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LOCAL PROPERTIES OF ACCESSIBLE INJECTIVE OPERATOR IDEALS

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Abstract. In addition to Pisier's counterexample of a non-accessible maximal Banach ideal, we will give a large class of maximal Banach ideals which *are accessible*. The first step is implied by the observation that a "good behaviour" of trace duality, which is canonically induced by conjugate operator ideals can be extended to adjoint Banach ideals, if and only if these adjoint ideals satisfy an accessibility condition (theorem 3.1). This observation leads in a natural way to a characterization of accessible injective Banach ideals, where we also recognize the appearance of the ideal of *absolutely summing operators* (prop. 4.1). By the famous *Grothendieck inequality*, every operator from L_1 to a Hilbert space is absolutely summing, and therefore our search for such ideals will be directed towards Hilbert space factorization—via an operator version of Grothendieck's inequality (lemma 4.2). As a consequence, we obtain a class of injective ideals, which are "quasi-accessible", and with the help of *tensor stability*, we improve the corresponding norm inequalities, to get accessibility (theorem 4.1 and 4.2). In the last chapter of this paper we give applications, which are implied by a non-trivial link of the above mentioned considerations to normed products of operator ideals.

Keywords: accessibility, Banach spaces, conjugate operator ideals, Hilbert space factorization, Grothendieck's inequality, tensor norms, tensor stability

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1. Introduction

Given Banach spaces E, F and a maximal Banach ideal $(\mathcal{A}, \mathbf{A})$, we are interested in reasonable sufficient conditions on E, F and $(\mathcal{A}, \mathbf{A})$ such that $(\mathcal{A}, \mathbf{A})$ is accessible. In general it is a nontrivial subject to prove accessibility of maximal Banach ideals since non-accessibility can only appear on Banach spaces without the metric approximation property, and in 1992, Pisier made use of such a Banach space (the Pisier space P) to construct a non-accessible maximal Banach ideal (cf. [3], 31.6).

On the other hand, accessible Banach ideals allow a suggestive (algebraic) calculus which leads to further results concerning the local structure of operator ideals (e.g. a transfer of the principle of local reflexivity from the operator norm to suitable ideal norms $\mathbf{A}$ (cf. [3], [11] and [12]).

This paper is mainly devoted to the description of a large class of maximal injective Banach ideals which are totally accessible. We will see a deep interplay between conjugates of Banach ideals, Hilbert space factorization, Grothendieck's inequality and tensor stable quasi-Banach ideals. We only deal with Banach spaces and most of our notations and definitions concerning Banach spaces and operator ideals are standard and can be found in the detailed monographs [3] and [13]. However, if $(\mathcal{A}, \mathbf{A})$ and $(\mathcal{B}, \mathbf{B})$ are given quasi-Banach ideals, we will use the shorter notation $(\mathcal{A}^d, \mathbf{A}^d)$ for the dual ideal (instead of $(\mathcal{A}^{dual}, \mathbf{A}^{dual})$) and the abbreviation $\mathcal{A} \stackrel{1}{=} \mathcal{B}$ for the equality $(\mathcal{A}, \mathbf{A}) = (\mathcal{B}, \mathbf{B})$. The inclusion $(\mathcal{A}, \mathbf{A}) \subseteq (\mathcal{B}, \mathbf{B})$ is often shortened by $\mathcal{A} \stackrel{1}{\subseteq} \mathcal{B}$, and if $T\colon\ E \longrightarrow F$ is an operator, we indicate that it is a metric injection by writing $T\colon\ E \stackrel{1}{\hookrightarrow} F$. Each section of this paper includes the more special terminology which is not so common.

2. On tensor norms and associated Banach ideals

At first we recall the basic notions of Grothendieck's metric theory of tensor products (cf., eg., [3], [4], [6], [9]), which will be used throughout this paper. A *tensor norm* α is a mapping which assigns to each pair (E, F) of Banach spaces a norm $\alpha(\cdot; E, F)$ on the algebraic tensor product $E \otimes F$ (shorthand: $E \otimes_\alpha F$ and $E \tilde{\otimes}_\alpha F$ for the completion) such that

(i) $\varepsilon \leqslant \alpha \leqslant \pi$,

(ii) α satisfies the metric mapping property: If $S \in \mathcal{L}(E, G)$ and $T \in \mathcal{L}(F, H)$, then $\|S \otimes T\colon\ E \otimes_\alpha F \longrightarrow G \otimes_\alpha H\| \leqslant \|S\|\,\|T\|$.

Well-known examples are the injective tensor norm ε, which is the smallest one, and the projective tensor norm π, which is the largest one. For other important examples we refer to [3], [4], or [9]. Each tensor norm α can be extended in two natural ways. For this, denote for given Banach spaces E and F

$$\mathrm{FIN}(E) := \{M \subseteq E \mid M \in \mathrm{FIN}\} \quad \textit{and} \quad \mathrm{COFIN}(E) := \{L \subseteq E \mid E/L \in \mathrm{FIN}\},$$

where FIN stands for the class of all finite dimensional Banach spaces. Let $z \in E \otimes F$. Then the *finite hull* $\overrightarrow{\alpha}$ is given by

$$\overrightarrow{\alpha}(z; E, F) := \inf\{\alpha(z; M, N) \mid M \in \mathrm{FIN}(E), N \in \mathrm{FIN}(F), z \in M \otimes N\}$$

and the *cofinite hull* $\overleftarrow{\alpha}$ of α is given by

$$\overleftarrow{\alpha}(z;E,F) := \sup\{\alpha(Q_K^E \otimes Q_L^F(z); E/K, F/L) \mid K \in \mathrm{COFIN}(E), L \in \mathrm{COFIN}(F)\}.$$

α is called *finitely generated* if $\alpha = \overrightarrow{\alpha}$, *cofinitely generated* if $\alpha = \overleftarrow{\alpha}$ (it is always true that $\overleftarrow{\alpha} \leqslant \alpha \leqslant \overrightarrow{\alpha}$). α is called *right-accessible* if $\overleftarrow{\alpha}(z;M,F) = \overrightarrow{\alpha}(z;M,F)$ for all $(M,F) \in \mathrm{FIN} \times \mathrm{BAN}$, *left-accessible* if $\overleftarrow{\alpha}(z;E,N) = \overrightarrow{\alpha}(z;E,N)$ for all $(E,N) \in \mathrm{BAN} \times \mathrm{FIN}$, and *accessible* if it is right- and left-accessible. α is called *totally accessible* if $\overleftarrow{\alpha} = \overrightarrow{\alpha}$. The injective norm ε is totally accessible, the projective norm π is accessible—but not totally accessible, and Pisier's counterexample implies the existence of a (finitely generated) tensor norm which is neither left- nor right-accessible (see [3], 31.6). There exists a powerful one-to-one correspondence between finitely generated tensor norms and maximal Banach ideals which links thinking in terms of operators with "tensorial" thinking and which allows to transfer notions in the "tensor-language" to the "operator-language" and conversely. We refer the reader to [3] and [11] for detailed informations concerning this subject.

Let E, F be Banach spaces and $z = \sum_{i=1}^{n} a_i \otimes y_i$ be an element in $E' \otimes F$. Then $T_z(x) := \sum_{i=1}^{n} \langle x, a_i \rangle y_i$ defines a finite operator $T_z \in \mathcal{F}(E,F)$ which is independent of the representation of z in $E' \otimes F$. Let α be a finitely generated tensor norm and $(\mathcal{A}, \mathbf{A})$ be a maximal Banach ideal. α and $(\mathcal{A}, \mathbf{A})$ are said to be *associated*, notation:

$$(\mathcal{A}, \mathbf{A}) \sim \alpha \quad \text{(shorthand: } \mathcal{A} \sim \alpha, \text{ resp. } \alpha \sim \mathcal{A})$$

if for all $M, N \in \mathrm{FIN}$

$$\mathcal{A}(M,N) = M' \otimes_\alpha N$$

holds isometrically: $\mathbf{A}(T_z) = \alpha(z; M', N)$.

