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SOME PROPERTIES OF THIRD ORDER DIFFERENTIAL OPERATORS

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Abstract. Consider the third order differential operator L given by

$$L(\cdot) \equiv \frac{1}{a_3(t)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{a_2(t)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{a_1(t)}\frac{\mathrm{d}}{\mathrm{d}t}(\cdot)$$

and the related linear differential equation $L(x)(t) + x(t) = 0$. We study the relations between L, its adjoint operator, the canonical representation of L, the operator obtained by a cyclic permutation of coefficients a_i, $i = 1,2,3$, in L and the relations between the corresponding equations.

We give the commutative diagrams for such equations and show some applications (oscillation, property A).

Keywords: Differential operators, linear differential equation of third order, canonical forms, adjoint equation, cyclic permutation, oscillatory solution, Kneser solution, property A

MSC 1991: 34C10, 34C20

Introduction

Many authors have studied the oscillatory, nonoscillatory and asymptotic behaviour for the linear third order differential equation in the normal form, that is

$$x'''(t) + p(t)x'(t) + q(t)x(t) = 0, \tag{1}$$

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where p, q are continuous functions for $t \geqslant 0$. Among the numerous results dealing with this subject, we refer to the books [13, 16, 20] and the references contained therein. Other contributions may be obtained, as a particular case, from many papers concerning equations of higher order, see, e.g., [1, 17, 20] and/or nonlinear equations, e.g. [2, 5]. Some authors (see, e.g., [12, 14]) consider the special forms

$$(r(t)x'(t))'' + q(t)x(t) = 0, \tag{2}$$

$$(r(t)x''(t))' + q(t)x(t) = 0. \tag{3}$$

Another often considered form of equation (1) arises when the second order differential equation

$$w''(t) + p(t)w(t) = 0 \tag{4}$$

is nonoscillatory. Let h be a solution of (4), $h(t) > 0$ for $t > T$; then, by standard computation, (1) may be written, for $t > T$, as

$$\left(h^2(t)\left(\frac{1}{h(t)}x'(t)\right)'\right)' + q(t)h(t)x(t) = 0. \tag{5}$$

The natural question which arises is the relation among (2), (3), (5). To understand the various forms of the third order linear differential equation, we consider the equation

$$\left(\frac{1}{a_2(t)}\left(\frac{1}{a_1(t)}x'(t)\right)'\right)' + a_3(t)x(t) = 0, \tag{E}$$

where

(Hp) $\quad a_i \in C^2([0,\infty), \mathbb{R}),\ i = 1,2,3,\ a_1(t) > 0,\ a_2(t) > 0,\ a_3(t) \neq 0$ on $[0,\infty)$.

We just remark that in the sequel the assumption "$a_i \in C^2([0,\infty), \mathbb{R})$" is not always necessary, but it is considered for the sake of simplicity. For instance, if $a_i \in C^{3-i}([0,\infty), \mathbb{R})$, $i = 1, 2$, then (E) may be written in the normal form

$$x'''(t) + r(t)x''(t) + p(t)x'(t) + q(t)x(t) = 0.$$

If $a_i \in C([0,\infty), \mathbb{R})$, $i = 1, 2, 3$, then (E) may be interpreted as a first order linear differential system.

The assumption (Hp), as regards the sign of functions a_i, $i = 1, 2, 3$, is motivated by the following fact: concerning the possible sign of coefficients of an equation of type (E), there are eight possible cases, but the significative ones with respect to the equation considered are only two, namely, for instance, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ or $a_1 > 0$, $a_2 > 0$, $a_3 < 0$.

Denote by (E_+) [(E_-)] the equation (E) when $a_3(t) > 0$ [$a_3(t) < 0$] on $[0, \infty)$.

Equation (E) is related to the third order differential operator L given by

$$L(x)(t) = \frac{1}{a_3(t)} \left(\frac{1}{a_2(t)} \left(\frac{1}{a_1(t)} x'(t) \right)' \right)' . \tag{6}$$

Then (E) can be written as

$$L(x)(t) + x(t) = 0.$$

In a recent paper [7], other operators, which are obtained from L by an ordered cyclic permutation of the functions a_i, have been considered. Further, for equations of higher order, in [7] some properties of such operators have been investigated and applied to the study of the oscillation as well as to the classification of solutions.

Together with L, other operators have often been considered in literature: in particular, the adjoint operator (see, e.g., [8]) and the operator in the canonical form ([22]). They are of particular interest in a large number of problems arising in the study of the qualitative behavior of solutions of (E), especially with regard to disconjugacy. Among numerous monographs dealing with this topic, we choose to refer to [13, 19].

The aim of this paper is to examine the relationships between the above quoted operators. Further, coming back to the related differential equations, we prove the commutation among them (Part I). In this way, we get some new criteria for (E) to have property A as well as the connection between (2) and (3).

Some definitions and notation are as follows:

A nontrivial solution of (E) is said to be *oscillatory* if it has infinitely large zeros and *nonoscillatory* otherwise. Equation (E) is said to be *oscillatory* if there exists at least one oscillatory solution, and *nonoscillatory* if all its solutions are nonoscillatory.

Denote by $\mathcal{S}$(E) the space of all solutions of (E), by $\mathcal{N}$(E) the set of all nonoscillatory solutions and by $\mathcal{O}$(E) the set of all oscillatory solutions. By $x^{[i]}$, $i = 1, 2$, we denote the quasiderivatives of a solution x of (E). That is, if $x \in \mathcal{S}$(E), then $x^{[1]} = \frac{1}{a_1} x'$, $x^{[2]} = \frac{1}{a_2} (x^{[1]})'$.

Finally, let u_1, u_2 be two continuous functions different from zero on $[0,\infty)$ and define

$$I(u_i) = \int_0^\infty |u_i(t)|\,\mathrm{d}t, \quad I(u_i,u_j) = \int_0^\infty |u_i(t)| \int_0^t |u_j(s)|\,\mathrm{d}s\,\mathrm{d}t, \quad i,j = 1,2.$$

The functions u_1, u_2 can be classified according to the convergence or divergence of the four integrals $I(u_1)$, $I(u_2)$, $I(u_1,u_2)$, $I(u_2,u_1)$ into six possible cases:

$$(7)\qquad \begin{array}{ll} (\mathrm{K}_1) & I(u_1) = I(u_2) = \infty, \\ (\mathrm{K}_2) & I(u_1) < \infty,\ I(u_2) = I(u_1,u_2) = \infty, \\ (\mathrm{K}_3) & I(u_2) < \infty,\ I(u_1) = I(u_2,u_1) = \infty, \\ (\mathrm{K}_4) & I(u_2) < \infty,\ I(u_2,u_1) < \infty,\ I(u_1) = \infty, \\ (\mathrm{K}_5) & I(u_1) < \infty,\ I(u_1,u_2) < \infty,\ I(u_2) = \infty, \\ (\mathrm{K}_6) & I(u_1) < \infty,\ I(u_2) < \infty. \end{array}$$

PART I. RELATIONS BETWEEN DIFFERENTIAL OPERATORS

Equation (E) is determined by the vector of functions

$$(8)\qquad a = (a_1, a_2, a_3).$$

Moreover, (E) is related to the third order differential operator L given in (6). We will say that L is *the operator associated with vector* a and (E) is *the equation associated with vector* a. From now on we introduce the operator "*cyclic permutation*" of L, the "*semi-adjoint operator*" to L and the "*canonical form*" of L.

I-1 Cyclic permutation.

