Czechoslovak Mathematical Journal

Dariusz Idczak

Distributional derivatives of functions of two variables of finite variation and their application to an impulsive hyperbolic equation

Czechoslovak Mathematical Journal, Vol. 48 (1998), No. 1, 145–171

Persistent URL: http://dml.cz/dmlcz/127406

Czechoslovak Mathematical Journal, 48 (123) (1998), 145–171

DISTRIBUTIONAL DERIVATIVES OF FUNCTIONS OF TWO VARIABLES OF FINITE VARIATION AND THEIR APPLICATION TO AN IMPULSIVE HYPERBOLIC EQUATION

Dariusz Idczak, Łódź

(Received June 23, 1995)

Abstract. We give characterizations of the distributional derivatives $D^{1,1}$, $D^{1,0}$, $D^{0,1}$ of functions of two variables of locally finite variation. Then we use these results to prove the existence theorem for the hyperbolic equation with a nonhomogeneous term containing the distributional derivative determined by an additive function of an interval of finite variation. An application of the above theorem to a hyperbolic equation with an impulse effect is also given.

MSC 2000: 26A99, 26A21

Introduction

The necessity of considering, in certain physical and technical problems, differential equations whose solutions may be discontinuous functions brought the development of the theory of impulse differential equations.

The study of such equations was initiated by J. Kurzweil in [7, 8, 9].

In literature, different approaches to the investigation of such equations are known. Paper [10] shows an approach consisting in the preassignment of values of impulses of a solution by help of a family of operators acting in a state space. In the second approach equations with distributional derivatives of functions with locally finite variation as coefficients are considered (in particular, a linear combination of Dirac's deltas concentrated at different moments). The linear equation

$$\dot{x} = Ax + b$$

in the case when A is a function and b—a distribution is studied, for example, in [3, 11, 20]. The case when both A and b are distributions is investigated, among other things, in [12, 14, 20].

Basic results, applications to the control theory and more extensive literature concerning the ordinary impulsive equations can be found in [19].

The aim of the present paper is to generalize some theorems concerning the distributional derivative of a function of one variable with locally finite variation as well as the existence of a solution to an ordinary differential equation containing such a derivative in the nonhomogeneous term to the case of functions of two variables and a partial differential equation of hyperbolic type.

The definition of a function of two variables with finite variation (cf. [6]), equivalent to that given by Hardy-Krause (cf. [2]), which we adopt in the paper, is analogous to the definition of an absolutely continuous function of two variables from [18]. On account of the fact that these definitions are based on the notion of a function of an interval, characterizations of the distributional derivatives of functions with finite variation are also based on the notion of a function of an interval.

In Chapter I we give certain facts from the theory of real functions. Some of them, which can be found in monographs [13, 17], are given without proofs. Those which were not accessible to the author in literature are presented together with their proofs.

In Chapter II we introduce the notion of functions with locally finite variation and, next, give a characterization of the distributional derivatives $D^{1,1}$, $D^{1,0}$, $D^{0,1}$ of such functions. Theorems 2.3, 2.5 are analogues of Lemmas IV.1.1, IV.1.2 proved in the monograph [3]. Lemma 2.2 is an analogue of the theorem on integration by parts (in the sense of Lebesgue-Stieltjes) proved in the monograph [11].

In Chapter III we prove the existence and uniqueness of a solution to a partial differential equation of hyperbolic type containing a function of an interval with finite variation (or, equivalently, a distributional derivative of a function of locally finite variation) in the class of functions with locally finite variation. Theorem 3.3 is an analogue of Theorem IV.1.2 proved in the monograph [3].

The considerations included in the present paper and, in particular, Corollary 2.4 constitute the starting point for the investigation of impulse hyperbolic equations of the form

$$\frac{\partial^2 \varphi}{\partial x \partial y} = Az + B\frac{\partial z}{\partial x} + C\frac{\partial z}{\partial y} + D^{1,1}\Lambda_h$$

where h is a linear combination $\sum_s \alpha_s h_s$ of two-dimensional Heaviside functions, that is, functions of the form

$$h_s\colon\ (x,y) \mapsto \begin{cases} 1\colon\ x > x_s \wedge y > y_s, \\ 0\colon\ \text{otherwise} \end{cases}$$

where (x_s, y_s) for an arbitrary index s is a fixed point.

I. Some facts from the theory of real functions (cf. [13, 17])

Let $\Omega = \,]a,b[\, \times \,]c,d[$ be an open interval (may be unbounded) contained in $\mathbb{R}^2$ and F an additive real function of an interval, defined on the collection of all closed bounded intervals contained in Ω, having a finite variation on each of them.

The symbols μ_{F^+}, μ_{F^-} will denote the measures determined by the upper variation F^+ of F and the lower variation F^- of F, respectively.

The symbols $\mathscr{M}_{F^+}$, $\mathscr{M}_{F^-}$ will denote the σ-additive algebras the measures μ_{F^+}, μ_{F^-}, respectively, are defined on.

It is known that if R is a closed bounded interval contained in Ω and $(P_n)_{n\in\mathbb{N}}$, $(R_n)_{n\in\mathbb{N}}$ are sequences of closed bounded intervals contained in Ω, such that

$$\begin{aligned} P_n &\subset \operatorname{Int} R, & R &\subset \operatorname{Int} R_n, \\ \operatorname{Int} P_n &\to \operatorname{Int} R, & R_n &\to R, \end{aligned}$$

then

$$\begin{aligned} \mu_{F^+}(\operatorname{Int} R) &= \lim_{n\to\infty} F^+(P_n), \\ \mu_{F^+}(R) &= \lim_{n\to\infty} F^+(R_n). \end{aligned}$$

Of course, equalities of this type can be written also for the measure determined by the lower variation F^- of F.

We will say that a function g is integrable on the set $A \subset \mathscr{M}_{F^+} \cap \mathscr{M}_{F^-}$ with respect to F if g is integrable on A with respect to μ_{F^+} and μ_{F^-}.

In this case, we shall adopt

$$\int_A g \,\mathrm{d}F := \int_A g \,\mathrm{d}\mu_{F^+} - \int_A g \,\mathrm{d}\mu_{F^-}.$$

An additive function F of an interval will be called $\nearrow$-continuous on Ω if, for any closed bounded intervals $P_0 = [x_1^0, x_2^0] \times [y_1^0, y_2^0] \subset \Omega$, $P_n = [x_1^n, x_2^n] \times [y_1^n, y_2^n] \subset \Omega$,

$n \in \mathbb{N}$, such that $(P_n)_{n\in\mathbb{N}}$ is $\nearrow$-convergent to P_0 ($P_n \nearrow P_0$), i.e.

$$\begin{aligned}
&x_1^n < x_1^0, \quad n \in \mathbb{N}, &&\lim_{n\to\infty} x_1^n = x_1^0,\\
&x_2^n < x_2^0, \quad n \in \mathbb{N}, &&\lim_{n\to\infty} x_2^n = x_2^0,\\
&y_1^n < y_1^0, \quad n \in \mathbb{N}, &&\lim_{n\to\infty} y_1^n = y_1^0,\\
&y_2^n < y_2^0, \quad n \in \mathbb{N}, &&\lim_{n\to\infty} y_2^n = y_2^0,
\end{aligned}$$

the equality

$$\lim_{n\to\infty} F(P_n) = F(P_0)$$

holds.

Remark 1.1. It is easy to see that the $\nearrow$-continuity of F implies the following continuity: if $P_0 = [x_1^0, x_2^0] \times [y_1^0, y_2^0] \subset \Omega$, $P_n = [x_1^n, x_2^n] \times [y_1^n, y_2^n] \subset \Omega$, $n \in \mathbb{N}$, are closed bounded intervals such that

$$\begin{aligned}
&x_1^n \leqslant x_1^0, \quad n \in \mathbb{N}, &&\lim_{n\to\infty} x_1^n = x_1^0,\\
&x_2^n \leqslant x_2^0, \quad n \in \mathbb{N}, &&\lim_{n\to\infty} x_2^n = x_2^0,\\
&y_1^n \leqslant y_1^0, \quad n \in \mathbb{N}, &&\lim_{n\to\infty} y_1^n = y_1^0,\\
&y_2^n \leqslant y_2^0, \quad n \in \mathbb{N}, &&\lim_{n\to\infty} y_2^n = y_2^0,
\end{aligned}$$

then

$$\lim_{n\to\infty} F(P_n) = F(P_0).$$

From the above definitions and the remark, one can directly obtain

Lemma 1.2. *If an additive function F of an interval has a finite variation on a closed bounded interval P, then the $\nearrow$-continuity of F in* $\operatorname{Int} P$ *implies the $\nearrow$-continuity of F^+ and F^- in* $\operatorname{Int} P$.

Now, we recall that the family $\mathscr{K}$ of closed bounded subintervals of Ω is called dense in Ω if any closed interval contained in Ω is the limit of a descending sequence of intervals from $\mathscr{K}$.

We have

Lemma 1.3. *If the family $\mathscr{K}$ is dense in Ω, then any closed bounded interval contained in Ω is the limit of an $\nearrow$-convergent sequence of intervals from $\mathscr{K}$.*

P r o o f. The assertion follows directly from the fact that the density of the family $\mathscr{K}$ in Ω is equivalent to that of the set

$$\{(a_1, a_2, b_1, b_2) \in \mathbb{R}^4\colon\ [a_1, a_2] \times [b_1, b_2] \in \mathscr{K}\}$$

in the set

$$\{(a_1, a_2, b_1, b_2) \in \mathbb{R}^4\colon\ [a_1, a_2] \times [b_1, b_2] \in \Omega\}.$$

□

We will use the above lemma in the proof of

Lemma 1.4. *Let F be an additive function of an interval of finite variation on each closed bounded interval contained in Ω, $\nearrow$-continuous in Ω. Then if*

$$\int_\Omega \varphi \,\mathrm{d}F = 0$$

for any $\varphi \in \mathscr{D}(\Omega)$ ($\mathscr{D}(\Omega)$—the set of test functions), then

$$F(S) = 0$$

for any closed bounded interval $S \subset \Omega$.