Besides the maximal Banach ideal $(\mathcal{L}, \|\cdot\|) \sim \varepsilon$ we will mainly be concerned with $(\mathcal{I}, \mathbf{I}) \sim \pi$ (integral operators), $(\mathcal{L}_2, \mathbf{L}_2) \sim w_2$ (Hilbertian operators), $(\mathcal{D}_2, \mathbf{D}_2) \stackrel{1}{=} (\mathcal{L}_2^*, \mathbf{L}_2^*) \sim w_2^*$ (2-dominated operators), $(\mathcal{P}_p, \mathbf{P}_p) \sim g_p \backslash = g_q^*$ (absolutely p-summing operators), $1 \leqslant p \leqslant \infty, \frac{1}{p} + \frac{1}{q} = 1$, $(\mathcal{L}_\infty, \mathbf{L}_\infty) \stackrel{1}{=} (\mathcal{P}_1^*, \mathbf{P}_1^*) \sim w_\infty$ and $(\mathcal{L}_1, \mathbf{L}_1) \stackrel{1}{=} (\mathcal{P}_1^{*d}, \mathbf{P}_1^{*d}) \sim w_1$. Since it is important for us, we recall the notion of the conjugate operator ideal (cf. [5], [8]): let $(\mathcal{A}, \mathbf{A})$ be a quasi-Banach ideal. Let $\mathcal{A}^\Delta(E,F)$ be the set of all $T \in \mathcal{L}(E,F)$ for which

$$\mathbf{A}^\Delta(T) := \sup\{\mathrm{tr}(TL) \mid L \in \mathcal{F}(F,E), \mathbf{A}(L) \leqslant 1\} < \infty.$$

Then a Banach ideal is obtained. It is called the *conjugate ideal* of $(\mathcal{A}, \mathbf{A})$.

$(\mathcal{A}, \mathbf{A})$ is called *right-accessible*, if for all $(M, F) \in \mathrm{FIN} \times \mathrm{BAN}$, operators $T \in \mathcal{L}(M, F)$ and $\varepsilon > 0$ there are $N \in \mathrm{FIN}(F)$ and $S \in \mathcal{L}(M, N)$ such that $T = J_N^F S$ and $\mathbf{A}(S) \leqslant (1+\varepsilon)\mathbf{A}(T)$. It is called *left-accessible*, if for all $(E, N) \in \mathrm{BAN} \times \mathrm{FIN}$, operators $T \in \mathcal{L}(E, N)$ and $\varepsilon > 0$ there are $L \in \mathrm{COFIN}(E)$ and $S \in \mathcal{L}(E/L, N)$ such that $T = SQ_L^E$ and $\mathbf{A}(S) \leqslant (1+\varepsilon)\mathbf{A}(T)$. A left- and right-accessible ideal is called *accessible*. $(\mathcal{A}, \mathbf{A})$ is *totally accessible*, if for every finite rank operator $T \in \mathcal{F}(E, F)$ between Banach spaces and $\varepsilon > 0$ there are $(L, N) \in \mathrm{COFIN}(E) \times \mathrm{FIN}(F)$ and $S \in \mathcal{L}(E/L, N)$ such that $T = J_N^F S Q_L^E$ and $\mathbf{A}(S) \leqslant (1+\varepsilon)\mathbf{A}(T)$. Every injective quasi-Banach ideal is right-accessible (every surjective ideal is left-accessible) and, if it is left-accessible, it is totally accessible. A finitely generated tensor norm is right-accessible (resp. left-accessible, accessible, totally accessible) if and only if its associated maximal Banach ideal is.

3. Accessible conjugate operator ideals

Let $(\mathcal{A}, \mathbf{A})$ be a p-Banach ideal $(0 < p \leqslant 1)$.[1] Suppose $\mathcal{A}$ is right-accessible. If we apply the cyclic composition theorem (see [3], 25.4) to $\mathcal{A} \circ \mathcal{L} \overset{1}{\subseteq} \mathcal{A}$, it follows that $\mathcal{A}^* \circ \mathcal{A} \overset{1}{\subseteq} \mathcal{I}$. If $\mathcal{A}$ is totally accessible, an easy calculation shows that $\mathcal{A}^* \overset{1}{=} \mathcal{A}^{\triangle}$. For $p = 1$, these properties of $\mathcal{A}$ characterize accessibility in the following sense:

Theorem 3.1. *Let $(\mathcal{A}, \mathbf{A})$ be a Banach ideal. Then $(\mathcal{A}^{*\triangle}, \mathbf{A}^{*\triangle})$ is always right-accessible. $(\mathcal{A}, \mathbf{A})$ is right-accessible if and only if $\mathcal{A}^* \circ \mathcal{A} \overset{1}{\subseteq} \mathcal{I}$. If in addition $(\mathcal{A}, \mathbf{A})$ is maximal then $(\mathcal{A}, \mathbf{A})$ is totally accessible if and only if $\mathcal{A}^* \overset{1}{=} \mathcal{A}^{\triangle}$.*

P r o o f. To prove the right-accessibility of $\mathcal{A}^{*\triangle}$, we may assume that $\mathcal{A}$ is maximal (cf [13], 9.3). Let $\alpha \sim \mathcal{A}$ be associated and $(M, F) \in \mathrm{FIN} \times \mathrm{BAN}$. Then $\alpha^* \sim \mathcal{A}^*$. The representation theorem for minimal operator ideals (see [3], 22.2) gives

$$\mathcal{A}^{\min}(M, F) \overset{1}{=} M' \otimes_\alpha F \overset{1}{\hookrightarrow} (F \otimes_{\alpha^t} M')''.$$

Since α is finitely generated, the representation theorem for maximal operator ideals (see [3], 17.5) yields

$$(F \otimes_{\alpha^t} M')' \cong \mathcal{A}^*(F, M).$$

On the other hand, by canonical trace duality, it follows that

$$\mathcal{A}^{*\triangle}(M, F) \overset{1}{\hookrightarrow} (\mathcal{A}^*(F, M))'.$$

[1] Most of the results in this paragraph first appeared in the author's doctoral thesis (see [11]). However, we now are using different proofs which give a better insight into the underlying structures.

Hence $\mathcal{A}^{\min}(M,F) \stackrel{1}{=} \mathcal{A}^{*\triangle}(M,F)$. Since $\mathcal{A}^{\min}$ is always right-accessible (see [3], 25.3), $\mathcal{A}^{*\triangle}$ is right-accessible. Let $(\mathcal{A},\ \mathbf{A})$ be an arbitrary Banach ideal such that $\mathcal{A}^* \circ \mathcal{A} \stackrel{1}{\subseteq} \mathcal{I}$. First we show that for all $(M,F) \in \mathrm{FIN} \times \mathrm{BAN}$, operators $T \in \mathcal{L}(M,F)$