Let us consider the map $\mathcal{C}\colon C^2([0,\infty),\mathbb{R}^3) \to C^2([0,\infty),\mathbb{R}^3)$ given by

$$\mathcal{C}\colon a = (a_1,a_2,a_3) \mapsto \mathcal{C}(a) = (a_2,a_3,a_1)$$

and let $L^{\mathcal{C}}$, $L^{\mathcal{CC}}$ be the operators associated with the vectors $\mathcal{C}(a)$, $\mathcal{C}(\mathcal{C}(a))$ respectively, that is the operators given by

$$L^{\mathcal{C}}(y)(t) = \frac{1}{a_1(t)}\left(\frac{1}{a_3(t)}\left(\frac{1}{a_2(t)}\,y'(t)\right)'\right)',$$

$$L^{\mathcal{CC}}(z)(t) = \frac{1}{a_2(t)}\left(\frac{1}{a_1(t)}\left(\frac{1}{a_3(t)}\,z'(t)\right)'\right)'.$$

We will call these operators *cyclic permutation operators of L*. Let ($\mathrm{E}^{\mathcal{C}}$) be the equation associated with the vector $\mathcal{C}(a)$, that is the equation associated with the first cyclic permutation of the vector a given in (8):

$$(\mathrm{E}^{\mathcal{C}}) \qquad L^{\mathcal{C}}(y)(t) + y(t) = \frac{1}{a_1(t)}\left(\frac{1}{a_3(t)}\left(\frac{1}{a_2(t)}y'(t)\right)'\right)' + y(t) = 0,$$

where $y = x^{[1]} = \frac{1}{a_1}x'$, with x a solution of (E).

The equation associated with the vector $\mathcal{C}(\mathcal{C}(a))$, i.e. the equation associated with the second cyclic permutation of (8), is

$$(\mathrm{E}^{\mathcal{CC}}) \qquad L^{\mathcal{CC}}(z)(t) + z(t) = \frac{1}{a_2(t)}\left(\frac{1}{a_1(t)}\left(\frac{1}{a_3(t)}z'(t)\right)'\right)' + z(t) = 0,$$

where $z = y^{[1]} = \frac{1}{a_2}y'$, with y a solution of ($\mathrm{E}^{\mathcal{C}}$). This phenomenon is studied in detail in [7] for the general n.

I-2 Semi-adjoint operator.

Consider now the map $\mathcal{A}\colon C^2([0,\infty),\mathbb{R}^3) \to C^2([0,\infty),\mathbb{R}^3)$ given by

$$\mathcal{A}\colon a = (a_1, a_2, a_3) \mapsto \mathcal{A}(a) = (a_2, a_1, -a_3)$$

and let $L^{\mathcal{A}}$ be the operator associated with the vector $\mathcal{A}(a)$, that is the operator given by

$$L^{\mathcal{A}}(u)(t) = -\frac{1}{a_3(t)}\left(\frac{1}{a_1(t)}\left(\frac{1}{a_2(t)}u'(t)\right)'\right)'.$$

The equation associated with vector $\mathcal{A}(a)$ is $L^{\mathcal{A}}(u)(t) + u(t) = 0$, i.e.,

$$(\mathrm{E}^{\mathcal{A}}) \qquad \frac{1}{a_3(t)}\left(\frac{1}{a_1(t)}\left(\frac{1}{a_2(t)}u'(t)\right)'\right)' - u(t) = 0,$$

where $u = x^{[1]}y - xy^{[1]}$, x, y being two independent solutions of (E). A similar definition holds for the operator $L^{\mathcal{AA}}$, that is the operator associated with $\mathcal{A}(\mathcal{A}(a))$, and for the equation ($\mathrm{E}^{\mathcal{AA}}$).

We will call the operator $L^{\mathcal{A}}$ *the semi-adjoint operator to L* since it is related to the operator L^* adjoint to the operator L defined as (see, e.g., [8])

$$L^* \equiv -\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{a_1}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{a_2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\cdot}{a_3}.$$

Indeed, the adjoint equation of (E) is the equation $L^*(v)(t) + v(t) = 0$ which can be written as ($\mathrm{E}^{\mathcal{A}}$) via the map $u = \frac{1}{a_3} v$.

I-3 Canonical form.

Let $u_i \in C^{3-i}([0,\infty), \mathbb{R})$, $i = 1,2,3$, with a fixed sign on $[0,\infty)$ and consider the third order differential operator D given by

$$D(x)(t) = \frac{1}{u_3(t)} \left(\frac{1}{u_2(t)} \left(\frac{1}{u_1(t)} x'(t) \right)' \right)' .$$

The operator D is disconjugate on $[0,\infty)$, that is every solution of the equation $D(x)(t) = 0$ has at most two zeros on $[0,\infty)$.

Hence by Theorem 1 in [22], D can be written in a certain *canonical form*, that is

$$D \equiv \frac{1}{u_3 b_3} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{b_2} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{b_1} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\cdot}{b_0}$$

where the functions b_i, $i = 0,1,2,3$, have a fixed sign, they are continuous, such that

$$\int^{\infty} |b_1| = \infty, \quad \int^{\infty} |b_2| = \infty,$$

and determined up to positive multiplicative constants with product 1.

The explicit forms of functions b_i depend on the convergence or divergence of the four integrals $I(u_1)$, $I(u_2)$, $I(u_1,u_2)$, $I(u_2,u_1)$, that is on the six cases (K_i), $i = 1,\dots,6$, given in (7). They may be calculated using the proof of Lemmas 1 and 2 in [22] and, for reader's convenience, they are given in Table 1.

Since the operator L given in (6) is disconjugate, it can be written in the canonical form, that is

$$L(\cdot) \equiv \frac{1}{a_3 b_3} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{b_2} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{b_1} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\cdot}{b_0} .$$

Moreover, (E) may be written as

$$(\mathrm{E}_1) \qquad \left(\frac{1}{b_2(t)} \left(\frac{1}{b_1(t)} \left(\frac{x(t)}{b_0(t)} \right)' \right)' \right)' + a_3(t) b_3(t) x(t) = 0.$$

Let us consider the map $\mathcal{K}$: $C^2([0,\infty), \mathbb{R}^3) \to C^2([0,\infty), \mathbb{R}^3)$ given by

$$\mathcal{K}_i \colon a = (a_1, a_2, a_3) \mapsto \mathcal{K}_i(a) = (b_1, b_2, a_3 b_0 b_3)$$

where b_j are the coefficients of the canonical form of L in case (K_i), $i = 2,3,4,5,6$ (see Table 1 with $u_1 = a_1$, $u_2 = a_2$).

Now let us consider the differential operator $L^{\mathcal{K}_i}$ associated with the vector $\mathcal{K}_i(a)$ given by

$$L^{\mathcal{K}_i}(v)(t) = \left(\frac{1}{a_3(t)b_0(t)b_3(t)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{b_2(t)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{b_1(t)}\frac{\mathrm{d}}{\mathrm{d}t}v(t)\right).$$

The equation associated with the vector $\mathcal{K}_i(a)$ is $L^{\mathcal{K}_i}(v)(t) + v(t) = 0$, i.e.,

$$(\mathrm{E}^{\mathcal{K}_i}) \qquad \left(\frac{1}{b_2(t)}\left(\frac{1}{b_1(t)}v'(t)\right)'\right)' + a_3(t)b_0(t)b_3(t)v(t) = 0,$$

which gives equation (E_1) via the map $v = \frac{1}{b_0}x$. If we do not need to specify the case (K_i), then we will use the notation ($\mathrm{E}^{\mathcal{K}}$) instead of ($\mathrm{E}^{\mathcal{K}_i}$).

In I-4 we focus our attention on cases (K_1)–(K_5); the case (K_6) is discussed in Part II, Remark 7.