P r o o f. Let f be any fixed function from the class C of all continuous (on Ω) functions with compact supports contained in Ω. So, there exist a closed bounded interval $P \subset \Omega$ and a sequence $(\varphi_n)_{n\in\mathbb{N}}$, $\varphi_n \in \mathscr{D}(\Omega)$, $n \in \mathbb{N}$, such that (φ_n) converges uniformly to f on Ω and $\operatorname{supp} f \subset \operatorname{Int} P$, $\operatorname{supp} \varphi_n \subset \operatorname{Int} P$, $n \in \mathbb{N}$. From this, on the basis of [17, XI.3.9], we have

$$\int_\Omega f \,\mathrm{d}F = \int_P f \,\mathrm{d}F = \int_P \lim_{n\to\infty} \varphi_n \,\mathrm{d}F = \lim_{n\to\infty} \int_P \varphi_n \,\mathrm{d}F = \lim_{n\to\infty} \int_\Omega \varphi_n \,\mathrm{d}F = 0.$$

Consequently,

$$\int_\Omega f \,\mathrm{d}\mu_{F^+} = \int_\Omega f \,\mathrm{d}\mu_{F^-}$$

for $f \in C$. This means, in view of the Riesz theorem (cf. [13, VII.5.4]), that

$$\mu_{F^+} = \mu_{F^-}$$

in the class of relatively compact Borel sets contained in Ω. So, in particular,

$$\begin{aligned} \mu_{F^+}(Q) &= \mu_{F^-}(Q), \\ \mu_{F^+}(\operatorname{Int} Q) &= \mu_{F^-}(\operatorname{Int} Q) \end{aligned} \tag{1}$$

for any closed bounded interval $Q \subset \Omega$.

Now, let R be an interval of continuity of F (and, consequently, of F^+ and F^-), that is,

$$\mu_{F^+}(\operatorname{Int} R) = F^+(R) = \mu_{F^+}(R),$$
$$\mu_{F^-}(\operatorname{Int} R) = F^-(R) = \mu_{F^-}(R).$$

This means, in view of (1), that

$$F^+(R) = F^-(R).$$

Since the family of intervals of continuity of F is dense in Ω, we get from Lemma 1.3 that any closed bounded interval $S \subset \Omega$ is the limit of an $\nearrow$-convergent sequence $(R_n)_{n\in\mathbb{N}}$ of intervals of continuity of F. From this, on the ground of Lemma 1.2, we have

$$\begin{aligned} F^+(S) &= F^+(\lim_{n\to\infty} R_n) = \lim_{n\to\infty} F^+(R_n) = \lim_{n\to\infty} F^-(R_n) \\ &= F^-(\lim_{n\to\infty} R_n) = F^-(S) \end{aligned}$$

for any closed bounded interval $S \subset \Omega$. So, from the Jordan decomposition of F it follows that

$$F(S) = 0$$

for any closed bounded interval $S \subset \Omega$. □

II. Functions of two variables of locally finite variation and their distributional derivatives

A function $f\colon [a,b]\times[c,d] \to \mathbb{R}$, where $[a,b]\times[c,d]$ is a closed bounded interval in $\mathbb{R}^2$, is called an absolutely continuous function on $[a,b]\times[c,d]$ (cf. [18]) if the functions $f(a,\cdot)$, $f(\cdot,c)$ of one variable are absolutely continuous on $[c,d]$, $[a,b]$, respectively, and a function F^f_{xy} of an interval, associated with f and given by the formula

$$F^f_{xy}(P = [x_1,x_2]\times[y_1,y_2]) = f(x_2,y_2) - f(x_1,y_2) - f(x_2,y_1) + f(x_1,y_1)$$

for $P \subset [a,b]\times[c,d]$, is an absolutely continuous function of an interval on $[a,b]\times[c,d]$.

In a similar way, a function $f\colon [a,b]\times[c,d] \to \mathbb{R}$ is called a function of finite variation on $[a,b]\times[c,d]$ if the functions $f(a,\cdot)$, $f(\cdot,c)$ have finite variation on $[c,d]$, $[a,b]$, respectively, and the function F^f_{xy} of an interval has a finite variation on $[a,b]\times[c,d]$.

Let $\Omega =]a,b[\times]c,d[\subset \mathbb{R}^2$ be an open interval (possibly unbounded).

A function $f\colon \Omega \to \mathbb{R}$ is called locally absolutely continuous on Ω if it is absolutely continuous on each closed bounded interval contained in Ω.

Similarly, a function $f\colon \Omega \to \mathbb{R}$ is called a function of locally finite variation if it has a finite variation on each closed bounded interval contained in Ω.

Now, let $(x_0, y_0) \in \Omega$ be a fixed point. The above definition directly implies that a function $f\colon \Omega \to \mathbb{R}$ is locally absolutely continuous on Ω iff the functions $f(x_0, \cdot)$, $f(\cdot, y_0)$ of one variable are locally absolutely continuous on $]c,d[\,,]a,b[$, respectively (i.e. they are absolutely continuous on each closed bounded interval contained in $]c,d[\,,]a,b[$, respectively), and the function F^f_{xy} of an interval is absolutely continuous on each closed bounded interval contained in Ω.

Similarly, a function $f\colon \Omega \to \mathbb{R}$ is a function of locally finite variation iff the functions $f(x_0, \cdot)$, $f(\cdot, y_0)$ of one variable are functions of locally finite variation on $]c,d[\,,]a,b[$, respectively (i.e. they have a finite variation on each closed bounded interval contained in $]c,d[\,,]a,b[$, respectively), and a function F^f_{xy} of an interval has a finite variation on each closed bounded interval contained in Ω.

It is easy to see (cf. [5, Th. 5.2]) that a function f of locally finite variation on Ω has at any point $(\bar{x}, \bar{y}) \in \Omega$ the following limits:

$$\begin{aligned}
{}_{\swarrow}f(\bar{x}, \bar{y}) &= \lim_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ x>\bar{x},\ y>\bar{y}}} f(x,y), \\
{}^{\nwarrow}f(\bar{x}, \bar{y}) &= \lim_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ x>\bar{x},\ y<\bar{y}}} f(x,y), \\
f^{\nearrow}(\bar{x}, \bar{y}) &= \lim_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ x<\bar{x},\ y<\bar{y}}} f(x,y), \\
f_{\searrow}(\bar{x}, \bar{y}) &= \lim_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ x<\bar{x},\ y>\bar{y}}} f(x,y).
\end{aligned}$$

We say that a function f is $\nearrow$-continuous at a point $(\bar{x}, \bar{y}) \in \Omega$ if

$$f(\bar{x}, \bar{y}) = f^{\nearrow}(\bar{x}, \bar{y}).$$

By the $\nearrow$-continuity of f on Ω we mean the $\nearrow$-continuity of f at any point $(\bar{x}, \bar{y}) \in \Omega$.

The fact that the $\nearrow$-continuity of f on Ω implies the $\nearrow$-continuity of F^f_{xy} on Ω follows at once from the definition of F^f_{xy}. Moreover, from Remark 1.1 we immediately have

Lemma 2.1. *If F is an additive function of an interval, $\nearrow$-continuous in Ω, then the function $f^F\colon \Omega \to \mathbb{R}$ given by the formula*

$$f^F(x,y) = \begin{cases} F([x_0,x]\times[y_0,y]), & x_0 \leqslant x \wedge y_0 \leqslant y, \\ F([x,x_0]\times[y,y_0]), & x \leqslant x_0 \wedge y \leqslant y_0, \\ -F([x_0,x]\times[y,y_0]), & x_0 \leqslant x \wedge y \leqslant y_0, \\ -F([x,x_0]\times[y_0,y]), & x \leqslant x_0 \wedge y_0 \leqslant y, \end{cases}$$

where $(x_0,y_0)\in\Omega$ is a fixed point, is $\nearrow$-continuous in Ω.

The function f^F described in the above lemma will be called the function of two variables associated with F. Of course,

$$F_{xy}^{f^F} = F$$

in Ω.

Now, we shall prove a lemma which is an analogue, in the case of two variables, of the theorem on integration by parts (in the Lebesgue-Stieltjes sense) proved in [13, VII.5.9] for functions of one variable.

