\mathscr{R}^0)$, then $x + h = h + x = xh = hx = h$.

It is easy to show that $\mathscr{R}^0$ is an inverse semiring with $\operatorname{card} E(\mathscr{R}^0) = 2$.

For every $X \in \operatorname{Con}(\mathscr{R})$ we put $X^0 = X \cup \{(h,h)\}$. Clearly we have $X^0 \in \operatorname{Con}(\mathscr{R}^0)$.

Lemma 6.3. *Let $\mathscr{R}$ be a ring. Then* $\operatorname{Con}(\mathscr{R}^0) = \{X^0;\ X \in \operatorname{Con}(\mathscr{R})\} \cup \{S(\mathscr{R}^0) \times S(\mathscr{R}^0)\}$.

Proof. The former statement "$\supseteq$" is obvious. To prove the latter statement "$\subseteq$", let $Y \in \operatorname{Con}(\mathscr{R}^0)$ and $(h,z) \in Y$ for some $z \in S(\mathscr{R})$. Then for arbitrary $x, y \in S(\mathscr{R}^0)$ we have $(h,x) = (h,z) - (h,z) + (x,x) \in Y$ and similarly $(h,y) \in Y$. Therefore $(x,y) \in Y$, which means that $Y = S(\mathscr{R}^0) \times S(\mathscr{R}^0)$. □

Theorem 6.1. *Let $\mathscr{S}$ be an inverse semiring. Then $\operatorname{Con}(\mathscr{S})$ is a chain if and only if $\mathscr{S}$ is isomorphic to either $\mathscr{R}$ or $\mathscr{R}^0$, where $\mathscr{R}$ is a ring whose $\operatorname{Con}(\mathscr{R})$ is a chain.*

The proof follows from Lemmas 6.1, 6.2 and 6.3. □

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