In accordance with the above notation, the identity "$\equiv$" between two equations means that the associated vectors are the same. Obviously, if (K_1) holds, then ($\mathrm{E}^{\mathcal{K}_1}$) $\equiv$ (E). If (K_2) holds, then the canonical form of L yields the equation

$$(\mathrm{E}^{\mathcal{K}_2}) \qquad \left(\frac{1}{a_2(t)\int_t^\infty a_1}\left(\frac{(\int_t^\infty a_1)^2}{a_1(t)}v'\right)'\right)' + a_3(t)\left(\int_t^\infty a_1\right)v = 0,$$

where $v = (\int_t^\infty a_1)^{-1}x$ with x a solution of (E). If (K_3) holds, then ($\mathrm{E}^{\mathcal{K}}$) has the form

$$(\mathrm{E}^{\mathcal{K}_3}) \qquad \left(\frac{(\int_t^\infty a_2)^2}{a_2(t)}\left(\frac{x'}{a_1(t)\int_t^\infty a_2}\right)'\right)' + a_3(t)\left(\int_t^\infty a_2\right)x = 0.$$

Similarly we obtain ($\mathrm{E}^{\mathcal{K}_4}$) and ($\mathrm{E}^{\mathcal{K}_5}$).

Remark 1. Coefficient b_0 in the canonical forms is either the constant function $b_0(t) \equiv 1$ (in cases (K_1), (K_3)) or a positive decreasing function in all other cases. Indeed, in cases (K_i), $i = 2,4,5$ we have $b_0(t) = \int_t^\infty f(s)\,\mathrm{d}s$, where $f > 0$, $\int^\infty f < \infty$ and thus $b_0'(t) = -f(t) < 0$. Similarly, by routine computation we get the conclusion in case (K_6).

I-4 Relations between the corresponding equations.

Let us examine all equations under consideration:

$$\text{(E)} \qquad \frac{1}{a_3(t)}\left(\frac{1}{a_2(t)}\left(\frac{1}{a_1(t)}x'(t)\right)'\right)' + x(t) = 0,$$

$$(\mathrm{E}^{\mathcal{C}}) \qquad \frac{1}{a_1(t)}\left(\frac{1}{a_3(t)}\left(\frac{1}{a_2(t)}y'(t)\right)'\right)' + y(t) = 0,$$

$$(\mathrm{E}^{\mathcal{A}}) \qquad \frac{1}{a_3(t)}\left(\frac{1}{a_1(t)}\left(\frac{1}{a_2(t)}u'(t)\right)'\right)' - u(t) = 0,$$

$$(\mathrm{E}^{\mathcal{K}_i}) \qquad \left(\frac{1}{b_2(t)}\left(\frac{1}{b_1(t)}v'(t)\right)'\right)' + b_0(t)b_3(t)a_3(t)v(t) = 0$$

where b_j are the coefficients of the canonical form in case (K_i) (see Table 1 with $u_1 = a_1$, $u_2 = a_2$). Our aim here is to describe some relations between these differential equations. In the sequel we will say that (E) *yields, via the map* $\mathcal{C}$, ($\mathrm{E}^{\mathcal{C}}$) (*in the sense of associated vectors*) and make use of the notation

$$\text{(E)} \xrightarrow{\mathcal{C}} (\mathrm{E}^{\mathcal{C}}).$$

Similarly for the maps $\mathcal{A}$, $\mathcal{K}_i$ and their composition. Hence

$$\text{(E)} \xrightarrow{\mathcal{A}} (\mathrm{E}^{\mathcal{A}}), \quad \text{(E)} \xrightarrow{\mathcal{K}} (\mathrm{E}^{\mathcal{K}}).$$

In what follows we denote by $L^{\mathcal{AC}}$ [($\mathrm{E}^{\mathcal{AC}}$)] the operator [equation] associated with the vector $\mathcal{C}(\mathcal{A}(a))$ and similarly for the other composition of the maps $\mathcal{C}$, $\mathcal{A}$, $\mathcal{K}_i$.

Concerning the explicit forms of the coefficients b_i, defining the canonical forms in cases under consideration, they may be obtained, by standard calculation, from Table 1. For instance, the explicit form of coefficients b_i which define the operator $L^{\mathcal{CK}_i}$ associated with the vector $\mathcal{K}_i(\mathcal{C}(a))$ are given in Table 2.

Theorems 1–2 below present other results related to these equations and Theorem 3 below gives some relations between the corresponding spaces of solutions.

Theorem 1. *The maps $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{K}_i$, $i = 2, 3, 4, 5$ make the following diagrams commutative*

$$\begin{tikzcd}
(\mathrm{E}) \arrow[r, leftrightarrow, "\mathcal{A}"] \arrow[d, "\mathcal{C}"'] & (\mathrm{E}^{\mathcal{A}}) \\
(\mathrm{E}^{\mathcal{C}}) \arrow[r, leftrightarrow, "\mathcal{A}"] & (\mathrm{E}^{\mathcal{CA}}) \arrow[u, "\mathcal{C}"']
\end{tikzcd}$$

Diagram 1

$$\begin{tikzcd}
(\mathrm{E}^{\mathcal{K}_3}) \arrow[r, leftrightarrow, "\mathcal{A}"] & (\mathrm{E}^{\mathcal{AK}_2}) \\
(\mathrm{E}) \arrow[u, "\mathcal{K}_3"] \arrow[r, leftrightarrow, "\mathcal{A}"] \arrow[d, "\mathcal{K}_2"'] & (\mathrm{E}^{\mathcal{A}}) \arrow[u, "\mathcal{K}_2"'] \arrow[d, "\mathcal{K}_3"] \\
(\mathrm{E}^{\mathcal{K}_2}) \arrow[r, leftrightarrow, "\mathcal{A}"] & (\mathrm{E}^{\mathcal{AK}_3})
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathrm{E}^{\mathcal{K}_4}) \arrow[r, leftrightarrow, "\mathcal{A}"] & (\mathrm{E}^{\mathcal{AK}_5}) \\
(\mathrm{E}) \arrow[u, "\mathcal{K}_4"] \arrow[r, leftrightarrow, "\mathcal{A}"] \arrow[d, "\mathcal{K}_5"'] & (\mathrm{E}^{\mathcal{A}}) \arrow[u, "\mathcal{K}_5"'] \arrow[d, "\mathcal{K}_4"] \\
(\mathrm{E}^{\mathcal{K}_5}) \arrow[r, leftrightarrow, "\mathcal{A}"] & (\mathrm{E}^{\mathcal{AK}_4})
\end{tikzcd}$$

Diagram 2 Diagram 3

Proof. Equation (E) yields, via the map $\mathcal{K}_i \circ \mathcal{A}$, the equation

$$(\mathrm{E}^{\mathcal{KA}}) \qquad \left(\frac{1}{b_1(t)}\left(\frac{1}{b_2(t)}u'(t)\right)'\right)' - a_3(t)b_0(t)b_3(t)u(t) = 0.$$

Similarly, the equation (E) yields, via the map $\mathcal{C} \circ \mathcal{A}$, the equation

$$(\mathrm{E}^{\mathcal{CA}}) \qquad \left(\frac{1}{a_2(t)}\left(\frac{1}{a_3(t)}y'(t)\right)'\right)' - a_1(t)y(t) = 0.$$

Diagram 1: We have $(\mathrm{E}) \equiv (\mathrm{E}^{\mathcal{CACA}})$, $(\mathrm{E}^{\mathcal{A}}) \equiv (\mathrm{E}^{\mathcal{AACAC}})$, $(\mathrm{E}^{\mathcal{A}}) \equiv (\mathrm{E}^{\mathcal{CAC}})$.

Diagram 2: Using Table 1 we get $(\mathrm{E}^{\mathcal{K}_3\mathcal{A}}) \equiv (\mathrm{E}^{\mathcal{AK}_2})$ and $(\mathrm{E}^{\mathcal{K}_2\mathcal{A}}) \equiv (\mathrm{E}^{\mathcal{AK}_3})$.