Lemma 2.2. *Let $P=[p_1,p_2]\times[q_1,q_2]\subset\Omega$ and let $f\colon \Omega\to\mathbb{R}$, $g\colon\Omega\to\mathbb{R}$ be functions of locally finite variation. Then, if one of the integrals below exists, the other exits as well and the following equality holds:*

$$\begin{aligned}\int_{\operatorname{Int} P} (f^{\nearrow}(p_2,q_2) -{}^{\nwarrow} f(\bar{x},q_2) - f_{\searrow}(p_2,\bar{y}) +_{\swarrow} f(\bar{x},\bar{y}))\,\mathrm{d}F_{xy}^g \\ = \int_{\operatorname{Int} P} (g^{\nearrow}(x,y) -{}^{\nwarrow} g(p_1,y) - g_{\searrow}(x,q_1) +_{\swarrow} g(p_1,q_1))\,\mathrm{d}F_{xy}^f.\end{aligned}$$

P r o o f. Suppose that the first of the above integrals exists, i.e. the following integrals exist and are finite:

$$\int_{\operatorname{Int} P} (f^{\nearrow}(p_2,q_2) -{}^{\nwarrow} f(\bar{x},q_2) - f_{\searrow}(p_2,\bar{y}) +_{\swarrow} f(\bar{x},\bar{y}))\,\mathrm{d}\mu_{(F_{xy}^g)^+},$$
$$\int_{\operatorname{Int} P} (f^{\nearrow}(p_2,q_2) -{}^{\nwarrow} f(\bar{x},q_2) - f_{\searrow}(p_2,\bar{y}) +_{\swarrow} f(\bar{x},\bar{y}))\,\mathrm{d}\mu_{(F_{xy}^g)^-}$$

and

$$\begin{aligned}&\int_{\operatorname{Int} P} (f^{\nearrow}(p_2,q_2) -{}^{\nwarrow} f(\bar{x},q_2) - f_{\searrow}(p_2,\bar{y}) +_{\swarrow} f(\bar{x},\bar{y}))\,\mathrm{d}F_{xy}^g \\ &\quad= \int_{\operatorname{Int} P} (f^{\nearrow}(p_2,q_2) -{}^{\nwarrow} f(\bar{x},q_2) - f_{\searrow}(p_2,\bar{y}) +_{\swarrow} f(\bar{x},\bar{y}))\,\mathrm{d}\mu_{(F_{xy}^g)^+} \\ &\quad- \int_{\operatorname{Int} P} (f^{\nearrow}(p_2,q_2) -{}^{\nwarrow} f(\bar{x},q_2) - f_{\searrow}(p_2,\bar{y}) +_{\swarrow} f(\bar{x},\bar{y}))\,\mathrm{d}\mu_{(F_{xy}^g)^-}.\end{aligned}$$

Let us describe the following open set:

$$A = \{(x, y, \bar{x}, \bar{y}) \in \mathbb{R}^4 \colon \ a < \bar{x} < x < b, \quad c < \bar{y} < y < d\}.$$

Easy computations show that

$$\begin{aligned}
&\int_{\operatorname{Int} P} (f^{\nearrow}(p_2, q_2) - {}^{\nwarrow} f(\bar{x}, q_2) - f_{\searrow}(p_2, \bar{y}) + {}_{\swarrow} f(\bar{x}, \bar{y}))\, \mathrm{d}\mu_{(F^g_{xy})^+} \\
&\quad = \int_{\operatorname{Int} P} \biggl(\int_{\operatorname{Int} P} \chi_A \, \mathrm{d}\mu_{(F^f_{xy})^+} \biggr) d\mu_{(F^g_{xy})^+} - \int_{\operatorname{Int} P} \biggl(\int_{\operatorname{Int} P} \chi_A d\mu_{(F^f_{xy})^-} \biggr) \mathrm{d}\mu_{(F^g_{xy})^+}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\operatorname{Int} P} (f^{\nearrow}(p_2, q_2) - {}^{\nwarrow} f(\bar{x}, q_2) - f_{\searrow}(p_2, \bar{y}) + {}_{\swarrow} f(\bar{x}, \bar{y}))\, \mathrm{d}\mu_{(F^g_{xy})^-} \\
&\quad = \int_{\operatorname{Int} P} \biggl(\int_{\operatorname{Int} P} \chi_A \, \mathrm{d}\mu_{(F^f_{xy})^+} \biggr) \mathrm{d}\mu_{(F^g_{xy})^-} - \int_{\operatorname{Int} P} \biggl(\int_{\operatorname{Int} P} \chi_A d\mu_{(F^f_{xy})^-} \biggr) \mathrm{d}\mu_{(F^f_{xy})^-},
\end{aligned}$$

where χ_A denotes the characteristic function of the set A. So,

$$\begin{aligned}
&\int_{\operatorname{Int} P} (f^{\nearrow}(p_2, q_2) - {}^{\nwarrow} f(\bar{x}, q_2) - f_{\searrow}(p_2, \bar{y}) + {}_{\swarrow} f(\bar{x}, \bar{y}))\, \mathrm{d}F^g_{xy} \\
&\quad = \int_{\operatorname{Int} P} \biggl(\int_{\operatorname{Int} P} \chi_A \, \mathrm{d}\mu_{(F^f_{xy})^+} \biggr) \mathrm{d}\mu_{(F^g_{xy})^+} - \int_{\operatorname{Int} P} \biggl(\int_{\operatorname{Int} P} \chi_A \, \mathrm{d}\mu_{(F^f_{xy})^-} \biggr) \mathrm{d}\mu_{(F^g_{xy})^+} \\
&\quad - \int_{\operatorname{Int} P} \biggl(\int_{\operatorname{Int} P} \chi_A \, \mathrm{d}\mu_{(F^f_{xy})^+} \biggr) \mathrm{d}\mu_{(F^g_{xy})^-} - \int_{\operatorname{Int} P} \biggl(\int_{\operatorname{Int} P} \chi_A \, \mathrm{d}\mu_{(F^f_{xy})^-} \biggr) \mathrm{d}\mu_{(F^g_{xy})^-}.
\end{aligned}$$

Applying to each of the above components the Fubini theorem and reversing the above argument, we get the assertion. □

We say that a distribution Λ on Ω is determined by an additive and $\nearrow$-continuous function F of an interval of finite variation on each closed bounded subinterval of Ω if

$$\Lambda(\varphi) = \int_{\Omega} \varphi \, \mathrm{d}F$$

for $\varphi \in \mathscr{D}(\Omega)$.

Such a distribution will be denoted by Λ_F.

Now, we shall prove a theorem characterizing the distributional derivative $D^{1,1}\Lambda_f$ of the distribution Λ_f determined by an $\nearrow$-continuous function f of locally finite variation.

Theorem 2.3. *Let $(x_0, y_0) \in \Omega = \left]a, b\right[\times \left]c, d\right[$ and let $f \colon \Omega \to \mathbb{R}$ be an $\nearrow$-continuous and locally integrable function on Ω. Then f has a locally finite variation* *on Ω iff the distributional derivative $D^{1,1}\Lambda_f$ of the distribution determined by f is the distribution determined by an additive and $\nearrow$-continuous function of an interval of finite variation on each closed bounded interval contained in Ω and $f(x_0,\cdot)$, $f(\cdot,y_0)$ have locally finite variations on $]c,d[$, $]a,b[$, respectively. Then*

$$D^{1,1}\Lambda_f = \Lambda_{F^f_{xy}}$$

in Ω.

Proof. *Necessity.* Let $\varphi \in \mathscr{D}(\Omega)$ and let $\widetilde{P} = [\widetilde{a},\widetilde{b}] \times [\widetilde{c},\widetilde{d}] \subset \Omega$ be an interval such that

$$\varphi \equiv 0, \qquad \frac{\partial\varphi}{\partial x} \equiv 0, \qquad \frac{\partial\varphi}{\partial y} \equiv 0, \qquad \frac{\partial^2\varphi}{\partial x\partial y} \equiv 0$$

on $\Omega\backslash[\widetilde{a},\widetilde{b}] \times [\widetilde{c},\widetilde{d}]$.

Now, let us consider a closed bounded interval $P =]\overline{\overline{a}},\overline{\overline{b}}[\times]\overline{\overline{c}},\overline{\overline{d}}[\subset \Omega$ such that $\widetilde{P} \subset \operatorname{Int} P$.

On the basis of Lemma 2.2, [17, XI.3.11] and the fact that the derivative $\frac{\partial^2\varphi}{\partial x\partial y}$ of an absolutely continuous function φ is equal a.e. to the derivative DF^{φ}_{xy} of the function F^{φ}_{xy} of an interval, we obtain

$$\begin{aligned}
D^{1,1}\Lambda_f &= \Lambda_f\left(\frac{\partial^2\varphi}{\partial x\partial y}\right) = \int_\Omega \frac{\partial^2\varphi}{\partial x\partial y}(x,y)f(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\operatorname{Int} P} \frac{\partial^2\varphi}{\partial x\partial y}(x,y)f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\operatorname{Int} P} DF^{\varphi}_{xy}(x,y)f(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\operatorname{Int} P} DF^{\varphi}_{xy}(x,y)f^{\nearrow}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\operatorname{Int} P} f^{\nearrow}(x,y)dF^{\varphi}_{xy} \\
&= \int_{\operatorname{Int} P} {}_{\swarrow}\varphi(x,y)\,\mathrm{d}F^f_{xy} + \int_{\operatorname{Int} P} {}^{\nwarrow}f(\overline{\overline{a}},y)\,\mathrm{d}F^{\varphi}_{xy} + \int_{\operatorname{Int} P} f_{\searrow}(x,\overline{\overline{c}})dF^{\varphi}_{xy}.
\end{aligned}$$

Furthermore, for an interval $Q = [\bar{a},\bar{b}] \times [\bar{c},\bar{d}] \subset \Omega$ such that $Q \subset \operatorname{Int} P$, $\widetilde{P} \subset \operatorname{Int} P$, we have

$$\begin{aligned}
\int_{\operatorname{Int} P} {}^{\nwarrow}f(\overline{\overline{a}},y)\,\mathrm{d}F^{\varphi}_{xy} &= \int_{\operatorname{Int} P} DF^{\varphi}_{xy}(x,y)^{\nwarrow}f(\overline{\overline{a}},y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\operatorname{Int} P} \frac{\partial^2\varphi}{\partial x\partial y}(x,y)^{\nwarrow}f(\overline{\overline{a}},y)\,\mathrm{d}x\,\mathrm{d}y = \int_Q \frac{\partial^2\varphi}{\partial x\partial y}(x,y)^{\nwarrow}f(\overline{\overline{a}},y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\bar{c}}^{\bar{d}}\int_{\bar{a}}^{\bar{b}} \frac{\partial^2\varphi}{\partial x\partial y}(x,y)^{\nwarrow}f(\overline{\overline{a}},y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\bar{c}}^{\bar{d}} {}^{\nwarrow}f(\overline{\overline{a}},y)\int_{\bar{a}}^{\bar{b}} \frac{\partial^2\varphi}{\partial x\partial y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\bar{c}}^{\bar{d}} {}^{\nwarrow}f(\overline{\overline{a}},y)\left(\frac{\partial\varphi}{\partial y}(\bar{b},y) - \frac{\partial\varphi}{\partial y}(\bar{a},y)\right)\,\mathrm{d}y = 0.
\end{aligned}$$

Similarly,

$$\int_{\operatorname{Int} P} f_{\searrow}(x, \overline{\overline{c}})\, \mathrm{d}F^{\varphi}_{xy} = 0.$$

So,

$$D^{1,1}\Lambda_f(\varphi) = \int_{\operatorname{Int} P} \varphi(x,y)\, \mathrm{d}F^{f}_{xy} = \int_{\Omega} \varphi(x,y)\, \mathrm{d}F^{f}_{xy}.$$

The arbitrariness of $\varphi \in \mathscr{D}(\Omega)$ and the fact that the $\nearrow$-continuity of f implies the $\nearrow$-continuity of F^{f}_{xy} yield the assertion.