$$\mathbf{A}^{*\triangle}(T) \leqslant \mathbf{A}(T)$$

Let $L \in \mathcal{F}(F,M)$. Then $|\operatorname{tr}(TL)| = |\operatorname{tr}(LT\,\mathrm{id}_M)| \leqslant \mathbf{I}(LT)\cdot\|\,\mathrm{id}_M\,\| \leqslant \mathbf{A}^*(L)\cdot\mathbf{A}(T)\cdot 1$. Hence $\mathbf{A}^{*\triangle}(T) \leqslant \mathbf{A}(T)$. Since $\mathcal{A}$ is normed, $\mathbf{A}(S) = \mathbf{A}^{**}(S) = \mathbf{A}^{*\triangle}(S)$ for all *elementary* operators S—between finite dimensional spaces—(cf. [13], 9.2.2), and it follows that $\mathcal{A}$ is right-accessible. Now let $(\mathcal{A},\ \mathbf{A})$ be a Banach ideal such that it is maximal and $\mathcal{A}^* \stackrel{1}{=} \mathcal{A}^{\triangle}$. Let $\alpha \sim \mathcal{A}$ be associated, E, F be Banach spaces and $z \in E \otimes F$. Let $w := j_E \otimes \mathrm{id}_F(z)$ and T_w the associated operator in $\mathcal{F}(E',F)$. Since α is finitely generated, the above mentioned representation theorem for maximal operator ideals and a simple application of the Hahn-Banach theorem give

$$\begin{aligned}\alpha(z;E,F) &= \alpha^t(z^t;F,E)\\ &= \sup\{|\langle z^t,\varphi\rangle| \mid \varphi \in B_{(F\otimes_{\alpha^t}E)'}\}\\ &= \sup\{|\operatorname{tr}(ST_w)| \mid S \in B_{\mathcal{A}^*(F,E')}\}\\ &= \sup\{|\operatorname{tr}(ST_w)| \mid S \in B_{\mathcal{A}^{\triangle}(F,E')}\}.\end{aligned}$$

Hence $\alpha(z;E,F) \leqslant \mathbf{A}(T_w) = \overleftarrow{\alpha}(z;E,F)$ (this equality follows from the embedding lemma (see [3], 17.6)). Therefore $\alpha \sim \mathcal{A}$ is totally accessible, and the proof is finished. □

Given an arbitrary maximal Banach ideal $(\mathcal{A},\ \mathbf{A})$ we have shown that $\mathcal{A}^{\triangle}$ is right-accessible. The natural question whether $\mathcal{A}^{\triangle}$ is *left*-accessible is still open and leads to interesting results concerning the local structure of $\mathcal{A}$.[2] It is even true that $\mathcal{A}^{\triangle}$ is left-accessible *if and only if* the weak $\mathcal{A}$-local principle of reflexivity holds (i.e., in this case it is possible to transfer the estimation for the operator norm $\|\cdot\|$ to the ideal norm $\mathbf{A}$ (see [11] and [12] for further details)). The dual ideal $\mathcal{A}^d$ is also a maximal Banach ideal and therefore $\mathcal{A}^{d\triangle}$ is right-accessible. Since $\mathcal{A}^{\triangle d} \stackrel{1}{\subseteq} \mathcal{A}^{d\triangle}$ we obtain that both $\mathcal{A}^{\triangle d}$ and $\mathcal{A}^{\triangle dd}$ are accessible. These considerations imply a slight generalization of ([3], 25.11) which does not assume accessibility-conditions:

Proposition 3.1. *Let $(\mathcal{A},\mathbf{A})$ be Banach ideal and E, F be Banach spaces. If E' or F has the approximation property, then*

$$(\mathcal{A}^{\mathrm{sur}})^{\min}(E,F) \stackrel{1}{=} (\mathcal{A}^{\min})^{\mathrm{sur}}(E,F)$$

[2] Note that we cannot use the preceeding proof to verify the left-accessibility of the conjugate ideal, since for all $(M,F) \in \mathrm{FIN} \times \mathrm{BAN}$, we have $(M\otimes_{\alpha^t}F')' \cong \mathcal{A}^*(M,F'')$.

and

$$(\mathcal{A}^{\mathrm{inj}})^{\min}(E,F) \stackrel{1}{=} (\mathcal{A}^{\min})^{\mathrm{inj}}(E,F).$$

P r o o f. It is sufficient to prove the first isometric equality (for $\mathcal{A}^{\mathrm{sur}}$) since the second one can be proved analogously. Let $\mathcal{B} \in \{\mathcal{A}, \mathcal{A}^{*\triangle dd}\}$. By assumption $\mathcal{B}$ is a normed operator ideal and therefore $\mathcal{B}^{**} \stackrel{1}{=} \mathcal{B}^{\max}$ (see [13], 9.3.1). Using known hull operations (cf. [13], 8.7 and 9.3) it follows that

$$(\mathcal{B}^{\mathrm{sur}})^{\min} \stackrel{1}{=} (\mathcal{B}^{**sur})^{\min} \stackrel{1}{=} (\mathcal{A}^{\mathrm{sur}})^{\min}$$

and

$$(\mathcal{B}^{\min})^{\mathrm{sur}} \stackrel{1}{=} (\mathcal{B}^{**\min})^{\mathrm{sur}} \stackrel{1}{=} (\mathcal{A}^{\min})^{\mathrm{sur}}.$$

In particular these equalities are true for $\mathcal{B} := \mathcal{A}^{*\triangle dd}$. Since $\mathcal{B}$ is accessible (in particular right-accessible), the claim follows by ([3], 25.11). □

So far we have seen that conjugates of maximal Banach ideals play a key role in the investigation of accessibility. Their appropriateness will be strenghtened by illuminating accessibility via a calculus derived from specific quotient ideals which are canonical extensions of conjugate ideals and which appear in a natural manner by theorem 3.1. Let $(\mathcal{A},\, \mathbf{A})$ be a p-Banach ideal ($0 < p \leqslant 1$).

We put:[3]

$$(\mathcal{A}^{\dashv}, \mathbf{A}^{\dashv}) := (\mathcal{I} \circ \mathcal{A}^{-1}, \mathbf{I} \circ \mathbf{A}^{-1}) \quad \text{and} \quad (\mathcal{A}^{\vdash}, \mathbf{A}^{\vdash}) := (\mathcal{A}^{-1} \circ \mathcal{I}, \mathbf{A}^{-1} \circ \mathbf{I})$$

and omit the proof of the simple but useful

Lemma 3.1.

(i) $\mathcal{A} \stackrel{1}{\subseteq} \mathcal{A}^{\vdash\dashv}$ *and* $\mathcal{A} \stackrel{1}{\subseteq} \mathcal{A}^{\dashv\vdash}$,

(ii) $\mathcal{A}^{\triangle} \stackrel{1}{\subseteq} \mathcal{A}^{\vdash} \stackrel{1}{\subseteq} \mathcal{A}^{*}$ *and* $\mathcal{A}^{\triangle} \stackrel{1}{\subseteq} \mathcal{A}^{\dashv} \stackrel{1}{\subseteq} \mathcal{A}^{*}$,

(iii) *if* $\mathcal{A} \stackrel{1}{\subseteq} \mathcal{B}$ *then* $\mathcal{B}^{\vdash} \stackrel{1}{\subseteq} \mathcal{A}^{\vdash}$ *and* $\mathcal{B}^{\dashv} \stackrel{1}{\subseteq} \mathcal{A}^{\dashv}$.