By the similar symmetry between the cases (K_4) and (K_5) (see Table 1) we get Diagram 3. □

Theorem 2. *1. Let $I(a_2) < \infty$, $I(a_2, a_1) = I(a_2, a_3) = \infty$. Then the maps $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{K}_i$, $i = 2, 3$ make the following diagram commutative*

$$\begin{tikzcd}
(\mathrm{E}) \arrow[r, "\mathcal{K}_3"] \arrow[d, "\mathcal{C}"'] & (\mathrm{E}^{\mathcal{K}_3}) \arrow[r, leftrightarrow, "\mathcal{A}"] \arrow[d, "\mathcal{C}"'] & (\mathrm{E}^{\mathcal{K}_3\mathcal{A}}) \\
(\mathrm{E}^{\mathcal{C}}) \arrow[r, "\mathcal{K}_2"] & (\mathrm{E}^{\mathcal{CK}_2}) \arrow[r, leftrightarrow, "\mathcal{A}"] & (\mathrm{E}^{\mathcal{CK}_2\mathcal{A}}) \arrow[u, "\mathcal{C}"']
\end{tikzcd}$$

2. Let $I(a_3) < \infty$, $I(a_3, a_1) = I(a_3, a_2) = \infty$. Then the maps $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{K}_i$, $i = 2, 3$ make the following diagram commutative

$$\begin{tikzcd}
(\mathrm{E}^{\mathcal{C}}) \arrow[r, "\mathcal{K}_3"] \arrow[d, "\mathcal{C}"'] & (\mathrm{E}^{\mathcal{C}\mathcal{K}_3}) \arrow[r, leftrightarrow, "\mathcal{A}"] \arrow[d, "\mathcal{C}"'] & (\mathrm{E}^{\mathcal{C}\mathcal{K}_3\mathcal{A}}) \\
(\mathrm{E}^{\mathcal{C}\mathcal{C}}) \arrow[r, "\mathcal{K}_2"'] & (\mathrm{E}^{\mathcal{C}\mathcal{C}\mathcal{K}_2}) \arrow[r, leftrightarrow, "\mathcal{A}"] & (\mathrm{E}^{\mathcal{C}\mathcal{C}\mathcal{K}_2\mathcal{A}}) \arrow[u, "\mathcal{C}"]
\end{tikzcd}$$

3. Let $I(a_1) < \infty$, $I(a_1, a_3) = I(a_1, a_2) = \infty$. Then the maps $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{K}_i$, $i = 2, 3$ make the following diagram commutative

$$\begin{tikzcd}
(\mathrm{E}^{\mathcal{C}\mathcal{C}}) \arrow[r, "\mathcal{K}_3"] \arrow[d, "\mathcal{C}"'] & (\mathrm{E}^{\mathcal{C}\mathcal{C}\mathcal{K}_3}) \arrow[r, leftrightarrow, "\mathcal{A}"] \arrow[d, "\mathcal{C}"'] & (\mathrm{E}^{\mathcal{C}\mathcal{C}\mathcal{K}_3\mathcal{A}}) \\
(\mathrm{E}) \arrow[r, "\mathcal{K}_2"'] & (\mathrm{E}^{\mathcal{K}_2}) \arrow[r, leftrightarrow, "\mathcal{A}"] & (\mathrm{E}^{\mathcal{K}_2\mathcal{A}}) \arrow[u, "\mathcal{C}"]
\end{tikzcd}$$

Proof. To prove the whole diagram is commutative, it is sufficient to prove that two square diagrams commute. The first concerns the composed transformations $\mathcal{K}_2 \circ \mathcal{C}$ and $\mathcal{C} \circ \mathcal{K}_3$, the second $\mathcal{C} \circ \mathcal{A}$ and $\mathcal{A} \circ \mathcal{C}$.

Let $I(a_2) < \infty$, $I(a_2, a_1) = I(a_2, a_3) = \infty$. We calculate the canonical form of (E) [of $(\mathrm{E}^{\mathcal{C}})$] according to Table 1, where the case (K_3) [(K_2), respectively] is considered. By simple computation we get $(\mathrm{E}^{\mathcal{C}\mathcal{K}_2}) \equiv (\mathrm{E}^{\mathcal{K}_3\mathcal{C}})$. The second part of the diagram follows from Diagram 1 of Theorem 1, where (E) is replaced by $(\mathrm{E}^{\mathcal{K}_3})$.

The other two cases follow from the cyclic permutation of (E) and $(\mathrm{E}^{\mathcal{C}})$, respectively. □

Transformations $\mathcal{C}$ and $\mathcal{K}_i$ induce isomorphisms $T_{\mathcal{C}} \colon \mathcal{S}(\mathrm{E}) \to \mathcal{S}(\mathrm{E}^{\mathcal{C}})$, defined by

$$T_{\mathcal{C}} \colon x \in \mathcal{S}(\mathrm{E}) \mapsto x^{[1]} \in \mathcal{S}(\mathrm{E}^{\mathcal{C}}),$$

and $T_{\mathcal{K}} \colon \mathcal{S}(\mathrm{E}) \to \mathcal{S}(\mathrm{E}^{\mathcal{K}})$, defined by

$$T_{\mathcal{K}} \colon x \in \mathcal{S}(\mathrm{E}) \mapsto \frac{x}{b_0} \in \mathcal{S}(\mathrm{E}^{\mathcal{K}}). \tag{8}$$

These isomorphisms preserve oscillation, as the following theorem shows.

Theorem 3. *The maps $T_{\mathcal{C}}$ and $T_{\mathcal{K}}$ are isomorphisms which maintain the oscillatory properties of the solutions. That is,*

$$x \in \mathcal{O}(\mathrm{E}) \iff T_{\mathcal{C}}(x) \in \mathcal{O}(\mathrm{E}^{\mathcal{C}}), \quad x \in \mathcal{O}(\mathrm{E}) \iff T_{\mathcal{K}}(x) \in \mathcal{O}(\mathrm{E}^{\mathcal{K}}) .$$

Equation $(\mathrm{E}^{\mathcal{A}})$ *satisfies*

$$\mathcal{O}(\mathrm{E}) \neq \emptyset \Longleftrightarrow \mathcal{O}(\mathrm{E}^{\mathcal{A}}) \neq \emptyset .$$

Proof. The statement on $T_{\mathcal{C}}$ is proved in [7], Theorem 1 (Correspondence Principle). We will prove the statement for $T_{\mathcal{K}}$. According to (8), if $x \in \mathcal{S}(\mathrm{E})$, then $y = x/b_0 \in \mathcal{S}(\mathrm{E}^{\mathcal{K}})$. It is obvious that if $x_1 \neq x_2$ then $y_1 \neq y_2$ and if $y \in \mathcal{S}(\mathrm{E}^{\mathcal{K}})$, then the function $x = b_0 y$ is from $\mathcal{S}(\mathrm{E})$, i.e. $T_{\mathcal{K}}$ is isomophism. Further, x is oscillatory iff y is oscillatory. The statement for the adjoint equation is proved in [10]. □

Remark 2. (i) The map $\mathcal{A}$ does not induce an isomorphism between spaces of solutions because these spaces have different structure with respect to the oscillation (see, e.g., [6, 10]).

(ii) The existence of an oscillatory solution of any equation in the diagrams derived in Theorems 1 and 2, ensures by Theorem 3 that all other equations in these diagrams are oscillatory, too. Similarly this holds for nonoscillatory (disconjugacy) criteria for (E). ((E) is nonoscillatory if and only if it is eventually disconjugate, i.e., there exists $T \in \mathbb{R}$ such that every nontrivial solution has at most two zeros on $[T, \infty)$, see [10].)

(iii) From the fact that $b_0(t) \to 0$ as $t \to \infty$ in cases (K_2), (K_{4-6}) of canonical forms and from (8) it follows that in all cases (K_i) $(i = 2, \dots, 6)$, $T_{\mathcal{K}}^{-1}$ preserves solutions tending to zero and $T_{\mathcal{K}}$ preserves unbounded solutions.