Sufficiency. Assume that the distribution $D^{1,1}\Lambda_f$ is determined by an additive $\nearrow$-continuous function F of an interval of finite variation on each closed bounded interval contained in Ω, and $f(x_0,\cdot)$, $f(\cdot,y_0)$ have locally finite variation on $]c,d[$, $]a,b[$, respectively. So,

$$D^{1,1}\Lambda_f(\varphi) = \int_{\Omega} \varphi\, \mathrm{d}F. \tag{2}$$

Let $f^F\colon\ \Omega \to \mathbb{R}$ be the function of two variables associated with F (cf. Lemma 2.1). Since

$$f^F(x_0,\cdot) \equiv 0, \qquad f^F(\cdot,y_0) \equiv 0, \qquad F^{f^F}_{xy} = F,$$

we assert that f^F has a locally finite variation on Ω. It is $\nearrow$-continuous on the basis of Lemma 2.1. Consequently, from the proved part of the theorem we have

$$D^{1,1}\Lambda_{f^F}(\varphi) = \int_{\Omega} \varphi\, \mathrm{d}F^{f^F}_{xy} \tag{3}$$

for $\varphi \in \mathscr{D}(\Omega)$. Equalities (2) and (3) give

$$D^{1,1}\Lambda_{f-f^F} = 0$$

in Ω. Now, using [1, 4.5.2], we conclude that, for any open bounded interval $I =]i_1,i_2[\times]j_1,j_2[$ such that $\bar{I} \subset \Omega$, there exist functions $g\colon I \to \mathbb{R}$, $h\colon I \to \mathbb{R}$ locally integrable on I, such that

$$D^{0,1}\Lambda_g = 0$$

in I,

$$D^{1,0}\Lambda_h = 0$$

in I and

$$f(x,y) - f^F(x,y) = g(x,y) + h(x,y)$$

for $(x,y) \in I$. The fact that the above equality holds for all points from I follows from the proof of [1, 4.5.2].

Moreover, from [1, 4.3.2] (see also [16]) it follows that there exist locally integrable functions of one variable $\bar{g}\colon\ ]i_1, i_2[\ \to \mathbb{R}$, $\bar{h}\colon\ ]j_1, j_2[\ \to \mathbb{R}$ such that

$$g(x, y) = \bar{g}(x)$$

for a.a. $(x, y) \in I$,

$$h(x, y) = \bar{h}(y)$$

for a.a. $(x, y) \in I$. Thus

$$f(x, y) = f^F(x, y) + \bar{g}(x) + \bar{h}(y) \tag{4}$$

for a.a. $(x, y) \in I$.

To complete the proof, we shall show that $F_{xy}^f = F$ in I.

Let $P = [x_1, x_2] \times [y_1, y_2] \subset I$ and $P_n = [x_1^n, x_2^n] \times [y_1^n, y_2^n] \subset I$, $n \in \mathbb{N}$, be intervals such that $P_n \nearrow P$ and the vertices of P_n, $n \in \mathbb{N}$, belong to the set (of full measure) on which equality (4) is satisfied (it is easy to see that, for any interval P, one can choose a sequence $(P_n)_{n\in\mathbb{N}}$ satisfying the above conditions). We have

$$\begin{aligned}
F_{xy}^f(P) &= f(x_2, y_2) - f(x_1, y_2) - f(x_2, y_1) + f(x_1, y_1) \\
&= \lim_{n\to\infty} (f(x_2^n, y_2^n) - f(x_1^n, y_2^n) - f(x_2^n, y_1^n) + f(x_1^n, y_1^n)) \\
&= \lim_{n\to\infty} (f^F(x_2^n, y_2^n) + \bar{g}(x_2^n) + \bar{h}(y_2^n) - f^F(x_1^n, y_2^n) - \bar{g}(x_1^n) - \bar{h}(y_2^n) \\
&\quad - f^F(x_2^n, y_1^n) - \bar{g}(x_2^n) - \bar{h}(y_1^n) + f^F(x_1^n, y_1^n) + \bar{g}(x_1^n) + \bar{h}(y_1^n)) \\
&= \lim_{n\to\infty} F_{xy}^{f^F}(P_n) = \lim_{n\to\infty} F(P_n) = F(P).
\end{aligned}$$

So, from the assumptions of the theorem and the fact that, for any closed bounded interval $P \subset \Omega$, there exists an open bounded interval I such that $P \subset I \subset \bar{I} \subset \Omega$, it follows that the function f has a locally finite variation on Ω. $\square$

The above theorem directly implies

Corollary 2.4. *A distribution Λ is determined by an additive $\nearrow$-continuous in Ω function of an interval of finite variation on each closed bounded interval contained in Ω iff it is a derivative of order $(1, 1)$ of a distribution determined by an $\nearrow$-continuous function of two variables of locally finite variation in Ω.*

On the basis of Theorem 2.3, one can easily characterize the derivative $D^{1,1}\Lambda_f$ of a distribution determined by a locally absolutely continuous function. We have

Theorem 2.5. *Let $(x_0, y_0) \in \Omega =]a, b[\times]c, d[$ and let $f\colon\ \Omega \to \mathbb{R}$ be a function $\nearrow$-continuous and locally integrable on Ω. Then the function f is locally absolutely* *continuous on Ω iff the distributional derivative $D^{1,1}\Lambda_f$ of the distribution determined by f is the distribution determined by a locally integrable function, and $f(x_0,\cdot)$, $f(\cdot,y_0)$ are locally integrable on $]c,d[\,,]a,b[$, respectively. Then*

$$D^{1,1}\Lambda_f = \Lambda_{\frac{\partial^2\varphi}{\partial x\partial y}}$$

in Ω.

Remark 2.6. In the paper [18] it was shown that a locally absolutely continuous function f has an integral representation of the form

$$f(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} l + \int_{x_0}^{x} l^1 + \int_{y_0}^{y} l^2 + c$$

for $(x,y) \in \Omega$, where $l\colon \Omega \to \mathbb{R}$ is locally integrable on Ω and $l^1\colon\ ]a,b[\ \to \mathbb{R}$, $l^2\colon\ ]c,d[\ \to \mathbb{R}$ are locally integrable on $]a,b[$, $]c,d[$, respectively. On the basis of this fact, the existence a.e. on Ω of the classical partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x\partial y}$ is proved.

Before we give a characterization of the derivative $D^{1,0}\Lambda_f$ of the distribution determined by a function f of locally finite variation on Ω, we shall examine this derivative in the case when f is a locally absolutely continuous function on Ω. So, integrating by parts and using the integral representation of an absolutely continuous function, one can easily show that

$$D^{1,0}\Lambda_f = \Lambda_{\frac{\partial f}{\partial x}}$$

in Ω, where $\frac{\partial f}{\partial x}$ is the classical partial derivative of f.

Now, notice that if we put

$$F_x^f(P) := \int_P \frac{\partial f}{\partial x}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

for closed bounded intervals $P \subset \Omega$, then

$$\begin{aligned} D^{1,0}\Lambda_f(\varphi) &= \int_\Omega \varphi(x,y)\frac{\partial f}{\partial x}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_\Omega \varphi(x,y) DF_x^f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \varphi(x,y)\,\mathrm{d}F_x^f. \end{aligned}$$

Thus, in this case, we can treat the derivative $D^{1,0}\Lambda_f$ as the distribution determined by an absolutely continuous function F_x^f of an interval. This function of an interval can be represented as follows:

$$F_x^f(P = [x_1, x_2] \times [y_1, y_2]) = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{\partial f}{\partial x}(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_{y_1}^{y_2} (f(x_2, y) - f(x_1, y))\,\mathrm{d}y$$

for closed bounded intervals $P \subset \Omega$.

The above facts constitute the prerequisite for determining the partial derivative $D^{1,0}\Lambda_f$ in the case when the function f has a locally finite variation on Ω.

Theorem 2.7. *Let $f\colon \Omega \to \mathbb{R}$ be a function of locally finite variation. Then*

$$D^{1,0}\Lambda_f = \Lambda_{F_x^f}$$

in Ω, where

$$F_x^f([x_1, x_2] \times [y_1, y_2]) = \int_{y_1}^{y_2} (f(x_2, y) - f(x_1, y))\,\mathrm{d}y$$

for $[x_1, x_2] \times [y_1, y_2] \subset \Omega$.

Remark 2.8. The function F_x^f described in the above theorem is, of course, additive and has, by virtue of Jordan decompositions of a function of an interval and a function of two variables of finite variation (cf. [6]), a locally finite variation on Ω.

P r o o f o f T h e o r e m 2.7. Let $\varphi \in \mathscr{D}(\Omega)$. We shall show that

$$-\int_\Omega f(x, y) \frac{\partial \varphi}{\partial x}(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \varphi(x, y)\,\mathrm{d}F_x^f.$$

Let intervals $P = [a, b] \times [c, d]$, Q be such that

- P is the interval of continuity of $F_x^f|_Q$,
- $P \subset \operatorname{Int} Q \subset Q \subset \Omega$,
- $\operatorname{supp} \varphi \subset \operatorname{Int} P$.

