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NONUNIQUENESS RESULTS FOR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In the present paper we give general nonuniqueness results which cover most of the known nonuniqueness criteria. In particular, we obtain a generalization of the nonuniqueness theorem of Chr. Nowak, of Samimi's nonuniqueness theorem and of Stettner's nonuniqueness criterion.

1. Introduction

In the recent paper of Chr. Nowak [5] the following criterion is given:

Theorem. *Assume that*

(i) $f \in C[R_0, \mathbb{R}^n]$, *where* $R_0 = \{(t,x)\colon\ 0 < t \leqslant a, |x - x_0| \leqslant b\}$ *and* $x_0(t)$ *is a solution of*

$$x' = f(t,x), \qquad x(0) = x_0 \tag{$*$}$$

on $[0,a]$;

(ii) $g(t,u)$ *is continuous on* $0 < t \leqslant a$, $0 \leqslant u \leqslant 2b$, $g(t,u)$ *is nondecreasing in* u *for* $t > 0$, *and* $u(t)$ *is a solution of*

$$u' = g(t,u), \qquad 0 < t \leqslant t_1,$$

such that $u(t_1) > 0$ *for some* t_1, $0 < t_1 \leqslant a$ *with* $u(0) = 0$ *and* $\lim_{t\to 0} u(t)/B(t) = 0$, *where* $B \in C[[0,a], \mathbb{R}^+]$ *with* $B(t) > 0$ *for* $t > 0$, $\mathbb{R}^+$ *being the interval* $[0,\infty)$;

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(iii) $v \in C[(0,a] \times \mathbb{R}^n, \mathbb{R}^+]$, *$v(t,x)$ is locally Lipschitzian in x, $v(t,x)=0 \Leftrightarrow x=0$ and*

$$\mathrm{D}^+ v_f(t, x-x_0(t)) = \limsup_{h\to 0+} \frac{1}{h}\{v(t+h, x-x_0(t)+h[f(t,x)-f(t,x_0(t))])- \\ -v(t,x-x_0(t))\} \geqslant g(t, v(t,x-x_0(t))) \qquad \text{on } \Omega,$$

where $\Omega = \{(t,x)\colon\ u(t) < v(t,x-x_0(t))$ for $0<t<t_1$, $|x-x_0| \leqslant b\}$;

(iv) $\exists x_1 \neq x_0$, $|x_1 - x_0| < \frac{1}{4}b$: $v(t_1, x_1 - x_0(t_1)) < u(t_1)$.

Then there exists a solution $x_1(t) \not\equiv x_0(t)$ of $(*)$ on $0 \leqslant t \leqslant a$ such that

$$\lim_{t\to 0} \frac{v(t, x_1(t)-x_0(t))}{B(t)} = 0.$$

Tracing the proof of this theorem, we observe two controversible points. First, the set R_0 is bounded with respect to x for fixed t, however the proof works with a solution $x_1(t)$ such that $|x_1(t)| \to \infty$ as $t \to \bar{t}+$, where $\bar{t} > 0$. Moreover, neither is the replacement of R_0 by $R_0 = \{(t,x)\colon\ 0 < t \leqslant a, x \in \mathbb{R}^n\}$ sufficient to ensure the existence of a solution $x_1(t)$ of

$$x' = f(t,x), \qquad x(t_1) = x_1$$

on $(0, t_1]$, because the function v can be small for large x. In our opinion, the theorem should be supplemented by a condition which ensures that the solution $x_1(t)$ exists on $(0, t_1]$. Such a condition is the condition (31) of our Corollary 2.

Secondly, the relation

$$\lim_{t\to 0} v(t, x_1(t) - x_0(t)) = 0$$

does not imply $\lim_{t\to 0} x_1(t) = x_0$ since $v(t, x_1(t)-x_0(t)) \to 0$ can be caused by $t \to 0$ and not by $x_1(t) - x_0(t) \to 0$. Thus the theorem should be supplemented by a condition such as our condition (32) in Corollary 2.

It is not difficult to give an example which shows that Nowak's theorem is not valid without additional conditions:

Example. Consider the initial value problem

$$x' = x, \qquad x(0) = 0.$$

This problem has the unique solution $x_0(t) \equiv 0$; the other solutions of the equation $x' = x$ are $x(t) = C\mathrm{e}^t$, $C \neq 0$, and do not satisfy the initial condition $x(0) = 0$. Put $v(t,x) = tx^2$, $g(t,u) = t^{-1}(2t+1)u$. Let $B(t)$, $t \geqslant 0$ be any continuous function such that $B(t) > 0$ for $t > 0$ and $\lim_{t\to 0} t/B(t) = 0$. Since the solutions $u = Cte^{2t}$ of $u' = g(t,u)$ are positive for $C > 0$ on $(0,\infty)$ and

$$\mathrm{D}^+v_f(t, x - x_0(t)) = \mathrm{D}^+v_f(t,x) = (2t+1)x^2 = g(t, v(t, x - x_0(t))) \quad \text{for } t > 0,\ x \in \mathbb{R},$$

all the assumptions of the theorem are satisfied, which is a contradiction with the uniqueness of $x_0(t)$.