Part II. Applications: Property A

In [14, Th. 5.10; 18, Th. 1.2] it has been observed that if $q > 0$, $p \leqslant 0$ and equation (1) has an oscillatory solution then every nonoscillatory solution x satisfies

$$\operatorname{sgn} x \operatorname{sgn} x' < 0, \quad \operatorname{sgn} x \operatorname{sgn} x'' > 0 \quad \text{on } [0, \infty).$$

Such a nonoscillatory solution is called a *Kneser solution* and the property that every solution of equation (1) is either an oscillatory solution or a Kneser solution tending to zero for $t \to \infty$ is called *property A*, see [16].

For this reason we consider equation (E_+) and study the problem how the transformations $T_{\mathcal{K}}^{-1}$, $T_{\mathcal{C}}$ preserve property A of (E_+).

The set $\mathcal{N}(\mathrm{E}_+)$ of all nontrivial nonoscillatory solutions of (E_+) can be divided into the following four classes:

$$N_0 = \{x \in \mathcal{N}(\mathrm{E}_+),\ \exists\, t_x : x(t)x^{[1]}(t) < 0,\ x(t)x^{[2]}(t) > 0 \quad \text{for} \quad t \geqslant t_x\}$$
$$N_1 = \{x \in \mathcal{N}(\mathrm{E}_+),\ \exists\, t_x : x(t)x^{[1]}(t) > 0,\ x(t)x^{[2]}(t) < 0 \quad \text{for} \quad t \geqslant t_x\}$$
$$N_2 = \{x \in \mathcal{N}(\mathrm{E}_+),\ \exists\, t_x : x(t)x^{[1]}(t) > 0,\ x(t)x^{[2]}(t) > 0 \quad \text{for} \quad t \geqslant t_x\}$$
$$N_3 = \{x \in \mathcal{N}(\mathrm{E}_+),\ \exists\, t_x : x(t)x^{[1]}(t) < 0,\ x(t)x^{[2]}(t) < 0 \quad \text{for} \quad t \geqslant t_x\}$$

Analogously to equation (1), solutions in the class N_0 are called *Kneser solutions* (see, e.g., [16]).

Definition. Equation (E_+) is said to have *property A* if each of its solutions either is oscillatory or is a Kneser solution tending to zero for $t \to \infty$.

Remark 3. When equations (E_+), $(\mathrm{E}_+^{\mathcal{C}})$ and $(\mathrm{E}_+^{\mathcal{CC}})$ have property A, then any Kneser solution x of any of these equations satisfies $\lim\limits_{t\to\infty} x^{[i]} = 0$, $i = 0, 1, 2$, where $x^{[0]} = x$. This property coincides with property A defined for the system of differential equations, see [15].

Lemma 1. *If* $(\mathrm{E}_+^{\mathcal{K}})$ *has property A then* $(\mathrm{E}_+^{\mathcal{K}})$ *and* (E_+) *are oscillatory.*

Proof. We have $\mathcal{N}(\mathrm{E}_+^{\mathcal{K}}) = N_0 \cup N_2$; $N_2 \neq \emptyset$ if and only if $(\mathrm{E}_+^{\mathcal{K}})$ is nonoscillatory, see Lemmas 2.1 and 2.2 in [16]. Hence if $(\mathrm{E}_+^{\mathcal{K}})$ has property A, then $(\mathrm{E}_+^{\mathcal{K}})$ is oscillatory and so, by Theorem 3, (E_+) is oscillatory, too. □

Lemma 2. *If* $I(a_1) = \infty$ *or* $I(a_1, a_2) = \infty$ *then* $N_3 = \emptyset$.

Proof. Let $N_3 \neq \emptyset$. Suppose that

$$x(t) < 0, \quad x^{[1]}(t) > 0, \quad x^{[2]}(t) > 0 \quad \text{for large } t.$$

Thus $x^{[3]}(t) > 0$ and we have

1. $x < 0$ and $x^{[1]} > 0$ implies that $x < 0$ is increasing, so x is bounded.
2. $x^{[1]} > 0$ and $x^{[2]} > 0$ implies that $x^{[1]} > 0$ is increasing, so

$$x(t) - x(T) > x^{[1]}(T) \int_T^t a_1(s)\,\mathrm{d}s \quad \text{for } t > T.$$

If $I(a_1) = \infty$, then this is a contradiction with the fact that x is bounded.

3. $x^{[2]} > 0$ and $x^{[3]} > 0$ implies that $x^{[2]} > 0$ is increasing. Thus we get, integrating twice from T to t, $t > T$,

$$\begin{gathered} x^{[2]}(t) > x^{[2]}(T) > 0 \\ x^{[1]}(t) > x^{[1]}(T) + x^{[2]}(T) \int_T^t a_2(s)\,\mathrm{d}s \\ x(t) - x(T) > x^{[1]}(T) + x^{[2]}(T) \int_T^t a_1(s) \int_T^s a_2(s)\,\mathrm{d}s\,\mathrm{d}t \to \infty, \end{gathered}$$

which contradicts the boundedness of x. □

Proposition 1. (i) *If $I(a_i) = \infty$, $i = 1,2,3$, then* (E_+) *has property A.*

(ii) *The maps induced by cyclic permutations preserve Kneser solutions, i.e., Kneser solutions of* (E_+) *are transformed to Kneser solutions of* $(\mathrm{E}_+^{\mathcal{C}})$ *and so on.*

(iii) *Let $I(a_1) = I(a_2) = \infty$. Then*

$$\int_0^\infty a_3(t) \int_0^t a_2(s) \int_0^s a_1(u) \,\mathrm{d}u \,\mathrm{d}s \,\mathrm{d}t = \infty \tag{9}$$

if and only if every Kneser solution of (E_+) *tends to zero for $t \to \infty$.*

(iv) *If*

$$\int_0^\infty a_3(t) \int_0^t a_1(s) \int_0^s a_2(u) \,\mathrm{d}u \,\mathrm{d}s \,\mathrm{d}t < \infty, \tag{10}$$

then (E_+) *is nonoscillatory.*

Proof. (i) See, e.g., [15, 16].

(ii) See [7, Collorary 3].

(iii) See [21, Theorem 7].

(iv) See [6, Theorem 5]. □

Lemma 3. *Let* (E_+) *have the canonical form* $(\mathrm{E}_+^{\mathcal{K}})$*. In cases* (K_i)*, $i = 2,3,4$, the map $T_{\mathcal{K}}^{-1}\colon \mathcal{S}(\mathrm{E}_+^{\mathcal{K}}) \to \mathcal{S}(\mathrm{E}_+)$ preserves property A, i.e., if* $(\mathrm{E}_+^{\mathcal{K}})$ *has property A then* (E_+) *has property A.*

In the other cases (K_i)*, $i = 5,6$, if* $(\mathrm{E}_+^{\mathcal{K}})$ *has property A then $\mathcal{N}(\mathrm{E}_+) = N_0 \cup N_3$ and every nonoscillatory solution of* (E_+) *tends to zero for $t \to \infty$.*

Proof. Let $(\mathrm{E}_+^{\mathcal{K}})$ have property A. Then every nonoscillatory solution u of $(\mathrm{E}_+^{\mathcal{K}})$ is a Kneser solution tending to zero. Then $x = b_0 u$ is a solution of (E_+) and using Remark 1 we get

$$x^{[1]} = \frac{1}{a_1}(b_0 u)' = \frac{1}{a_1}(b_0' u + b_0 u') < 0,$$

hence $x \in N_0 \cup N_3$. In addition, using Remark 2 (iii), x tends to zero. If case (K_i), $i = 2,3,4$, occurs, then using Lemma 2 we get $N_3 = \emptyset$. □

Lemma 4. *Let $I(a_1) = \infty$. Then $T_{\mathcal{C}}\colon \mathcal{S}(\mathrm{E}_+) \to \mathcal{S}(\mathrm{E}_+^{\mathcal{C}})$ preserves property A, i.e., if* (E_+) *has property A then* $(\mathrm{E}_+^{\mathcal{C}})$ *has property A.*