On the basis of [13, I.6.7, I.6.8], we get

$$\begin{aligned}
D^{1,0}\Lambda_f(\varphi) &= -\Lambda_f\Big(\frac{\partial \varphi}{\partial x}\Big) = -\int_\Omega f(x, y) \frac{\partial \varphi}{\partial x}(x, y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_P f(x, y) \frac{\partial \varphi}{\partial x}(x, y)\,\mathrm{d}x\,\mathrm{d}y = -\int_c^d \Big(\int_a^b f(x, y) \frac{\partial \varphi}{\partial x}(x, y)\,\mathrm{d}x\Big)\,\mathrm{d}y \\
&= -\int_c^d \Big(\int_a^b f(x, y) d_x \varphi(x, y)\Big)\,\mathrm{d}y = -\int_c^d \Big(-\int_a^b \varphi(x, y) d_x f(x, y)\Big)\,\mathrm{d}y \\
&= \int_c^d \Big(\int_a^b \varphi(x, y) d_x f(x, y)\Big)\,\mathrm{d}y
\end{aligned}$$

where the symbols $\int_a^b f(x,y)d_x\varphi(x,y)$, $\int_a^b \varphi(x,y)d_x f(x,y)$, for a fixed point $y \in [c,d]$, denote the Riemann-Stieltjes integrals of the function $f(\cdot,y)$ with respect to the function $\varphi(\cdot,y)$ and of the function $\varphi(\cdot,y)$ with respect to the function $f(\cdot,y)$, respectively.

Now, let us consider a sequence $(\mathscr{P}_n)_{n\in\mathbb{N}}$ of partitions of the interval P such that the partition $\mathscr{P}_n$ is obtained by the partitions of the interval $[a,b]$ into n equal parts and of the interval $[c,d]$ into n^2 equal parts. Let us choose a subsequence of this sequence that is indexed by the powers of the number 2. So, the partition $\mathscr{P}_{k+1}$ is a subpartition of $\mathscr{P}_k$.

Using [13, I.5.3], we state that, for sufficiently great indices k and any $y \in [c,d]$,

$$|S_k(y) - S(y)| < \operatorname*{var}_{[a,b]} f(\cdot,y) \leqslant \operatorname*{var}_{P} F^f_{xy} + \operatorname*{var}_{[a,b]} f(\cdot,c)$$

where

$$S(y) = \int_a^b \varphi(x,y)d_x f(x,y)$$

and

$$S_k = \sum_{i=1}^{2^k} \varphi(\xi_i^k, y)(f(x_i^k,y) - f(x_{i-1}^k,y)), \qquad \xi_i^k \in [x_{i-1}^k, x_i^k],$$

for $k \in \mathbb{N}$ are the approximative sums for $S(y)$ corresponding to the partitions $a = x_0^k < x_1^k < \ldots < x_{2^k}^k = b$ of the interval $[a,b]$ into 2^k equal parts. The symbols $\operatorname{var}_{[a,b]} f(\cdot,y)$, $\operatorname{var}_P F^f_{xy}$ denote the variations of $f(\cdot,c)$ on $[a,b]$, and of F^f_{xy} on P, respectively.

In view of the above, we have

$$\begin{aligned}
\int_c^d S(y)\,\mathrm{d}y &= \int_c^d \lim_{k\to\infty} S_k(y)\,\mathrm{d}y = \lim_{k\to\infty} \int_c^d S_k(y)\,\mathrm{d}y \\
&= \lim_{k\to\infty} \sum_{j=1}^{(2^k)^2} \int_{y_{j-1}^k}^{y_j^k} \sum_{i=1}^{2^k} \varphi(\xi_i^k,y)(f(x_i^k,y) - f(x_{i-1}^k,y))\,\mathrm{d}y \\
&= \lim_{k\to\infty} \sum_{j=1}^{(2^k)^2} \sum_{i=1}^{2^k} \int_{y_{j-1}^k}^{y_j^k} \varphi(\xi_i^k,y)(f(x_i^k,y) - f(x_{i-1}^k,y))\,\mathrm{d}y
\end{aligned}$$

where $c = y_0^k < y_1^k < \ldots < y_{(2^k)^2}^k = d$ is the partition of $[c,d]$ into $(2^k)^2$ equal parts.

Moreover,

$$\int_{\Omega} \varphi(x,y)\,\mathrm{d}F_x^f = \int_{[a,b]\times[c,d]} \varphi(x,y)\,\mathrm{d}F_x^f = (R-S)\int_{[a,b]\times[c,d]} \varphi(x,y)\,\mathrm{d}F_x^f|_Q$$
$$= \lim_{k\to\infty} \sum_{j=1}^{(2^k)^2} \sum_{i=1}^{2^k} \varphi(\xi_i^k, \zeta_j^k) F_x^f([x_{i-1}^k, x_i^k] \times [y_{j-1}^k, y_j^k])$$
$$= \lim_{k\to\infty} \sum_{j=1}^{(2^k)^2} \sum_{i=1}^{2^k} \varphi(\xi_i^k, \zeta_j^k) \int_{y_{j-1}^k}^{y_j^k} (f(x_i^k, y) - f(x_{i-1}^k, y))\,\mathrm{d}y$$

where $\zeta_j^k \in [y_{j-1}^k, y_j^k]$ and the symbol $(R-S)\int_{[a,b]\times[c,d]} \varphi(x,y)\,\mathrm{d}F_x^f|_Q$ denotes the Riemann-Stieltjes integral of the function φ with respect to the function $F_x^f|_Q$ of an interval of finite variation on Q.

So, if L is a Lipschitz constant for φ and M the boundedness of f on $[a,b]\times[c,d]$, then

$$\left| -\int_{\Omega} f(x,y)\frac{\partial \varphi}{\partial x}(x,y)\,\mathrm{d}x\,\mathrm{d}y - \int_{\Omega} \varphi(x,y)\,\mathrm{d}F_x^f \right|$$
$$\leqslant \lim_{k\to\infty} \sum_{j=1}^{(2^k)^2} \sum_{i=1}^{2^k} \int_{y_{j-1}^k}^{y_j^k} |\varphi(\xi_i^k, y) - \varphi(\xi_i^k, \zeta_j^k)||f(x_i^k, y) - f(x_{i-1}^k, y)|\,\mathrm{d}y$$
$$\leqslant \lim_{k\to\infty} \sum_{j=1}^{(2^k)^2} \sum_{i=1}^{2^k} \int_{y_{j-1}^k}^{y_j^k} L|y - \zeta_j^k|\,2M\,\mathrm{d}y$$
$$\leqslant \lim_{k\to\infty} \sum_{j=1}^{(2^k)^2} \sum_{i=1}^{2^k} \int_{y_{j-1}^k}^{y_j^k} L|y_j^k - y_{j-1}^k|\,2M\,\mathrm{d}y$$
$$= 2LM \lim_{k\to\infty} \sum_{j=1}^{(2^k)^2} \sum_{i=1}^{2^k} |y_j^k - y_{j-1}^k|^2$$
$$= 2LM \lim_{k\to\infty} \sum_{j=1}^{(2^k)^2} \sum_{i=1}^{2^k} |y_j^k - y_{j-1}^k||x_i^k - x_{i-1}^k| \frac{|y_j^k - y_{j-1}^k|}{|x_i^k - x_{i-1}^k|}$$
$$= 2LM \lim_{k\to\infty} \sum_{j=1}^{(2^k)^2} \sum_{i=1}^{2^k} |[x_i^k - x_{i-1}^k] \times [y_j^k - y_{j-1}^k]| \frac{|[c,d]|}{(2^k)^2} \frac{2k}{|[a,b]|}$$
$$= 2LM \frac{|[c,d]|}{|[a,b]|} \lim_{k\to\infty} \frac{1}{2^k} |[a,b]\times[c,d]| = 2LM|[c,d]|^2 \lim_{k\to\infty} \frac{1}{2^k} = 0,$$

which completes the proof. $\square$

In an analogous way one can prove that

$$D^{0,1}\Lambda_f = \Lambda_{F_y^f}$$

where

$$F_y^f([x_1,x_2]\times[y_1,y_2]) = \int_{x_1}^{x_2} (f(x,y_2) - f(x,y_1))\,\mathrm{d}x$$

for $[x_1,x_2]\times[y_1,y_2]\subset\Omega$.

To conclude the considerations of this part of the paper, we shall prove the following

Theorem 2.9. *If $(x_0,y_0)\in\Omega =]a,b[\times]c,d[$, and $f\colon\ \Omega\to\mathbb{R}$ is an $\nearrow$-continuous function of locally finite variation such that the functions $f(x_0,\cdot)$, $f(\cdot,y_0)$ are left-hand continuous, then the function F_x^f of an interval is $\nearrow$-continuous.*

Remark 2.10. In the proof of the above theorem we shall use the notion of a nondecreasing function of two variables (cf. [6]). We recall that a function $f\colon$ $[a',b']\times[c',d']\to\mathbb{R}$ is nondecreasing if the functions $f(a',\cdot)$, $f(\cdot,c')$ are nondecreasing functions of one variable on $[c',d']$, $[a',b']$, respectively, and the function F_{xy}^f of an interval is additive and nonnegative.