For $p = 1$, theorem 3.1 therefore implies that $\mathcal{A}$ is right-accessible if and only if $\mathcal{A}^{*} \stackrel{1}{=} \mathcal{A}^{\dashv}$. If $\mathcal{A}^{*\triangle}$ is left-accessible or $\mathcal{A}$ maximal, then the left-accessibility of $\mathcal{A}$ is equivalent to the statement $\mathcal{A}^{*} \stackrel{1}{=} \mathcal{A}^{\vdash}$. Note that $\mathcal{A}^{\triangle} \stackrel{1}{=} \mathcal{A}^{\vdash}$ if $\mathcal{A}$ is injective and $\mathcal{A}^{\triangle} \stackrel{1}{=} \mathcal{A}^{\dashv}$ if $\mathcal{A}$ is surjective (see [8], 2.6). Since $\mathbf{A}^{\triangle}$ and $\mathbf{A}^{*}$ coincide on the space of all elementary operators it follows that always $\mathcal{A}^{\triangle *} \stackrel{1}{=} \mathcal{A}^{**}$, hence $\mathcal{A}^{\vdash *} \stackrel{1}{=} \mathcal{A}^{**}$. Therefore, if $\mathcal{A}$ is a *maximal* Banach ideal, then lemma 3.1 implies that $\mathcal{A} \stackrel{1}{=} \mathcal{A}^{\vdash *} \stackrel{1}{=} \mathcal{A}^{\vdash\dashv}$ and we have obtained the:

Corollary 3.1. *Let* $(\mathcal{A}, \mathbf{A})$ *be a maximal Banach ideal. Then* $(\mathcal{A}^{\vdash}, \mathbf{A}^{\vdash})$ *is right-accessible. If* $\mathcal{A}^{\triangle}$ *is left-accessible, then* $(\mathcal{A}^{\dashv}, \mathbf{A}^{\dashv})$ *is left-accessible.*

[3] In [11], $\mathcal{I} \circ \mathcal{A}^{-1}$ was abbreviated as $\mathcal{A}^{\varepsilon}$ and $\mathcal{A}^{-1} \circ \mathcal{I}$ as $\mathcal{A}_{\varepsilon}$.

Now we turn our attention to *injective* maximal Banach ideals; in particular we are interested in aspects concerning the left-accessibility of those ideals.

4. Totally accessible injective operator ideals

Let $(\mathcal{A}, \mathbf{A})$ be a maximal Banach ideal and $(\mathcal{A}^{\mathrm{inj}}, \mathbf{A}^{\mathrm{inj}})$ the injective hull of $(\mathcal{A}, \mathbf{A})$. Let $\alpha \sim \mathcal{A}$ be associated. Then $\alpha\backslash \sim \mathcal{A}^{\mathrm{inj}}$ and $\backslash\alpha^* \sim \mathcal{A}^{\mathrm{inj}\,*}$. Since $\backslash\alpha^* \sim \backslash\mathcal{A}^*$ and $\backslash\mathcal{L} \overset{1}{=} \mathcal{L}_\infty$ it follows that

$$\mathcal{A}^{\mathrm{inj}\,*} \overset{1}{=} \backslash\mathcal{A}^* \overset{1}{=} (\mathcal{A}^* \circ \backslash\mathcal{L})^{\mathrm{reg}} \overset{1}{=} (\mathcal{A}^* \circ \mathcal{L}_\infty)^{\mathrm{reg}} \sim \backslash\alpha^*$$

is the adjoint of $\mathcal{A}^{\mathrm{inj}}$, hence a *maximal Banach ideal* (see [3], 25.9). In particular we obtain $\mathcal{L}_\infty \overset{1}{=} (\mathcal{L}_\infty \circ \mathcal{L}_\infty)^{\mathrm{reg}}$ (since $\mathcal{P}_1 \overset{1}{=} \mathcal{L}_\infty^*$ is injective) and $\mathcal{I} \overset{1}{=} (\mathcal{P}_1 \circ \mathcal{L}_\infty)^{\mathrm{reg}}$ (since $\mathcal{L}_\infty^{\mathrm{inj}} \overset{1}{=} \mathcal{L}$ (cf. [3], 20.14)) which implies that both $(\mathcal{L}_\infty \circ \mathcal{L}_\infty)^{\mathrm{reg}}$ and $(\mathcal{P}_1 \circ \mathcal{L}_\infty)^{\mathrm{reg}}$ are normed operator ideals—a fact which is not obvious.

Lemma 4.1. *Let* $(\mathcal{A}, \mathbf{A})$ *be a* p*-Banach ideal* $(0 < p \leqslant 1)$ *and* $(\mathcal{B}, \mathbf{B})$ *be a* q*-Banach ideal* $(0 < q \leqslant 1)$*. If* $(\mathcal{A}, \mathbf{A}) \overset{1}{\subseteq} (\mathcal{A}^{dd}, \mathbf{A}^{dd})$ *then*

$$\mathcal{A} \circ \mathcal{B}^{\mathrm{reg}} \overset{1}{\subseteq} (\mathcal{A} \circ \mathcal{B})^{\mathrm{reg}}.$$

P r o o f. Let E, F be Banach spaces, $\varepsilon > 0$ and $T \in \mathcal{A} \circ \mathcal{B}^{\mathrm{reg}}(E, F)$. Then there are a Banach space G and operators $R \in \mathcal{A}(G, F)$ and $S \in \mathcal{B}^{\mathrm{reg}}(E, G)$ such that $T = RS$ and $\mathbf{A}(R)\mathbf{B}^{\mathrm{reg}}(S) < (1+\varepsilon)(\mathbf{A} \circ \mathbf{B})^{\mathrm{reg}}(T)$. Hence $j_F T = R'' j_G S \in \mathcal{A} \circ \mathcal{B}(E, F'')$ and $\mathbf{A} \circ \mathbf{B}(j_F T) \leqslant \mathbf{A}(R'')\mathbf{B}^{\mathrm{reg}}(S) < (1+\varepsilon)(\mathbf{A} \circ \mathbf{B})^{\mathrm{reg}}(T)$. $\square$

Now we have prepared all tools to prove

Proposition 4.1. *Let* $(\mathcal{A}, \mathbf{A})$ *be a maximal Banach ideal. Then the following statements are equivalent:*

(a) $\mathcal{A} \circ \mathcal{A}^* \overset{1}{\subseteq} \mathcal{P}_1$,

(b) $\mathcal{A}^{\mathrm{inj}}$ *is totally accessible.*

P r o o f. Let (a) be valid. To prove (b), it is enough to show that $\mathcal{A}^{\mathrm{inj}\,*}$ is right accessible. Since $\mathcal{P}_1$ is right-accessible, theorem 3.1 implies $\mathcal{A} \circ \mathcal{A}^* \circ \mathcal{L}_\infty \overset{1}{\subseteq} \mathcal{I}$ and hence $\mathcal{A}^* \circ \mathcal{L}_\infty \overset{1}{\subseteq} \mathcal{A}^{\vdash}$. Since $\mathcal{L}_\infty \overset{1}{=} (\mathcal{L}_\infty \circ \mathcal{L}_\infty)^{\mathrm{reg}}$, it follows by lemma 4.1 that $\mathcal{A}^* \circ \mathcal{L}_\infty \overset{1}{\subseteq} (\mathcal{A}^* \circ \mathcal{L}_\infty \circ \mathcal{L}_\infty)^{\mathrm{reg}} \overset{1}{\subseteq} (\mathcal{A}^{\vdash} \circ \mathcal{L}_\infty)^{\mathrm{reg}}$. Hence

$$\mathcal{A}^{\mathrm{inj}\,*} \overset{1}{=} (\mathcal{A}^{\vdash} \circ \mathcal{L}_\infty)^{\mathrm{reg}}.$$

Now we apply theorem 3.1 to the Banach ideal $(\mathcal{A}^{\vdash} \circ \mathcal{L}_\infty)^{\mathrm{reg}}$: since $\mathcal{A}$ is assumed to be a maximal Banach ideal and $\mathcal{L}_\infty$ is right-accessible, $\mathcal{A}^{\vdash} \circ \mathcal{L}_\infty =: \mathcal{B}$ is also right-accessible. Hence $\mathcal{B}^* \circ \mathcal{B} \overset{1}{\subseteq} \mathcal{I}$ and therefore

$$(\mathcal{B}^{\mathrm{reg}})^* \circ \mathcal{B}^{\mathrm{reg}} \overset{1}{\subseteq} \mathcal{B}^* \circ \mathcal{B}^{\mathrm{reg}} \overset{1}{\subseteq} (\mathcal{B}^* \circ \mathcal{B})^{\mathrm{reg}} \overset{1}{\subseteq} \mathcal{I}.$$