Similarly, if $I(a_2) = \infty$ and $(\mathrm{E}_+^{\mathcal{C}})$ *has property A, then* $(\mathrm{E}_+^{\mathcal{CC}})$ *has property A; if $I(a_3) = \infty$ and* $(\mathrm{E}_+^{\mathcal{CC}})$ *has property A, then* (E_+) *has property A.*

Proof. Let $I(a_1) = \infty$. Suppose that (E_+) has property A. Then by Theorem 3 and Proposition 1(ii) we get that every solution of $(\mathrm{E}_+^{\mathcal{C}})$ is either oscillatory or a

Kneser solution. Suppose that y is a Kneser solution of $(E_+^{\mathcal{C}})$ tending to nonzero constant $c > 0$. Since $y = x^{[1]} > 0$, with x a solution of (E_+) and $x < 0$, we have for large t and $\varepsilon > 0$ sufficiently small

$$0 > x(t) = x(0) + \int_0^t a_1(s)x^{[1]}(s)\,\mathrm{d}s > x(0) + (c-\varepsilon)\int_0^t a_1(s)\,\mathrm{d}s \to \infty \text{ for } t \to \infty,$$

which is a contradiction. Similarly we get the other two conclusions. □

Lemma 5. *Let* (E_+) *have the canonical form* $(E_+^{\mathcal{K}})$. *Suppose*

$$\int^\infty a_3(t)b_0(t)b_3(t)\,\mathrm{d}t = \infty. \tag{11}$$

Then $(E_+^{\mathcal{K}})$ *has property A.*

Proof. Since $\int^\infty b_i = \infty$, $i = 1, 2$, and (11) holds, the conclusion follows from Proposition 1(i). □

Theorem 4. *Let one of the following three conditions hold:*

$$\begin{aligned}
&\text{(i)} \quad I(a_1) = I(a_3) = I(a_2, a_3) = I(a_2, a_1) = \infty\\
&\text{(ii)} \quad I(a_1) = I(a_2) = I(a_3, a_1) = I(a_3, a_2) = \infty\\
&\text{(iii)} \quad I(a_2) = I(a_3) = I(a_1, a_2) = I(a_1, a_3) = \infty.
\end{aligned}$$

Then (E_+), $(E_+^{\mathcal{C}})$ *and* $(E_+^{\mathcal{CC}})$ *are oscillatory and have property A.*

Proof. First, let us remark that if $I(a_1) = I(a_2) = I(a_3) = \infty$, the conclusion holds in view of Proposition 1(i) and Lemma 1.

Hence, suppose that $I(a_2) < \infty$ and (i) holds. Then (E_+) may be written in the canonical form $(E_+^{\mathcal{K}_3})$. Since $I(a_2, a_3) = \infty$, the condition (11) is satisfied and we get from Lemma 5 that $(E_+^{\mathcal{K}_3})$ has property A. Hence, by Lemma 3, (E_+) has property A and, in addition, by Lemma 1, it has an oscillatory solution. By Theorem 3, $(E_+^{\mathcal{C}})$ and $(E_+^{\mathcal{CC}})$ are oscillatory, too. Since $I(a_2, a_1) = \infty$, we have $I(a_1) = \infty$ and in view of Lemma 4 the equation $(E_+^{\mathcal{C}})$ has property A.

It remains to prove that also $(E_+^{\mathcal{CC}})$ has property A. From the fact that $(E_+^{\mathcal{C}})$ has property A it follows, in view of Proposition 1(ii), that every nontrivial solution of $(E_+^{\mathcal{CC}})$ is either oscillatory or a Kneser solution. Since $I(a_2, a_1) = \infty$, we have

$$\int_0^\infty a_2(t)\int_0^t a_1(s)\int_0^s a_3(u)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t = \infty.$$

Using Proposition 1(iii) for $(\mathrm{E}_+^{\mathcal{CC}})$, we get that every Kneser solution of $(\mathrm{E}_+^{\mathcal{CC}})$ tends to zero.

Similarly, if $I(a_3) < \infty$ and (ii) [or $I(a_1) < \infty$ and (iii)] hold, we apply Lemma 5 for $(\mathrm{E}_+^{\mathcal{C}\mathcal{K}_3})$ [for $(\mathrm{E}_+^{\mathcal{CC}\mathcal{K}_3})$]. Then we use Lemma 3 and get the conclusion for $(\mathrm{E}_+^{\mathcal{C}})$ [or $(\mathrm{E}_+^{\mathcal{CC}})$]. Using Lemma 4 we get the conclusion for $(\mathrm{E}_+^{\mathcal{CC}})$ [for (E_+)] and finally, using Proposition 1(iii), for (E_+) [$(\mathrm{E}_+^{\mathcal{C}})$], respectively. □

Theorem 5. *Let one of the following three conditions hold:*

(i) $I(a_1) = \infty$, $I(a_2, a_1) < \infty$ *and*

$$\int_0^\infty a_3(t)\left(\int_t^\infty a_2(s)\,\mathrm{d}s\right)\left(\int_t^\infty a_1(s)\int_s^\infty a_2(\tau)\,d\tau\,\mathrm{d}s\right)\mathrm{d}t = \infty$$

(ii) $I(a_2) = \infty$, $I(a_3, a_2) < \infty$ *and*

$$\int_0^\infty a_1(t)\left(\int_t^\infty a_3(s)\,\mathrm{d}s\right)\left(\int_t^\infty a_2(s)\int_s^\infty a_3(\tau)\,d\tau\,\mathrm{d}s\right)\mathrm{d}t = \infty$$

(iii) $I(a_3) = \infty$, $I(a_1, a_3) < \infty$ *and*

$$\int_0^\infty a_2(t)\left(\int_t^\infty a_1(s)\,\mathrm{d}s\right)\left(\int_t^\infty a_3(s)\int_s^\infty a_1(\tau)\,d\tau\,\mathrm{d}s\right)\mathrm{d}t = \infty.$$

Then (E_+), $(\mathrm{E}_+^{\mathcal{C}})$, $(\mathrm{E}_+^{\mathcal{CC}})$ *are oscillatory and have property A.*

Proof. The idea of the proof is similar to that of Theorem 4.

Suppose (i). Then (E_+) may be written in the canonical form $(\mathrm{E}_+^{\mathcal{K}_4})$. By Lemma 5, $(\mathrm{E}_+^{\mathcal{K}_4})$ has property A and so, by Lemma 3, also (E_+) has property A. In addition, by Lemma 1, (E_+) is oscillatory. Using Lemma 4 we get the conclusion for $(\mathrm{E}_+^{\mathcal{C}})$.

Consider $(\mathrm{E}_+^{\mathcal{CC}})$. Since $(\mathrm{E}_+^{\mathcal{C}})$ has property A, it follows, in view of Proposition 1(ii) and Theorem 3, that every nontrivial solution of $(\mathrm{E}_+^{\mathcal{CC}})$ is either oscillatory or a Kneser solution. The assumptions of the theorem imply that

$$\infty = \int_0^\infty a_3(t)\int_t^\infty a_1(s)\int_s^\infty a_2(u)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t$$
$$= \int_0^\infty a_3(t)\int_t^\infty a_2(s)\int_0^s a_1(u)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t = \int_0^\infty a_2(t)\int_0^t a_1(s)\int_0^s a_3(u)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t,$$

so, using Proposition 1(iii) as in the proof of Theorem 4, every Kneser solution of $(\mathrm{E}_+^{\mathcal{CC}})$ tends to zero. Then $(\mathrm{E}_+^{\mathcal{CC}})$ is oscillatory and has property A. In the two remaining cases we proceed similarly. □

Remark 4. Theorems 4 and 5 generalize for $n = 3$ Corollary 5 of [7] where it is proved that if $I(a_i) = \infty$, $i = 1, 2, 3$, then (E_+), $(\mathrm{E}_+^{\mathcal{C}})$ and $(\mathrm{E}_+^{\mathcal{CC}})$ have property A.