P r o o f o f T h e o r e m 2.9. To begin with, we notice that the assumptions of the theorem imply the left-hand continuity of $f(x,\cdot)$, $f(\cdot,y)$ for any $x\in\ ]a,b[$, $y\in\ ]c,d[$. Indeed, let $x\in\ ]a,b[$, $y\in\ ]c,d[$ be such that $x_0<x$, $y_0<y$. If $(y_n)_{n\in\mathbb{N}}$ is a sequence such that $y_n<y$ and $\lim\limits_{n\to\infty} y_n = y$, then from Remark 1.1 we have

$$f(x,y) - f(x,y_n) = F_{xy}^f([x_0,x]\times[y_n,y]) + f(x_0,y) - f(x_0,y_n) \underset{n\to\infty}{\longrightarrow} 0.$$

In the remaining cases, the reasoning is analogous.

Now, let $(P_n)_{n\in\mathbb{N}}$ be a sequence of intervals $P_n = [x_1^n,x_2^n]\times[y_1^n,y_2^n]\subset\Omega$ $\nearrow$-convergent to an interval $P_0 = [x_1^0,x_2^0]\times[y_1^0,y_2^0]\subset\Omega$, and $Q = [\bar{a},\bar{b}]\times[\bar{c},\bar{d}]\subset\Omega$ an interval such that $P_n\subset \operatorname{Int} Q$, $n=0,1,\ldots$.

Since the function $f|_Q$ has finite variation on Q, therefore, using the Jordan decomposition (cf. [6, Th. 4]), we have

$$f|_Q(x,y) = g(x,y) - h(x,y)$$

for $(x,y)\in Q$, where g,h are nondecreasing functions of two variables on Q. These functions are given by the formulae

$$g(x,y) = g^1(x) + g^2(y) - \frac{1}{2}f(\bar{a},\bar{c}) + (F_{xy}^f|_Q)^+([\bar{a},x]\times[\bar{c},y]),$$
$$h(x,y) = h^1(x) + h^2(y) + \frac{1}{2}f(\bar{a},\bar{c}) + (F_{xy}^f|_Q)^-([\bar{a},x]\times[\bar{c},y])$$

for $(x,y)\in Q$, where

$$f(x,\bar{c})=g^1(x)-h^1(x)$$

for $x\in[\bar{a},\bar{b}]$,

$$f(\bar{a},y)=g^2(y)-h^2(y)$$

for $y\in[\bar{c},\bar{d}]$ are the Jordan decompositions of $f(\cdot,\bar{c})\,,f(\bar{a},\cdot)$, respectively. The left-hand continuity of $f(\cdot,\bar{c})\,,f(\bar{a},\cdot)$ implies (cf. [13, I.4.1]) the left-hand continuity of g^1,h^1,g^2,h^2. So, on the basis of Lemma 1.2, the functions $g\colon Q\to\mathbb{R}$, $h\colon Q\to\mathbb{R}$ are $\nearrow$-continuous. Also, if M is the boundedness of f on Q, then the fact that a nondecreasing function of two variables is nondecreasing with respect to each variable separately directly yields

$$\begin{aligned}|F_x^f(P_n)-F_x^f(P_0)|&=|F_x^f([x_1^n,x_1^0]\times[y_1^0,y_2^n])+F_x^f([x_1^n,x_1^0]\times[y_1^n,y_1^0])\\&+F_x^f([x_1^0,x_2^n]\times[y_1^n,y_1^0])-F_x^f([x_2^n,x_2^0]\times[y_1^0,y_2^n])\\&-F_x^f([x_2^n,x_2^0]\times[y_2^n,y_2^0])-F_x^f([x_1^0,x_2^n]\times[y_2^n,y_2^0])|\\&\leqslant(g(x_1^0,y_2^0)-g(x_1^n,y_2^n))+(h(x_1^0,y_2^0)-h(x_1^n,y_2^n))\\&+4M(y_1^0-y_1^n)+4M(y_2^0-y_2^n)+(g(x_2^0,y_2^0)\\&-g(x_2^n,y_2^n))+(h(x_2^0,y_2^0)-h(x_2^n,y_2^0))\xrightarrow[n\to\infty]{}0\end{aligned}$$

and the proof is completed. □

III. On the existence of a solution of a partial differential equation of hyperbolic type

Let $P=[a,b]\times[c,d]$ be a closed bounded interval contained in $\mathbb{R}^2$. Denote by

- $\mathrm{BV}(P)$— the set of all functions of two variables of finite variation on P,
- $\mathrm{BV}_0^{\nearrow}(P)$— the set of all functions from $\mathrm{BV}(P)$ that are $\nearrow$-continuous on $]a,b]\times]c,d]$ and $f(a,\cdot)\equiv 0$, $f(\cdot,c)\equiv 0$
- $\mathrm{BVFI}(P)$— the set of all additive functions of an interval of finite variation on P,
- $\mathrm{BV}([a,b])$— the set of all functions of one variable of finite variation on $[a,b]$.

It is well known that the space $\mathrm{BVFI}(P)$ with the norm

$$\|\cdot\|_{\mathrm{BVFI}(P)}\colon\ \mathrm{BVFI}(P)\ni F\mapsto\operatorname*{var}_P F\in\mathbb{R}_0^+$$

and the space $\mathrm{BV}([a,b])$ with the norm

$$\|\cdot\|_{\mathrm{BV}([a,b])}\colon\ \mathrm{BV}([a,b])\ni f\mapsto|f(a)|+\operatorname*{var}_{[a,b]} f\in\mathbb{R}_0^+$$

are complete.

Using these facts, one can prove in an elementary way that $\mathrm{BV}(P)$ with the norm

$$\|f\| = |f(x_0, y_0)| + \operatorname*{var}_{[c,d]} f(x_0, \cdot) + \operatorname*{var}_{[a,b]} f(\cdot, y_0) + \operatorname*{var}_{P} F^f_{xy},$$

where $(x_0, y_0) \in P$ is a fixed point, is complete and, consequently, the space of functions f of finite variation on P, $\nearrow$-continuous on $]a,b] \times]c,d]$ and satisfying the conditions $f(x_0,\cdot) \equiv 0$, $f(\cdot, y_0) \equiv 0$ with the norm

$$\|f\| = \operatorname*{var}_{P} F^f_{xy}$$

is also complete.

Now, let $\Omega =]a,b[\times]c,d[\subset \mathbb{R}^2$ be an open interval (possibly unbounded) and $(x_0, y_0) \in \Omega$—a fixed point.

Let us consider the equation

$$\frac{\partial^2 z}{\partial x \partial y} = Az + B\frac{\partial z}{\partial x} + C\frac{\partial z}{\partial y} + G \tag{5}$$

in Ω, with the boundary conditions

$$\begin{aligned} z(x_0, \cdot) &\equiv 0, \\ z(\cdot, y_0) &\equiv 0, \end{aligned} \tag{6}$$

where G is an additive $\nearrow$-continuous function of an interval of finite variation on each closed bounded interval contained in Ω, and $A, B, C \in \mathbb{R}$.

By a solution of problem (5)–(6) we mean a function $z\colon \Omega \to \mathbb{R}$ of locally finite variation, $\nearrow$-continuous, satisfying boundary conditions (6) and such that

$$D^{1,1}\Lambda_z = A\Lambda_z + BD^{1,0}\Lambda_z + CD^{0,1}\Lambda_z + \Lambda_G \tag{7}$$

in Ω.

This equation, in view of Theorems 2.3 and 2.7, is equivalent to

$$\Lambda_{F^z_{xy}} = \Lambda_{F^{Az}} + \Lambda_{F^{Bz}_x} + \Lambda_{F^{Cz}_y} + \Lambda_G \tag{8}$$

in Ω, where

$$F^{Az}(P) = \int_P Az$$

for closed bounded intervals $P \subset \Omega$.

On the basis of Lemma 1.4, equation (8) is equivalent to

$$F^z_{xy} = F^{Az} + F^{Bz}_x + F^{Cz}_y + G \tag{9}$$

in Ω, i.e.

$$F_{xy}^{z}(P) = F^{Az}(P) + F_{x}^{Bz}(P) + F_{y}^{Cz}(P) + G(P)$$

for closed bounded intervals $P \subset \Omega$.

The above equation with boundary conditions (6), in view of additivity of the functions appearing in it, is equivalent to the following system of equations:

$$F_{xy}^{z} = F^{Az} + F_{x}^{Bz} + F_{y}^{Cz} + G$$

in $[x_0, b[\times[y_0, d[$,

$$F_{xy}^{z} = F^{Az} + F_{x}^{Bz} + F_{y}^{Cz} + G$$

in $[x_0, b[\times\,]c, y_0]$,

$$F_{xy}^{z} = F^{Az} + F_{x}^{Bz} + F_{y}^{Cz} + G$$

in $]a, x_0] \times [y_0, d[$,

$$F_{xy}^{z} = F^{Az} + F_{x}^{Bz} + F_{y}^{Cz} + G$$

in $]a, x_0] \times\,]c, y_0]$,

with boundary conditions (6).

To find a function $z\colon\ \Omega \to \mathbb{R}$ of locally finite variation, $\nearrow$-continuous, satisfying boundary conditions (6) and the above system, it is sufficient to find the following functions:

- an $\nearrow$-continuous function $z_1\colon\ [x_0, b[\times[y_0, d[\to \mathbb{R}$ of finite variation on each closed bounded interval contained in $[x_0, b[\times[y_0, d[$, satisfying boundary conditions (6) and equation (9) in $[x_0, b[\times[y_0, d[$,
- an $\nearrow$-continuous function $z_2\colon\ [x_0, b[\times\,]c, y_0] \to \mathbb{R}$ of finite variation on each closed bounded interval contained in $[x_0, b[\times\,]c, y_0]$, satisfying boundary conditions (6) and equation (9) in $[x_0, b[\times\,]c, y_0]$,
- an $\nearrow$-continuous function $z_3\colon\ ]a, x_0] \times [y_0, d[\to \mathbb{R}$ of finite variation on each closed bounded interval contained in $]a, x_0] \times [y_0, d[$, satisfying boundary conditions (6) and equation (9) in $]a, x_0] \times [y_0, d[$,
- an $\nearrow$-continuous function $z_4\colon\ ]a, x_0] \times\,]c, y_0] \to \mathbb{R}$ of finite variation on each closed bounded interval contained in $]a, x_0] \times\,]c, y_0]$, satisfying boundary conditions (6) and equation (9) in $]a, x_0] \times\,]c, y_0]$,

and then put them together.