In other words: $(\mathcal{B}^{\mathrm{reg}})^* \overset{1}{=} (\mathcal{B}^{\mathrm{reg}})^{\dashv}$. Since $\mathcal{B}^{\mathrm{reg}} \overset{1}{=} \mathcal{A}^{\mathrm{inj}\,*}$ is normed, theorem 3.1 implies that $\mathcal{A}^{\mathrm{inj}\,*}$ is right-accesible. Now let $\mathcal{A}^{\mathrm{inj}}$ be totally accessible, hence left-accessible. Then $\mathcal{A}^{\mathrm{inj}} \circ \mathcal{A}^{\mathrm{inj}\,*} \overset{1}{\subseteq} \mathcal{I}$ and therefore $\mathcal{A} \circ \mathcal{A}^* \circ \mathcal{L}_\infty \overset{1}{\subseteq} \mathcal{A} \circ \mathcal{A}^{\mathrm{inj}\,*} \overset{1}{\subseteq} \mathcal{I}$. Hence $\mathcal{A} \circ \mathcal{A}^* \overset{1}{\subseteq} \mathcal{L}_\infty^{\dashv} \overset{1}{=} \mathcal{L}_\infty^* \overset{1}{=} \mathcal{P}_1$. □

We don't know if there exists a maximal Banach ideal $\mathcal{A}$ such that $\mathcal{A}^{\mathrm{inj}}$ is totally accessible and $\mathcal{A} \circ \mathcal{A}^* \overset{1}{\not\subseteq} \mathcal{I}$ (hence $\mathcal{A}$ is not left-accessible).

Corollary 4.1. *Let $(\mathcal{A}, \mathbf{A})$ be a maximal Banach ideal. Then $\mathcal{A} \circ \mathcal{L}_\infty$ is left-accessible and*

$$\mathcal{A} \circ \mathcal{L}_\infty \circ \mathcal{A}^* \overset{1}{\subseteq} \mathcal{I}.$$

If $\mathcal{A} \circ \mathcal{L}_\infty$ is right-accessible, then $\mathcal{A}^{*inj}$ is totally accessible.

Proof. Since $\mathcal{A} \circ \mathcal{L}_\infty \overset{1}{\subseteq} (\mathcal{A} \circ \mathcal{L}_\infty)^{\mathrm{reg}} \overset{1}{=} \backslash\mathcal{A}$, and $\mathbf{A} \circ \mathbf{L}_\infty$ coincides with $(\mathbf{A} \circ \mathbf{L}_\infty)^{\mathrm{reg}}$ on the space of all elementary operators, $\mathcal{A} \circ \mathcal{L}_\infty$ is left-accessible and

$$\mathcal{A} \circ \mathcal{L}_\infty \circ \mathcal{A}^* \overset{1}{\subseteq} (\mathcal{A} \circ \mathcal{L}_\infty) \circ (\mathcal{A} \circ \mathcal{L}_\infty)^* \overset{1}{\subseteq} \mathcal{I}.$$

Now let $\mathcal{A} \circ \mathcal{L}_\infty$ be right-accessible. We have to show that

$$\mathcal{A}^* \circ \mathcal{A} \overset{1}{\subseteq} \mathcal{P}_1.$$

But this follows from $\mathcal{A}^* \circ \mathcal{A} \circ \mathcal{L}_\infty \overset{1}{\subseteq} (\mathcal{A} \circ \mathcal{L}_\infty)^* \circ (\mathcal{A} \circ \mathcal{L}_\infty) \overset{1}{\subseteq} \mathcal{I}$. □

Now we will recognize that prop. 4.1 leads to interesting consequences concerning the characterization of a class of injective maximal Banach ideals which are totally accessible. Since $\mathcal{P}_1$ is included, *Grothendieck's inequality* in operator form implies a non-trivial relation to $\mathcal{L}_2$ and $\mathcal{L}_1$ in the following sense (with Grothendieck constant K_G):

Lemma 4.2. $\mathcal{L}_2 \circ \mathcal{L}_1 \subseteq \mathcal{P}_1$ *and* $\mathbf{P}_1(T) \leqslant K_G \cdot (\mathbf{L}_2 \circ \mathbf{L}_1)(T)$ *for all* $T \in \mathcal{L}_2 \circ \mathcal{L}_1$.

P r o o f. Let E, F be Banach spaces, $\varepsilon > 0$ and $T \in \mathcal{L}_2 \circ \mathcal{L}_1(E, F)$. Then there exists a Banach space G and operators $R \in \mathcal{L}_2(G, F), S \in \mathcal{L}_1(E, G)$ such that $T = RS$ and $\mathbf{L}_2(R)\mathbf{L}_1(S) < (\mathbf{L}_2 \circ \mathbf{L}_1)(T)$. Since $\mathcal{L}_1 \overset{1}{=} \mathcal{L}/$ and $S \in \mathcal{L}_1(E, G)$, there exists a measure μ, operators $W \in \mathcal{L}(L_1(\mu), G'')$ and $Z \in \mathcal{L}(E, L_1(\mu))$ such that $j_G S = WZ$ and $\|W\|\|Z\| < (1+\varepsilon)\mathbf{L}_1(S)$. Since $R'' \in \mathcal{L}_2(G'', F'')$, there exists a Hilbert space H, operators $U \in \mathcal{L}(H, F'')$ and $V \in \mathcal{L}(G'', H)$ such that $R'' = UV$ and $\|U\|\|V\| < (1+\varepsilon)\mathbf{L}_2(R)$. Hence $j_F T = R'' j_G S = U(VW)Z$ and $VW \in \mathcal{L}(L_1(\mu), H)$. Since $L_1(\mu)$ is a $\mathcal{L}^g_{1,1}$—space, Grothendieck's inequality implies that $VW \in \mathcal{P}_1(L_1(\mu), H)$ and $\mathbf{P}_1(VW) \leqslant K_G \cdot \|VW\|$ (cf. [3], 23.10). Hence $j_F T \in \mathcal{P}_1(E, F'')$ and $\mathbf{P}_1(j_F T) \leqslant \|U\|\mathbf{P}_1(VW)\|Z\| \leqslant (1+\varepsilon)^2\ K_G\ \mathbf{L}_2(R)\mathbf{L}_1(S) < (1+\varepsilon)^3\ K_G\ (\mathbf{L}_2 \circ \mathbf{L}_1)(T)$. Since $\mathcal{P}_1$ is regular, the claim follows. $\square$

Now let $(\mathcal{A},\ \mathbf{A})$ be a maximal Banach ideal such that $\mathcal{D}_2 \subseteq \mathcal{A} \subseteq \mathcal{L}_1$ (since $\mathcal{P}_1 \overset{1}{\subseteq} \mathcal{L}_2 \overset{1}{=} \mathcal{L}_2^d$, it follows that $\mathcal{D}_2 \overset{1}{=} \mathcal{L}_2^* \overset{1}{\subseteq} \mathcal{P}_1^{d*} \overset{1}{=} \mathcal{L}_1$. Hence the class of such ideals $\mathcal{A}$ is not empty; consider e.g. $\mathcal{P}_2^d$). Then $\mathcal{A}^* \subseteq \mathcal{L}_2$ and therefore $\mathcal{A}^* \circ \mathcal{A} \subseteq \mathcal{L}_2 \circ \mathcal{L}_1 \subseteq \mathcal{P}_1$ by lemma 4.2. If $\mathcal{A}^* \circ \mathcal{A} \overset{1}{\subseteq} \mathcal{P}_1$, prop. 4.1 would imply that $\mathcal{A}^{*inj}$ is totally accessible. In general we don't know if this is the case. However there exists a beautiful "trick" to arrange $\mathbf{P}_1(T) \leqslant 1 \cdot (\mathbf{A}^* \circ \mathbf{A})(T)$ for all $T \in \mathcal{A}^* \circ \mathcal{A}$, which is given by *tensor stability*. Let γ be a fixed tensor norm. Remember that a given quasi Banach ideal $(\mathcal{A},\ \mathbf{A})$ is called *γ-tensorstable* (cf. [1], [3]), if

$$S \tilde{\otimes}_\gamma T \in \mathcal{A}(E \tilde{\otimes}_\gamma F, G \tilde{\otimes}_\gamma H) \text{ for all } S \in \mathcal{A}(E, G), T \in \mathcal{A}(F, H).$$