Remark 5. Concerning oscillatory properties, we have proved in [6, Theorems 8 and 10] a more general statement: (E_+), $(\mathrm{E}_+^{\mathcal{C}})$ and $(\mathrm{E}_+^{\mathcal{CC}})$ are oscillatory provided any of the following condition is satisfied:

$$\begin{aligned} &\text{(i)} \quad I(a_1) = I(a_3) = I(a_2, a_3) = \infty, \\ &\text{(ii)} \quad I(a_1) = I(a_2) = I(a_3, a_1) = \infty, \\ &\text{(iii)} \quad I(a_2) = I(a_3) = I(a_1, a_2) = \infty. \end{aligned} \tag{12}$$

Using (ii) and Proposition 1(iii) we can also prove the existence of an *unbounded* oscillatory solution for equation (E_+).

Corollary 1. *If*

$$I(a_1) = I(a_2) = I(a_3, a_1) = \infty \tag{13}$$

and

$$\int_0^\infty a_3(t) \int_0^t a_2(s) \int_0^s a_1(u)\, \mathrm{d}u\, \mathrm{d}s\, \mathrm{d}t < \infty, \tag{14}$$

then there exists an unbounded oscillatory solution of (E_+).

Proof. If (14) holds, then, by Proposition 1(iii), there exists a Kneser solution tending to a nonzero constant. By Remark 5, (13) ensures that there exists an oscillatory solution. Assume that every oscillatory solution is bounded. Then (E_+) has a weakly oscillatory solution y, i.e., $y \neq 0$, y' is oscillatory, which contradicts [7, Corollary 2]. □

Remark 6. As already proved (Lemma 1), if $I(a_1) = I(a_2) = \infty$ and (E_+) has property A then this equation possesses an oscillatory solution. From (10), (14) and Theorem 3 we obtain the well-known fact that the equation $x''' + q(t)x = 0$ is oscillatory if and only if it has property A (see, e.g., [16, Lemma 2.8]). But in general, the converse is not true. The following conclusion holds.

Corollary 2. *Let* $I(a_1) = I(a_2) = \infty$. *If* (E_+) *is oscillatory and has not property A, then* (E_-) *is nonoscillatory.*

Proof. Since (E_+) is oscillatory and has not property A, there exists a Kneser solution of (E_+) tending to a nonzero constant. By Proposition 1(iii) we have (14).

This condition ensures, in view of Proposition 1(iv), that the equation

$$\left(\frac{1}{a_1(t)}\left(\frac{1}{a_2(t)}x'(t)\right)'\right)' + a_3(t)x(t) = 0$$

is nonoscillatory. By Theorem 3 its adjoint equation, which is (E_-), is nonoscillatory, too. □

Corollaries 1 and 2 are illustrated by the following example.

Example. Let $\varepsilon \in (0,1)$ and $T > 1$. Consider the equation

$$\left(t\ln t\left(\frac{y'(t)}{\ln t}\right)'\right)' + \frac{y(t)}{t^2(\ln t)^{1+\varepsilon}} = 0, \quad t \in [T,\infty). \tag{15}$$

Then

$$\int_T^t a_1(s)\,\mathrm{d}s = t(\ln t - 1) + c_1 \to \infty, \quad \int_T^t a_2(s)\,\mathrm{d}s = \ln\ln t + c_2 \to \infty \ \text{ for } t \to \infty,$$
$$\int_T^t a_2(s)\int_T^s a_1(s)\,\mathrm{d}s\,\mathrm{d}t = t - \int_T^t \frac{\mathrm{d}s}{\ln s} - T - \int_T^t \frac{c\,\mathrm{d}s}{s\ln s} \to \infty \quad \text{for } t \to \infty,$$
$$\int_T^\infty a_3(t)\int_T^t a_2(s)\int_T^s a_1(u)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t \leqslant \int_T^\infty \frac{\mathrm{d}t}{t(\ln t)^{1+\varepsilon}} = \int_{\ln T}^\infty \frac{\mathrm{d}u}{u^{1+\varepsilon}} < \infty,$$
$$\int_T^\infty a_3(t)\int_T^t a_1(s)\,\mathrm{d}s\,\mathrm{d}t \geqslant \int_T^\infty \frac{\ln t - 1}{t(\ln t)^{1+\varepsilon}}\,\mathrm{d}t = \int_T^\infty \frac{\mathrm{d}t}{t(\ln t)^{\varepsilon}}\,\mathrm{d}t = \int_{\ln T}^\infty \frac{\mathrm{d}s}{s^\varepsilon}\,\mathrm{d}s = \infty,$$

i.e., (14) and (13) hold. Hence, by Corollary 1, equation (15) has an unbounded oscillatory solution and a Kneser solution tending to nonzero constant.

By Corollary 2 the equation

$$\left(t\ln t\left(\frac{y'(t)}{\ln t}\right)'\right)' - \frac{y(t)}{t^2(\ln t)^{1+\varepsilon}} = 0$$

is nonoscillatory.

Remark 7. Theorems 4 and 5 concern the cases (K_3) and (K_4). In case (K_2), i.e., if $I(a_1) < \infty$ and $I(a_1,a_2) = \infty$, we obtain, applying Lemmas 1, 3 and 5 to equation ($\mathrm{E}_+^{\mathcal{K}_2}$), the same conclusion as that given in Theorem 4 (iii).

In the two remaining cases (K_5) and (K_6) we get, applying Lemmas 1, 3 and 5 to the equation ($\mathrm{E}_+^{\mathcal{K}_5}$) or ($\mathrm{E}_+^{\mathcal{K}_6}$), the following conclusion:

Let one of the following two conditions hold:

(i) $I(a_2) = \infty$, $I(a_1, a_2) < \infty$ and

$$\int_0^\infty a_3(t) \int_t^\infty a_2(s)\,\mathrm{d}s \int_t^\infty a_1(s) \int_s^\infty a_2(\tau)\,d\tau\,\mathrm{d}s\,\mathrm{d}t = \infty,$$

(ii) $I(a_1) < \infty$, $I(a_2) < \infty$ and

$$\int_0^\infty a_3(t) \int_t^\infty a_1(s)\,\mathrm{d}s \int_t^\infty \nu(s)\,\mathrm{d}s \int_t^\infty a_2(s) \int_s^\infty a_1(\tau)\,d\tau\,\mathrm{d}s\,\mathrm{d}t = \infty,$$

where $\nu(t) = \left(\frac{1}{\int_t^\infty a_1}\right)' \int_t^\infty a_2 \int_s^\infty a_1$.

Then (E_+), $(\mathrm{E}_+^{\mathcal{C}})$ and $(\mathrm{E}_+^{\mathcal{CC}})$ are oscillatory and, by Theorem 3, also the adjoint equations $(\mathrm{E}_+^{\mathcal{A}})$, $(\mathrm{E}_+^{\mathcal{CA}})$ and $(\mathrm{E}_+^{\mathcal{CCA}})$ are oscillatory. In addition, every nonoscillatory solution of (E_+) is either a Kneser solution or of class N_3 and tends to zero for $t \to \infty$.

Analogous statements can be obtained by application of Lemmas 1, 3 and 5 to equations $(\mathrm{E}_+^{\mathcal{C}})$ and $(\mathrm{E}_+^{\mathcal{CC}})$.

Coming back to the equations (2), (3), if we apply Theorem 4 (iii) to equation (2), Theorem 4 (ii) to equation $(2^{\mathcal{C}})$ and Theorem 3 to the adjoint equations to (2) and (3), we obtain the following theorem.