Let us consider, for example, the first of the above problems. In the other cases, one proceeds in an analogous way.

To ascertain the existence of a unique solution of equation (9) in $[x_0, b[\times[y_0, d[$, satisfying boundary conditions (6), it is sufficient to state the existence, for sufficiently great numbers $n \in \mathbb{N}$, of a solution of equation (9) in the set $[x_0, b - \frac{1}{n}] \times [y_0, d - \frac{1}{n}]$, satisfying boundary conditions (6). Equation (9) in the set $[x_0, b-\frac{1}{n}] \times [y_0, d-\frac{1}{n}]$ is equivalent, in view of the additivity of the functions appearing in it, to

$$F^z_{xy}(P) = F^{Az}(P) + F^{Bz}_x(P) + F^{Cz}_y(P) + G(P)$$

for intervals $P \subset [x_0, b-\frac{1}{n}] \times [y_0, d-\frac{1}{n}]$ of type $P = [x_0, x] \times [y_0, y]$.

The above equation can be written as

$$\begin{aligned}(10)\quad & z(x,y) - z(x,y_0) - z(x_0,y) - z(x_0,y_0)\\ &= A\int_{x_0}^{x}\int_{y_0}^{y} z + C\int_{x_0}^{x}(z(s,y) - z(s,y_0))\,\mathrm{d}s + B\int_{y_0}^{y}(z(x,t) - z(x_0,t))\,\mathrm{d}t + l(x,y)\end{aligned}$$

for $(x,y) \in [x_0, b-\frac{1}{n}] \times [y_0, d-\frac{1}{n}]$, where

$$l(x,y) = G([x_0,x] \times [y_0,y])$$

is an $\nearrow$-continuous function of finite variation on the interval $[x_0, b-\frac{1}{n}] \times [y_0, d-\frac{1}{n}]$, satisfying the conditions

$$\begin{aligned} l(x_0,\cdot) &\equiv 0,\\ l(\cdot,y_0) &\equiv 0.\end{aligned}$$

Taking into account boundary conditions (6), we can write equation (10) in the form

$$(11)\qquad z(x,y) = A\int_{x_0}^{x}\int_{y_0}^{y} z + C\int_{x_0}^{x} z(s,y)\,\mathrm{d}s + B\int_{y_0}^{y} z(x,t)\,\mathrm{d}t + l(x,y)$$

for $(x,y) \in [x_0, b-\frac{1}{n}] \times [y_0, d-\frac{1}{n}]$.

So, finally, to show the existence of a unique solution of equation (5), satisfying boundary conditions (6) in the class of functions $z\colon\ \Omega \to \mathbb{R}$ of locally finite variation on Ω and $\nearrow$-continuous, it is enough to prove, for any $n \in \mathbb{N}$, the existence of a unique solution of equation (11), satisfying boundary conditions (6) in the class of functions $z\colon\ [x_0, b-\frac{1}{n}] \times [y_0, d-\frac{1}{n}] \to \mathbb{R}$ of finite variation on $[x_0, b-\frac{1}{n}] \times [y_0, d-\frac{1}{n}]$ and $\nearrow$-continuous on $\left]x_0, b-\frac{1}{n}\right] \times \left]y_0, d-\frac{1}{n}\right]$. Of course, we can replace the interval $[x_0, b-\frac{1}{n}] \times [y_0, d-\frac{1}{n}]$ with $P = [0,1] \times [0,1]$.

The existence of a unique solution of the equation

$$(12)\qquad z(x,y) = A\int_{0}^{x}\int_{0}^{y} z + C\int_{0}^{x} z(s,y_0)\,\mathrm{d}s + B\int_{0}^{y} z(x,t)\,\mathrm{d}t + l(x,y)$$

for $(x,y)\in P$, satisfying the boundary conditions

$$
\begin{aligned}
&z(0,\cdot)\equiv 0,\\
&z(\cdot,0)\equiv 0
\end{aligned}
\tag{13}
$$

in the class of functions $z\colon P\to\mathbb{R}$ of finite variation on P and $\nearrow$-continuous on $]0,1]\times]0,1]$, is equivalent to the existence of a unique fixed point of the operator

$$
\mathscr{H}\colon \mathrm{BV}_0^{\nearrow}(P)\to\mathrm{BV}_0^{\nearrow}(P),
$$

$$
(\mathscr{H}z)(x,y)=A\int_0^x\int_0^y z+C\int_0^x z(s,y)\,\mathrm{d}s+B\int_0^y z(x,t)\,\mathrm{d}t+l(x,y).
$$

We have

Theorem 3.1. *There exists a positive integer k such that the operator $\mathscr{H}^k=\mathscr{H}\circ\ldots\circ\mathscr{H}$ is a contraction.*

P r o o f. Notice that if $z\in\mathrm{BV}_0^{\nearrow}(P)$ is a nondecreasing function or a nonincreasing function (i.e. nondecreasing with the sign minus), then

$$
\|z\|_{\mathrm{BV}_0^{\nearrow}(P)}=\||z|\|_{\mathrm{BV}_0^{\nearrow}(P)}=\operatorname*{var}_P F_{xy}^{|z|}=F_{xy}^{|z|}(P)=|z(1,1)|.
$$

Moreover, if $z\in\mathrm{BV}_0^{\nearrow}(P)$, then

$$
\begin{aligned}
z(x,y)&=F_{xy}^z([0,x]\times[0,y])=(F_{xy}^z)^+([0,x]\times[0,y])-(F_{xy}^z)^-([0,x]\times[0,y])\\
&=z_1(x,y)-z_2(x,y)
\end{aligned}
$$

for $(x,y)\in P$, where

$$
\begin{aligned}
z_1(x,y)&=(F_{xy}^z)^+([0,x]\times[0,y]),\\
z_2(x,y)&=(F_{xy}^z)^-([0,x]\times[0,y])
\end{aligned}
$$

are nondecreasing functions of two variables satisfying boundary conditions (13) and such that

$$
\begin{aligned}
\|z\|_{\mathrm{BV}_0^{\nearrow}(P)}&=\operatorname*{var}_P F_{xy}^z=(F_{xy}^z)^+(P)+(F_{xy}^z)^-(P)\\
&=z_1(1,1)+z_2(1,1).
\end{aligned}
$$

Now, let us define the operator

$$
H\colon \mathrm{BV}_0^{\nearrow}(P)\to\mathrm{BV}_0^{\nearrow}(P),
$$

$$
(Hz)(x,y)=A\int_0^x\int_0^y z+C\int_0^x z(s,y_0)\,\mathrm{d}s+B\int_0^y (x,t)\,\mathrm{d}t.
$$

Of course,

$$\mathscr{H}z = Hz + l$$

for $z \in \mathrm{BV}_0^{\nearrow}(P)$.

Using the induction principle one can show that, for $k \geqslant 2$, $(x,y) \in P$ and $w, z \in \mathrm{BV}_0^{\nearrow}(P)$,

$$\begin{aligned}
(H^k z)(x,y) - (H^k w)(x,y) &= A^k \underbrace{\int_0^x \int_0^y \dots \int_0^x \int_0^y}_{k \quad \text{times}} (z-w) \\
&+ \sum_{l=1}^{k-1} \binom{k}{l} \sum_{j_1=1}^{2} \dots \sum_{j_l=1}^{2} A^{k-l} D_{j_1} \cdot \ldots \cdot D_{j_l} \underbrace{\int_0^x \int_0^y \dots \int_0^x \int_0^y}_{k-l \quad \text{times}} \int_0^{\Theta_{j_1}} \dots \int_0^{\Theta_{j_l}} (z-w) \\
&+ \sum_{j_1=1}^{2} \dots \sum_{j_k=1}^{2} D_{j_1} \cdot \ldots \cdot D_{j_k} \int_0^{\Theta_{j_1}} \dots \int_0^{\Theta_{j_k}} (z-w)
\end{aligned}$$

where $\Theta_1 = y$, $\Theta_2 = x$, $D_1 = B$, $D_2 = C$.

Now, let us fix an even number $k \geqslant 2$. We have

$$\begin{aligned}
\|\mathscr{H}^k z - \mathscr{H}^k w\|_{\mathrm{BV}_0^{\nearrow}(P)} &= \|H^k z - H^k w\|_{\mathrm{BV}_0^{\nearrow}(P)} = \|H^k(z-w)\|_{\mathrm{BV}_0^{\nearrow}(P)} \\
&= \|H^k((z-w)_1 - (z-w)_2)\|_{\mathrm{BV}_0^{\nearrow}(P)} \\
&= \|H^k((z-w)_1) - H^k((z-w)_2)\|_{\mathrm{BV}_0^{\nearrow}(P)} \\
&\leqslant \sum_{i=1}^{2} \|H^k((z-w)_i)\|_{\mathrm{BV}_0^{\nearrow}(P)} \\
&\leqslant (\max\{|A|, |B|, |C|\})^k \sum_{i=1}^{2} \|\widetilde{H}^k((z-w)_i)\|_{\mathrm{BV}_0^{\nearrow}(P)}
\end{aligned}$$

where

$$\begin{aligned}
&\widetilde{H}((z-w)_i)(x,y) \\
&\qquad = \int_0^x \int_0^y (z-w)_i + \int_0^x (z-w)_i(s,y)\,\mathrm{d}s + \int_0^y (z-w)_i(x,t)\,\mathrm{d}t
\end{aligned}$$