In this case there is a constant $c \geqslant 1$ satisfying

$$\mathbf{A}(S \tilde{\otimes}_\gamma T) \leqslant c\mathbf{A}(S)\mathbf{A}(T).$$

If $c = 1$, $\mathcal{A}$ is called *metrically γ-tensorstable.* If $c = 1$ and the above inequality is an equality, then $\mathcal{A}$ is called *strongly γ-tensorstable.* With the help of tensor stability, we will show how it is possible to improve the inequality $\mathcal{A}^* \circ \mathcal{A} \subseteq \mathcal{P}_1$ to obtain $\mathcal{A}^* \circ \mathcal{A} \overset{1}{\subseteq} \mathcal{P}_1$ and turn our attention to

Theorem 4.1. *Let $(\mathcal{A}, \mathbf{A})$ be a maximal Banach ideal. If*
(i) $\mathcal{D}_2 \subseteq \mathcal{A} \subseteq \mathcal{L}_1$ *and*
(ii) *both $(\mathcal{A}^*, \mathbf{A}^*)$ and $(\mathcal{A}, \mathbf{A})$ are metrically ε-tensorstable*
then $(\mathcal{A}^{*inj}, \mathbf{A}^{*inj})$ is totally accessible.

P r o o f. Since $\mathcal{D}_2 \subseteq \mathcal{A} \subseteq \mathcal{L}_1$, the adjoint $\mathcal{A}^*$ is contained in $\mathcal{L}_2$. Hence there exist constants $c \geqslant 0$ and $c^* \geqslant 0$ such that $\mathbf{L}_2(R) \leqslant c^*\mathbf{A}^*(R)$ for all $R \in \mathcal{A}^*$ and $\mathbf{L}_1(S) \leqslant c\mathbf{A}(S)$ for all $S \in \mathcal{A}$. Let E, F be Banach spaces, $\varepsilon > 0$ and $T \in$

$\mathcal{A}^* \circ \mathcal{A}(E, F)$. We must show that $T \in \mathcal{P}_1(E, F)$ and $\mathbf{P}_1(T) \leqslant 1 \cdot (\mathbf{A}^* \circ \mathbf{A})(T)$. By the previous considerations, there exists a Banach space D, operators $R \in \mathcal{A}^*(D, F)$ and $S \in \mathcal{A}(E, D)$ such that $T = RS \in \mathcal{L}_2 \circ \mathcal{L}_1$ and $\mathbf{L}_2(R)\ \mathbf{L}_1(S) < cc^*(1+\varepsilon)(\mathbf{A}^* \circ \mathbf{A})(T)$. Lemma 4.2 now implies that $T = RS \in \mathcal{P}_1(E, F)$ and

$$\mathbf{P}_1(T) \leqslant K_G \mathbf{L}_2(R) \mathbf{L}_1(S) < (1+\varepsilon) K_G cc^* (\mathbf{A}^* \circ \mathbf{A})(T).$$

At this point the improvement of this norm estimation will be realized by the assumed metric ε-tensor stability of $\mathcal{A}^*$ and $\mathcal{A}$ which implies in particular that $\mathcal{A}^* \circ \mathcal{A}$ is metrically ε-tensorstable (cf. [3], 34.4). Since $\mathcal{P}_1$ even is *strongly* ε-tensorstable (see [3], 34.5), it follows that

$$\begin{aligned} \mathbf{P}_1(T)^2 &= \mathbf{P}_1(T \tilde{\otimes}_\varepsilon T) \\ &\leqslant (1+\varepsilon) K_G cc^* (\mathbf{A}^* \circ \mathbf{A})(T \tilde{\otimes}_\varepsilon T) \\ &\leqslant (1+\varepsilon) K_G cc^* (\mathbf{A}^* \circ \mathbf{A})(T)^2. \end{aligned}$$

Hence: $\mathbf{P}_1(T) \leqslant ((1+\varepsilon)K_G cc^*)^{1/2}\ (\mathbf{A}^* \circ \mathbf{A})(T)$, and an obvious induction argument implies:

$$\forall n \in \mathbf{N}:\ \mathbf{P}_1(T) \leqslant ((1+\varepsilon)K_G cc^*)^{1/2^n}\ (\mathbf{A}^* \circ \mathbf{A})(T).$$

$n \to \infty$ now yields the desired improved norm estimate, and the proof is finished. $\square$

Next we will show that the statement of theorem 4.1 remains valid for arbitrary finitely generated tensor norms (not only for the injective tensor norm ε) if we assume a (slight) restriction of the tensor stability condition—with a completely different proof than the previous one. So, let $(\mathcal{A}, \mathbf{A})$ be a maximal Banach ideal with $\mathcal{D}_2 \subseteq \mathcal{A} \subseteq \mathcal{L}_1$. Then $\backslash\mathcal{A} \subseteq \backslash\mathcal{L}_1 \overset{1}{=} (\mathcal{P}_1^{\mathrm{sur}})^*$. By Grothendieck's inequality, $\mathcal{L}_2 \subseteq \mathcal{P}_1^{\mathrm{sur}}$ and $\mathbf{P}_1^{\mathrm{sur}}(T) \leqslant K_G \mathbf{L}_2(T)$ for all $T \in \mathcal{L}_2$ (cf. [3], 20.17). Hence $\backslash\mathcal{A} \subseteq \mathcal{D}_2$ and there is a constant $c \geqslant 0$ such that $\mathbf{D}_2(S) \leqslant cK_G \backslash\mathbf{A}(S)$ for all $S \in \backslash\mathcal{A}$. Since $\mathcal{L}_2$ is injective, it follows that $(\backslash\mathcal{A})^* \subseteq \mathcal{L}_2$, and there exists a constant c^* such that $\mathbf{L}_2(R) \leqslant c^*(\backslash\mathbf{A})^*(R)$ for all $R \in (\backslash\mathcal{A})^*$. Since $\mathcal{D}_2$ is right-accessible, hence $\mathcal{L}_2 \circ \mathcal{D}_2 \overset{1}{\subseteq} \mathcal{I}$, we have obtained the following statement which is of its own interest:

Lemma 4.3. *Let $(\mathcal{A}, \mathbf{A})$ be a maximal Banach ideal such that $\mathcal{D}_2 \subseteq \mathcal{A} \subseteq \mathcal{L}_1$. Then*

$$(\backslash\mathcal{A})^* \circ \backslash\mathcal{A} \subseteq \mathcal{I}$$

and there exist constants c, c^* such that

$$\mathbf{I}(T) \leqslant cc^* K_G ((\backslash\mathbf{A})^* \circ \backslash\mathbf{A})(T)$$

for all $T \in (\backslash\mathcal{A})^* \circ \backslash\mathcal{A}$.