Corollary 3. *Let $q \neq 0$, $r > 0$ and let one of the following conditions hold:*

(i) $\int^\infty t/r(t)\,\mathrm{d}t = \infty$ *and* $I(\frac{1}{r}, |q|) = \infty$,

(ii) $\int^\infty t\,|q(t)|\,\mathrm{d}t = \infty$ *and* $I(|q|, \frac{1}{r}) = \infty$.

Then equations (2), (3) are both oscillatory. In addition, if $q > 0$, then these equations have property A.

The results presented in Parts I and II can be used for proving integral criteria for (E_-) to have property B as well as for the nonlinear equation $L(x) + f(x) = 0$ to have property A or property B. This will be shown elsewhere.

Case	$b_0(t)$	$b_1(t)$	$b_2(t)$	$b_3(t)$
(K_1)	1	u_1	u_2	1
(K_2)	$\int_t^\infty u_1$	$\left(\frac{1}{\int_t^\infty u_1}\right)'$	$u_2 \int_t^\infty u_1$	1
(K_3)	1	$u_1 \int_t^\infty u_2$	$\left(\frac{1}{\int_t^\infty u_2}\right)'$	$\int_t^\infty u_2$
(K_4)	$\int_t^\infty u_1 \int_s^\infty u_2$	$\left(\frac{1}{\int_t^\infty u_1 \int_s^\infty u_2}\right)'$	$\left(\frac{1}{\int_t^\infty u_2}\right)' \int_t^\infty u_1 \int_s^\infty u_2$	$\int_t^\infty u_2$
(K_5)	$\int_t^\infty u_1$	$\left(\frac{1}{\int_t^\infty u_1}\right)' \int_t^\infty u_2 \int_s^\infty u_1$	$\left(\frac{1}{\int_t^\infty u_2 \int_s^\infty u_1}\right)'$	$\int_t^\infty u_2 \int_s^\infty u_1$
(K_6)	$\int_t^\infty u_1 \int_t^\infty \nu$	$\left(\frac{1}{\int_t^\infty \nu}\right)'$	$\int_t^\infty \nu \left(\frac{1}{\int_t^\infty u_2 \int_s^\infty u_1)}\right)'$	$\int_t^\infty u_2 \int_s^\infty u_1$

Table 1: Canonical forms for the operator D where $\nu(t) = \left(\frac{1}{\int_t^\infty u_1}\right)' \int_t^\infty u_2 \int_s^\infty u_1$.

Case	$b_0(t)$	$b_1(t)$	$b_2(t)$	$b_3(t)$
(K_1)	1	a_2	a_3	1
(K_2)	$\int_t^\infty a_2$	$\left(\frac{1}{\int_t^\infty a_2}\right)'$	$a_3 \int_t^\infty a_2$	1
(K_3)	1	$a_2 \int_t^\infty a_3$	$\left(\frac{1}{\int_t^\infty a_3}\right)'$	$\int_t^\infty a_3$
(K_4)	$\int_t^\infty a_2 \int_s^\infty a_3$	$\left(\frac{1}{\int_t^\infty a_2 \int_s^\infty a_3}\right)'.$	$\left(\frac{1}{\int_t^\infty a_3}\right)' \int_t^\infty a_2 \int_s^\infty a_3$	$\int_t^\infty a_3$
(K_5)	$\int_t^\infty a_2$	$\left(\frac{1}{\int_t^\infty a_2}\right)' \int_t^\infty a_3 \int_s^\infty a_2$	$\left(\frac{1}{\int_t^\infty a_3 \int_s^\infty a_2}\right)'$	$\int_t^\infty a_3 \int_s^\infty a_2$
(K_6)	$\int_t^\infty a_2 \int_t^\infty \mu$	$\left(\frac{1}{\int_t^\infty \mu}\right)'$	$\int_t^\infty \mu \left(\frac{1}{\int_t^\infty a_3 \int_s^\infty a_2)}\right)'$	$\int_t^\infty a_3 \int_s^\infty a_2$

Table 2: Canonical forms for the operator L^C where $\mu(t) = \left(\frac{1}{\int_t^\infty a_2}\right)' \int_t^\infty a_3 \int_s^\infty a_2$.

References

[1] *Bartušek M.*: Asymptotic properties of oscillatory solutions of differential equations of the n-th order. Folia Fac. Sci. Nat. Univ. Brunensis Masarykianae (1992).

[2] *Bartušek, M., Došlá Z.*: Oscillatory criteria for nonlinear third order differential equations with quasiderivatives. Diff. equation and Dynam. Syst. *3* (1995), 251–268.

[3] *Cecchi M.*: Oscillation criteria for a class of third order linear differential equations. Boll. Un. Mat. Ital., VI, 2–C (1983), 297–306.

[4] *Cecchi M.*: Sul comportamento delle soluzioni di una classe di equazioni differenziali lineari del terzo ordine in caso di oscillazione. Boll. Un. Mat. Ital., VI, 4–C *4* (1985), 71–85.

[5] *Cecchi M., Marini M.*: Oscillation properties of third order nonlinear differential equation. Nonlinear Analysis, Th. M. Appl. *15* (1990), 141–153.

[6] *Cecchi M., Došlá Z., Marini M., Villari G.*: On the qualitative behavior of solutions of third order differential equations. J. Math. Anal. Appl. *197* (1996), 749–766.

[7] *Cecchi M., Marini M., Villari G.*: On a cyclic disconjugated operator associated to linear differential equations. Annali Mat. Pura Appl. IV *CLXX* (1996), 297–309.

[8] *Coppel W.A.*: Disconjugacy. Springer-Verlag 1971, Lectures Notes in Math. 220.

[9] *Dolan J. M.*: On the relationship between the oscillatory behavior of a linear third-order differential equation and its adjoint. J. Diff. Equat. *7* (1970), 367–388.

[10] *Elias U.*: Nonoscillation and eventual disconjugacy. Proc. Amer.Math. Soc. *66* (1977), 269–275.

[11] *Erbe L.*: Oscillation, nonoscillation, and asymptotic behavior for third order nonlinear differential equations. Annali Mat. Pura Appl. *IV, 110* (1976), 373–391.

[12] *Gaudenzi M.*: On the Sturm-Picone theorem for nth–order differential equations. Siam J. Math. Anal. *21* (1990), 980–994.

[13] *Greguš M.*: Third Order Linear Differential Equations. D. Reidel Publ. Comp., Dordrecht, Boston, Lancaster, Tokyo, 1987.

[14] *Hanan M.*: Oscillation criteria for third-order linear differential equation. Pacific J. Math. *11* (1961), 919-944.

[15] *Chanturia T.A.*: On oscillatory properties of systems of nonlinear ordinary differential equations (Russian). Trudy universiteta prikladnoj matematiky, Tbilisi *14* (1983), 163–203.

[16] *Kiguradze I. T., Chanturia T.A.*: Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations. Kluwer Academic Publishers, Dordrecht-Boston-London (1993, 432 pp).

[17] *Kusano T., Naito M., Tanaka K.*: Oscillatory and asymptotic behavior of solutions of a class of linear ordinary differential equations. Proc. Royal Soc. Edinburgh *90A* (1981), 24–40.

[18] *Lazer A. C.*: The behaviour of solutions of the differential equation $y'''+p(x)y'+q(x)y=0$. Pacific J. Math. *17* (1966), 435–466.

[19] *Ohriska J.*: Oscillatory and asymptotic properties of third and fourth order linear differential equations. Czech. Math. J. *39 (114)* (1989), 215–224.

[20] *Swanson C. A.*: Comparison and Oscillation Theory of Linear Differential Equations. Acad. Press, New York, 1968.

[21] *Švec M.*: Behaviour of nonoscillatory solutions of some nonlinear differential equations. Acta Math. Univ. Comenianae *34* (1980), 115–130.

[22] *Trench W. F.*: Canonical forms and principal systems for general disconjugate equations. TAMS *189* (1974), 319–327.

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