for $(x,y)\in P$ and

$$
\begin{aligned}
&\sum_{i=1}^{2}\|\widetilde{H}^k((z-w)_i)\|_{\mathrm{BV}_0^{\nearrow}(P)}=\sum_{i=1}^{2}(\widetilde{H}^k((z-w)_i))(1,1)\\
&\leqslant \sum_{i=1}^{2}\Big(\ \frac{1}{k!k!}\int_0^1\int_0^1(z-w)_i+\sum_{\substack{j_1,\dots,j_{k+1}=1,2\\ 1<\prod_{s=1}^{k+1}j_s<2^{k+1}}}\frac{1}{(\frac{k}{2}-1)!}\int_0^1\int_0^1(z-w)_i\\
&\quad+\frac{1}{k!}\int_0^1(z-w)_i(1,t)\,\mathrm{d}t+\frac{1}{k!}\int_0^1(z-w)_i(s,1)\,\mathrm{d}s\\
&\quad+\sum_{l=1}^{k}\binom{k+1}{l}\sum_{j_1=1}^{2}\cdots\sum_{j_l=1}^{2}\frac{1}{(\frac{k+2}{2}-1)!}\int_0^1\int_0^1(z-w)_i\ \Big)\\
&\leqslant \sum_{i=1}^{2}\Big(\ \frac{1}{k!k!}(z-w)_i(1,1)+2^{k+1}\frac{1}{(\frac{k}{2}-1)!}(z-w)_i(1,1)\\
&\quad+\sum_{l=1}^{k}\binom{k+1}{l}2^l\frac{1}{(\frac{k}{2})!}(z-w)_i(1,1)\ \Big)\\
&=\left(\frac{1}{k!k!}+\frac{2^{k+1}}{(\frac{k}{2}-1)!}+\sum_{l=1}^{k}\binom{k+1}{l}2^l\frac{1}{(\frac{k}{2})!}\right)\|z-w\|_{\mathrm{BV}_0^{\nearrow}(P)}\\
&\leqslant\frac{1+2^{k+1}+3^{k+1}}{(\frac{k}{2}-1)!}\|z-w\|_{\mathrm{BV}_0^{\nearrow}(P)}.
\end{aligned}
$$

Since

$$\frac{1+2^{k+1}+3^{k+1}}{(\frac{k}{2}-1)!}\underset{k\to\infty}{\longrightarrow}0,$$

the proof is complete. □

Thus, in view of the general contraction principle, we state that there exists a unique solution of equation (12), satisfying boundary conditions (13) in the class of functions of finite variation on P and $\nearrow$-continuous. This means, as was shown, that the following theorem is valid:

Theorem 3.2. *There exists a unique solution of problem* (5)–(6) *in the class of $\nearrow$-continuous functions of locally finite variation on Ω.*

The above theorem implies

Theorem 3.3. *There exists a unique solution of equation* (5) *satisfying the boundary conditions*

$$z(x_0,\cdot)\equiv\psi(\cdot),\qquad z(\cdot,y_0)\equiv\varphi(\cdot), \tag{14}$$

where $\varphi\colon\ ]a,b[\ \to\mathbb{R}$, $\psi\colon\ ]c,d[\ \to\mathbb{R}$ *are left-hand continuous functions of one variable of locally finite variation and* $\varphi(x_0)=\psi(y_0)$, *in the class of* $\nearrow$*-continuous functions of locally finite variation on* Ω.

Proof. Let $\widetilde{z}\colon\ \Omega\to\mathbb{R}$ be an $\nearrow$-continuous function of locally finite variation on Ω such that

$$\frac{\partial^2\widetilde{z}}{\partial x\partial y}=A\widetilde{z}+B\frac{\partial\widetilde{z}}{\partial x}+C\frac{\partial\widetilde{z}}{\partial y}+G+F^{A(\varphi+\psi-c)}+F_x^{B\varphi}+F_x^{C\psi}$$

and

$$\widetilde{z}(x_0,\cdot)\equiv 0,\qquad \widetilde{z}(\cdot,y_0)\equiv 0,$$

where $c=\varphi(x_0)=\psi(y_0)$.

It is easy to see that the function

$$z\colon\ \Omega\ni(x,y)\mapsto\widetilde{z}(x,y)+\varphi(x)+\psi(y)-c$$

is a solution of equation (5) satisfying the boundary conditions (14).

The uniqueness of this solution follows from the uniqueness of $\widetilde{z}$. The proof is complete. □

Example. Let us consider the equation

$$\frac{\partial^2 z}{\partial x\partial y}=z+\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}+\delta_{(\frac34,\frac34)} \tag{15}$$

in $\Omega=\,]0,1[\,\times\,]0,1[$, with the boundary conditions

$$z\left(\frac12,y\right)=\begin{cases}1; & 0<y\leqslant\frac12,\\ 2; & \frac12<y<1,\end{cases}\qquad z\left(x,\frac12\right)=\begin{cases}1; & 0<x\leqslant\frac12,\\ 0; & \frac12<x<1,\end{cases} \tag{16}$$

where $\delta_{\left(\frac{3}{4},\frac{3}{4}\right)}$ is Dirac's delta concentrated at the point $\left(\frac{3}{4},\frac{3}{4}\right)$.

It is easily seen (cf. also [4]) that

$$\delta_{\left(\frac{3}{4},\frac{3}{4}\right)} = D^{1,1}\Lambda_h$$

in Ω with

$$h\colon\ \Omega \ni (x,y) \mapsto \begin{cases} 1; & \frac{3}{4} < x < 1,\ \ \frac{3}{4} < y < 1,\\ 0; & \text{otherwise} \end{cases}$$

(a Heaviside function). Since h is an $\nearrow$-continuous function of locally finite variation on Ω, Theorem 2.3 yields that $D^{1,1}\Lambda_h = \Lambda_{F^h_{xy}}$ in Ω, where

$$\begin{aligned} F^h_{xy}([x_1,x_2]\times[y_1,y_2]) &= h(x_2,y_2) - h(x_1,y_2) - h(x_2,y_1) + h(x_1,y_1)\\ &= \begin{cases} 1; & \frac{3}{4} < x_2 < 1,\ \ \frac{3}{4} < y_2 < 1,\\ 0; & \text{otherwise} \end{cases} \end{aligned}$$

for $[x_1,x_2]\times[y_1,y_2] \subset \Omega$.

So, we may write equation (15) in the form (5) with $G = F^h_{xy}$ being an additive $\nearrow$-continuous function of an interval of finite variation on each closed bounded interval contained in Ω. Consequently, Theorem 3.3 implies the existence of a unique solution of (15) satisfying boundary conditions (16), in the class of $\nearrow$-continuous functions of locally finite variation on Ω.

References

[1] *P. Antosik, J. Mikusiński, R. Sikorski*: Theory of Distributions—the Sequential Aproach. Elsevier Scientific Publishing Company, Amsterdam, Polish Scientific Publishers, Warsaw, 1973.

[2] *J.A. Clarkson, C.R. Adams*: On definitions of bounded variation for functions of two variables. Trans. Amer. Math. Soc. *35* (1933), 824–854.

[3] *A. Halanay, D. Wexler*: Qualitative Theory of Impulsive Systems. Moscow, 1971. (In Russian.)

[4] *I. Halparin*: Introduction to the Theory of Distributions. University of Toronto Press, Toronto, 1952. (In Russian.)

[5] *T.H. Hildebrandt*: Introduction to the Theory of Integration. Academic Press, New York-London, 1963.

[6] *D. Idczak*: Functions of several variables of finite variation and their differentiability. Annales Pol. Math. *60* (1994), no. 1, 47–56.

[7] *J. Kurzweil*: Generalized ordinary differential equations. Czechoslovak Math. J. *8* (1958), no. 83, 360–388.

[8] *J. Kurzweil*: On generalized ordinary differential equations with discontinuous solutions. Prikl. Mat. Mech. *22* (1958), 27–45.

[9] *J. Kurzweil*: Linear differential equations with distributions as coefficients. Bull. Acad. Pol. Sci., ser. Math. Astr. Phys. *7* (1959), 557–560.

[10] *V. Lakshmikantham*: Trends in the theory of impulsive differential equations. Proceedings of the International Conference on Theory and Applications of Differential Equations, Ohio University. 1988.

[11] *A. Lasota*: Remarks on linear differential equations with distributional perturbations. Ordinary differential equations (Proc. NRL–MRC Conf. Math. Res. Center, Naval Res. Lab., Washington, D.C., 1971). New York, Academic Press, 1972, pp. 489–495.

[12] *J. Ligęza*: On distributional solutions of some systems of linear differential equations. Čas. pro Pěst. Mat. *102* (1977), 37–41.

[13] *S. Łojasiewicz*: An Introduction to the Theory of Real Functions. John Willey and Sons, Chichester, 1988.

[14] *R. Pfaff*: Generalized systems of linear differential equations. Proc. of the Royal Sci. of Edinburgh, sect. A *89* (1981), 1–14.

[15] *W. W. Schmaedeke*: Optimal control theory for nonlinear vector-differential equations containing measures. SIAM Control *3* (1965), no. 2, 231–280.

[16] *L. Schwartz*: Théorie des Distributions I. Paris, 1950.

[17] *R. Sikorski*: Funkcje Rzeczywiste I. PWN, Warszawa, 1958.

[18] *S. Walczak*: Absolutely continuous functions of several variables and their application to differential equations. Bull. Polish Acad. Sci. Math. *35* (1987), no. 11–12, 733–744.

[19] *Z. Wyderka*: Linear Differential Equations with Measures as Coefficients and Control Theory. Siles. Univ. Press, Katowice, 1994.

[20] *Z. Wyderka*: Linear differential equations with measures as coefficients and the control theory. Čas. pro Pěst. Mat. *114* (1989), no. 1, 13–27.

Author's address: Institute of Mathematics, University of Łódź, Poland.