Now let in addition γ be an arbitrary finitely generated tensor norm and assume that $(\backslash\mathcal{A})^*$ as well as $\backslash\mathcal{A}$ are metrically γ-tensorstable. Let $(M,F) \in \mathrm{FIN} \times \mathrm{BAN}$ and $U \in (\backslash\mathcal{A})^{*\vdash}(M,F)$. Then U and $U\tilde{\otimes}_\gamma U$ are finite operators and lemma 4.3. implies that

$$(\backslash\mathbf{A})^{*\vdash}(U) \leqslant cc^* K_G \backslash\mathbf{A}(U).$$

This estimation now can be improved as follows: Let $\varepsilon > 0$. Then there is a Banach space D, an operator $V \in (\backslash\mathcal{A})^*(F,D)$ with $(\backslash\mathbf{A})^*(V) = 1$ such that $(\backslash\mathbf{A})^{*\vdash}(U) < (1+\varepsilon)\mathbf{I}(VU)$. Since $\mathcal{L} \stackrel{1}{=} \mathcal{I}^*$ as well as $\mathcal{I}$ are metrically γ- tensorstable (see [3], 34.5), $\mathcal{I}$ even is strongly γ- tensorstable (see [3], 34.2). Hence

$$\begin{aligned}
(\backslash\mathbf{A})^{*\vdash}(U)^2 &< (1+\varepsilon)^2\mathbf{I}(VU)^2 \\
&\leqslant (1+\varepsilon)^2\mathbf{I}((V\tilde{\otimes}_\gamma V) \circ (U\tilde{\otimes}_\gamma U)) \\
&\leqslant (1+\varepsilon)^2(\backslash\mathbf{A})^*(V\tilde{\otimes}_\gamma V)\ (\backslash\mathbf{A})^{*\vdash}(U\tilde{\otimes}_\gamma U) \\
&\leqslant (1+\varepsilon)^2(\backslash\mathbf{A})^{*\vdash}(U\tilde{\otimes}_\gamma U).
\end{aligned}$$

(The last inequality follows by the metric γ-tensor stability of $(\backslash\mathcal{A})^*$.) Since $\backslash\mathcal{A}$ also is metrically γ-tensorstable, we obtain

$$(\backslash\mathbf{A})^{*\vdash}(U)^2 \leqslant (\backslash\mathbf{A})^{*\vdash}(U\tilde{\otimes}_\gamma U) \leqslant cc^* K_G\ \backslash\mathbf{A}(U\tilde{\otimes}_\gamma U) \leqslant cc^* K_G\ \backslash\mathbf{A}(U)^2.$$

Hence, induction implies $(\backslash\mathbf{A})^{*\vdash}(U) \leqslant \backslash\mathbf{A}(U)$, and since $(\backslash\mathcal{A})^{*\vdash}$ is right-accessible, we have proved:

Theorem 4.2. *Let $(\mathcal{A},\mathbf{A})$ be a maximal Banach ideal and γ be a finitely generated tensor norm. If*

(i) *$\mathcal{D}_2 \subseteq \mathcal{A} \subseteq \mathcal{L}_1$ and*

(ii) *both $((\backslash\mathcal{A})^*, (\backslash\mathbf{A})^*)$ and $(\backslash\mathcal{A}, \backslash\mathbf{A})$ are metrically γ-tensorstable*

then $(\mathcal{A}^{*inj}, \mathbf{A}^{*inj}) \stackrel{1}{=} ((\backslash\mathcal{A})^*, (\backslash\mathbf{A})^*)$ is totally accessible.

To finish this paper we will give now interesting examples which also show that *normed products* of maximal Banach ideals will be of crucial significance concerning the investigation of accessibility. In particular we give a partial answer to a question of A. Defant and K. Floret whether the ideal $(\mathcal{L}_\infty, \mathbf{L}_\infty)$ is totally accessible or not (see [3], 21.12).

Let $(\mathcal{A},\ \mathbf{A})$ be a maximal Banach ideal such that $(\mathcal{A}, \mathbf{A}) \subseteq (\mathcal{L}_2, \mathbf{L}_2)$. Then $\mathcal{D}_2 \subseteq \mathcal{A}^*$, and related to the foregoing results, it is a natural question to ask for further properties of $\mathcal{A}$, which even imply that $\mathcal{D}_2 \subseteq \mathcal{A}^* \subseteq \mathcal{L}_1$. In order to arrange this inclusion, we consider now the product ideal $\mathcal{L}_2 \circ \mathcal{A}$:

Lemma 5.1. *Let $(\mathcal{A}, \mathbf{A})$ be a p-Banach ideal $(0 < p \leqslant 1)$. Then $(\mathcal{L}_2 \circ \mathcal{A}, \mathbf{L}_2 \circ \mathbf{A})$ is always an injective $\frac{p}{1+p}$-Banach ideal. In particular it is right-accessible and regular.*

Proof. Let $T \in (\mathcal{L}_2 \circ \mathcal{A})^{\mathrm{inj}}(E, F)$ and $\varepsilon > 0$. Then there are a Banach space G, operators $R \in \mathcal{L}_2(G, F^\infty)$ and $S \in \mathcal{A}(E, G)$ such that $J_F T = RS$ and $\mathbf{L}_2(R)\mathbf{A}(S) < (1+\varepsilon)\mathbf{L}_2 \circ \mathbf{A}(J_F T)$. Let H be a Hilbert space, $V \in \mathcal{L}(H, F^\infty)$ and $W \in \mathcal{L}(G, H)$ such that $R = VW$ and $\|V\|\|W\| < (1+\varepsilon)\mathbf{L}_2(R)$. Let C be the (closed) range of $J_F \colon F \overset{1}{\hookrightarrow} F^\infty$. Then $H_0 := V^{-1}(C)$ is a closed subspace of H, and consequently there exists a projection $P \in \mathcal{L}(H, H)$ from H onto H_0 such that the closure of the range of VP is contained in C. Since the range of WS is contained in H_0, it follows that $WS = PWS$. Now let $\gamma \colon C \longrightarrow F$ be defined canonically and let γ_0 be the restriction of γ to C_0, where C_0 is the closure of $VP(H)$. Let $B := \gamma_0 Z$, with $Z \colon H \longrightarrow C_0$, $z \mapsto VPz$, and let $D := WS$. Then, $B \in \mathcal{L}_2(H, F)$ and $\mathbf{L}_2(B) \leqslant \|V\|$, $D \in \mathcal{A}(E, H)$ and $\mathbf{A}(D) \leqslant \|W\|\mathbf{A}(S)$. In accordance with the construction,

$$BDx = \gamma(VPWSx) = \gamma(VWSx) = \gamma(RSx) = Tx \quad \text{for all } x \in E.$$

Hence $T = BD \in \mathcal{L}_2 \circ \mathcal{A}(E, F)$ and $\mathbf{L}_2(B)\mathbf{A}(D) < (1+\varepsilon)^2(\mathbf{L}_2 \circ \mathbf{A})^{\mathrm{inj}}(T)$. Injective p-Banach ideals are always right-accessible, and since F^∞ has the metric extension property, they are also regular. □

Which ideals $\mathcal{A}$ imply now the non-normability of $\mathcal{L}_2 \circ \mathcal{A}$? One answer is given by

Proposition 5.1. *Let $(\mathcal{A}, \mathbf{A})$ be a maximal Banach ideal such that $(\mathcal{L}_2 \circ \mathcal{A}, \mathbf{L}_2 \circ \mathbf{A})$ is normed. Then $(\mathcal{A}^*, \mathbf{A}^*) \subseteq (\mathcal{L}_\infty, \mathbf{L}_\infty)$.*

Proof. Since $\mathcal{L}_2 \circ \mathcal{A}$ is normed and injective (in particular regular), and as a product of two ultrastable Banach ideals again ultrastable (cf. [2], 3.4.5), it follows that

$$\mathcal{N} \overset{1}{\subseteq} \mathcal{L}_2 \circ \mathcal{A} \overset{1}{=} (\mathcal{L}_2 \circ \mathcal{A})^{\mathrm{reg}} \overset{1}{=} (\mathcal{L}_2 \circ \mathcal{A})^{\max}.$$

The last isometric identity is implied by ([13], 8.8.6). Hence $\mathcal{I} \stackrel{1}{=} \mathcal{N}^{\max} \stackrel{1}{\subseteq} \mathcal{L}_2 \circ \mathcal{A}$, and the injectivity further implies that $\mathcal{P}_1 \stackrel{1}{\subseteq} \mathcal{L}_2 \circ \mathcal{A}$ and therefore

$$\mathcal{A}^* \stackrel{1}{\subseteq} (\mathcal{L}_2 \circ \mathcal{A})^* \stackrel{1}{\subseteq} \mathcal{L}_\infty.$$

□

Combining the preceeding considerations with theorem 4.1 and 4.2 leads to the somehow surprising

Corollary 5.1. *Let $(\mathcal{A}, \mathbf{A})$ be a maximal Banach ideal such that $(\mathcal{A}, \mathbf{A}) \subseteq (\mathcal{L}_2, \mathbf{L}_2)$. Let $(\mathcal{A}^*, \mathbf{A}^*)$ as well as $(\mathcal{A}, \mathbf{A})$ be metrically ε-tensorstable, or let $(\backslash\mathcal{A}^*, \backslash\mathbf{A}^*)$ and $(\mathcal{A}^{\mathrm{inj}}, \mathbf{A}^{\mathrm{inj}})$ be metrically γ-tensorstable with respect to a given finitely generated tensor norm γ. If $(\mathcal{L}_2 \circ \mathcal{A}^d, \mathbf{L}_2 \circ \mathbf{A}^d)$ is normed, then $(\mathcal{A}^{\mathrm{inj}}, \mathbf{A}^{\mathrm{inj}})$ is totally accessible.*

P r o o f. Let $\mathcal{B} := \mathcal{A}^d$. Then

$$\mathcal{D}_2 \subseteq \mathcal{A}^* \stackrel{1}{=} \mathcal{B}^{*d} \stackrel{1}{\subseteq} \mathcal{L}_\infty^d \stackrel{1}{=} \mathcal{L}_1.$$

Now, theorems 4.1 and 4.2—applied to $\mathcal{A}^*$—yield the claim. □

Corollary 5.2. *Let $(\mathcal{A}, \mathbf{A})$ be a maximal injective Banach ideal such that $(\mathcal{A}, \mathbf{A}) \subseteq (\mathcal{L}_2, \mathbf{L}_2)$. Let $(\mathcal{A}^*, \mathbf{A}^*)$ as well as $(\mathcal{A}, \mathbf{A})$ be metrically γ-tensorstable with respect to a given finitely generated tensor norm γ. If $(\mathcal{L}_2 \circ \mathcal{A}^d, \mathbf{L}_2 \circ \mathbf{A}^d)$ is normed, then $(\mathcal{A}, \mathbf{A})$ is totally accessible.*

Proposition 5.2. *Let $(\mathcal{A}, \mathbf{A})$ be a maximal Banach ideal such that $(\mathcal{A}^*, \mathbf{A}^*)$ is γ-tensorstable for some injective tensor norm γ. If $(\mathcal{L}_2 \circ \mathcal{A}^d, \mathbf{L}_2 \circ \mathbf{A}^d)$ is normed, than there is no infinite dimensional Banach space E such that $\mathrm{id}_E \in \mathcal{A}^*$.*

P r o o f. By ([3], 23.3), the ideals $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_\infty$ are mutually uncomparable. Since γ is an injective tensor norm, an infinite dimensional Banach space in space$(\mathcal{A}^*)$ would lead to $\mathcal{L}_\infty \subseteq \mathcal{A}^*$ (cf. [3], 34.7). On the other hand, (since $\mathcal{L}_2 \circ \mathcal{A}^d$ is normed) the previous calculations show that $\mathcal{A}^* \stackrel{1}{\subseteq} \mathcal{L}_1$. Hence $\mathcal{L}_\infty \subseteq \mathcal{A}^* \stackrel{1}{\subseteq} \mathcal{L}_1$, which is a contradiction. □

Remark. Let $\mathcal{A} \subseteq \mathcal{D}_2$. Then $\mathcal{A} \subseteq \mathcal{L}_2$, and $\mathcal{L}_2 \circ \mathcal{A}^d$ is a *trace ideal* (cf. [2], 4.4). In particular $\mathcal{L}_2 \circ \mathcal{A}^d$ is not normed.

So far we have seen that there is an intimate relation between normed products of operator ideals and accessibility-conditions. We will finish this paper with another example which shows, again, that normed products of operator ideals have an impact on accessibility.

Proposition 5.3. *Let $(\mathcal{A}, \mathbf{A})$ be an injective, maximal Banach ideal, which is totally accessible. If $((\mathcal{A}^* \circ \mathcal{A})^{\mathrm{reg}}, (\mathbf{A}^* \circ \mathbf{A})^{\mathrm{reg}})$ is normed, then $(\mathcal{A}^*, \mathbf{A}^*)$ is not totally accessible.*

Proof. By assumption, both $\mathcal{A}$ and $\mathcal{A}^*$ are maximal, hence ultrastable, so is their product (see [2], 3.4.5), and it follows that

$$(\mathcal{A}^* \circ \mathcal{A})^{\mathrm{reg}} \stackrel{1}{=} (\mathcal{A}^* \circ \mathcal{A})^{\max}.$$

Since $(\mathcal{A}^* \circ \mathcal{A})^{\mathrm{reg}}$ is assumed to be normed, and $\mathcal{A}$ is right-accessible, we obtain

$$\mathcal{I} \stackrel{1}{=} \mathcal{N}^{\max} \stackrel{1}{\subseteq} (\mathcal{A}^* \circ \mathcal{A})^{\mathrm{reg}} \stackrel{1}{\subseteq} \mathcal{I}.$$

Since $\mathcal{A}$ is injective and $\mathcal{A}^*$ regular, an easy calculation shows that $\mathcal{A}^* \circ \mathcal{A}$ is regular as well, and therefore it follows that

$$\mathcal{A}^* \circ \mathcal{A} \stackrel{1}{=} \mathcal{I}.$$

Now, assume that $\mathcal{A}^*$ is totally accessible. Then, by the injectivity of the totally accessible ideal $\mathcal{A}$, $\mathcal{A}^* \circ \mathcal{A}$ is also *totally* accessible (see [3], 21.4), which is a contradiction, since $\mathcal{I}$ is not totally accessible. □

Corollary 5.3. *If $(\mathcal{L}_\infty \circ \mathcal{P}_1)^{\mathrm{reg}}$ is normed, then $\mathcal{L}_\infty$ is not totally accessible.*

Note, that $(\mathcal{P}_1 \circ \mathcal{L}_\infty)^{\mathrm{reg}} \stackrel{1}{=} \mathcal{I}$ is normed, as it was shown in section 4.

6. Questions and open problems

- Is the conjugate of a maximal Banach ideal always left-accessible? (Conjecture: no)
- Let $(\mathcal{A}, \mathbf{A})$ be a maximal Banach ideal. Does this even imply the validity of the $\mathcal{A}$-local principle of reflexivity? (Conjecture: no)
- What relations exist between tensorstable operator ideals, normed products of operator ideals and accessibility? How far *trace ideals* are involved?

- Assume that $\mathcal{A}$ is a maximal Banach ideal such that $\mathcal{A}^* \overset{1}{\subseteq} \mathcal{L}_\infty$. Does this condition even imply that $\mathcal{L}_2 \circ \mathcal{A}$ is a *normed* ideal? (The converse implication is true (see 5.1)).
- Is it possible to maintain the statement of corollary 5.2 *without* the property of $\mathcal{L}_2 \circ \mathcal{A}^d$ being normed?
- In general it seems to be more easy (via trace ideals) to prove that the product of two Banach ideals is *not* normed. Find criteria which show the normability.

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