In [2] (see also [1], page 197) we have given a nonuniqueness criterion which covers several special cases. The applicability of the results is illustrated by examples. In the present paper we attempt to generalize these results to a general form which covers most of the known nonuniqueness criteria. Our results make it possible to take the initial value t_0 of t at the point $-\infty$. Moreover, the estimates of the form

$$\begin{aligned} \mathrm{D}^+v_f(t, x - x_0(t)) &\geqslant g(t, v(t, x - x_0(t))), \\ \mathrm{D}^+v_f(t, x - y) &\geqslant g(t, v(t, x - y)), \\ |f(t,x) - f(t, x_0(t))| &\geqslant g(t, |x - x_0(t)|), \\ |f(t,x) - f(t,y)| &\geqslant g(t, |x - y|), \end{aligned}$$

where $x_0(t)$ is a solution of $x' = f(t,x)$, $x(t_0) = x_0$, can be replaced by estimates of the form

$$\begin{aligned} \mathrm{D}^+v_{fF}(t, x - z(t)) &\geqslant g(t, v(t, x - z(t))), \\ \mathrm{D}^+v_{fF}(t, x - y) &\geqslant g(t, v(t, x - y)), \\ |f(t,x) - F(t, z(t))| &\geqslant g(t, |x - z(t)|), \\ |f(t,x) - F(t,y)| &\geqslant g(t, |x - y|), \end{aligned}$$

where $z(t)$ is a solution of $z' = F(t,z)$, $z(t_0) = x_0$, and f, F may be different functions.

2. Results

Consider an equation

(1) $$x' = f(t,x),$$

where $f \in C[R_a, \mathbb{R}^n]$, $-\infty \leqslant a < A \leqslant \infty$, $R_a = \{(t,x) \in \mathbb{R}^{n+1} \colon a < t < A, |x - x_0| \leqslant b\}$, $x_0 \in \mathbb{R}^n$, $b > 0$. Here $|\cdot|$ is an arbitrary but fixed norm in $\mathbb{R}^n$. By the initial value problem

(2) $$x' = f(t,x), \qquad x(a) = x_0$$

we mean the problem to find solutions $x(t)$ of (1) such that $\lim_{t\to a} x(t) = x_0$. We say that (2) *has at least two different solutions*, if there exists a $T \in (a, A)$ such that (2) has solutions $x_1(t)$, $x_2(t)$ defined on $(a, T]$ and $x_1(t) \not\equiv x_2(t)$ on $(a, T]$. In this case we also say that (2) *has at least two different solutions on* $(a, T]$. The problem (2) is said to be *nonunique*, if there is a $T_0 \in (a, A)$ such that for any $T \in (a, T_0]$, (2) has at least two different solutions on $(a, T]$.

If V is a continuous real-valued function for $a < t < A$, $|x - x_0| \leqslant b$, we define

$$\begin{aligned} \mathrm{D}^+ V_f(t,x) &= \limsup_{h\to 0+} \frac{V(t+h, x+hf(t,x)) - V(t,x)}{h}, \\ \mathrm{D}_+ V_f(t,x) &= \liminf_{h\to 0+} \frac{V(t+h, x+hf(t,x)) - V(t,x)}{h} \end{aligned}$$

for $(t,x) \in R_a$, $|x - x_0| < b$. If v is a continuous real-valued function for $a < t < A$, $x \in \mathbb{R}^n$, and $F \in C[R_a, \mathbb{R}^n]$, we define

$$\mathrm{D}^+ v_{fF}(t, x-z) = \limsup_{h\to 0+} \frac{v(t+h, x-z+h[f(t,x) - F(t,z)]) - v(t, x-z)}{h}$$

for $a < t < A$, $x \in \mathbb{R}^n$, $z \in \mathbb{R}^n$, $|x - x_0| < b$, $|z - x_0| < b$. Particularly, if $z(t)$ is a solution of $z' = F(t,z)$ such that $|z(t) - x_0| < b$, we have

$$\begin{aligned} &\mathrm{D}^+ v_{fF}(t, x - z(t)) \\ &\qquad = \limsup_{h\to 0+} \frac{v(t+h, x-z(t)+h[f(t,x) - F(t,z(t))]) - v(t, x-z(t))}{h}. \end{aligned}$$

Theorem 1. *Let $t_1 \in (a, A)$. Assume that*

(i) *there exist functions $g, h \in C[(a, t_1] \times \mathbb{R}, \mathbb{R}]$ nondecreasing in the second variable and such that there are solutions $\varphi(t)$, $t \in (a, t_1]$ of*

$$u' = g(t,u) \tag{3}$$

and $\psi(t)$, $t \in (a, t_1]$ of

$$u' = h(t,u), \tag{4}$$

satisfying conditions $\psi(t_1) < \varphi(t_1)$,

$$\lim_{t\to a} \frac{\varphi(t)}{B(t)} = 0, \qquad \lim_{t\to a} \frac{\psi(t)}{B(t)} = 0,$$

where $B \in C[(a, t_1], \mathbb{R}]$ is positive;

(ii) $V \in C[R_a, \mathbb{R}]$ *is such that*

$$\psi(t_1) < V(t_1, y_0) < \varphi(t_1) \quad \text{for some } y_0 \in \mathbb{R}^n,\ |y_0 - x_0| < b; \tag{5}$$

$$V(t,x) > \varphi(t) \quad \text{or} \quad V(t,x) < \psi(t) \qquad \text{for } a < t < t_1,\ |x - x_0| = b; \tag{6}$$

(iii) *there exists a positive function* $\varepsilon \in C[(a,t_1), \mathbb{R}^+]$ *such that* $V(t,x)$ *satisfies locally the Lipschitz condition with respect to* x *for* $(t,x) \in \Omega_\varphi \cup \Omega_\psi$, *where*

$$\Omega_\varphi = \{(t,x)\colon\ \varphi(t) < V(t,x) < \varphi(t) + \varepsilon(t),\ a < t < t_1,\ |x - x_0| < b\}, \tag{7}$$

$$\Omega_\psi = \{(t,x)\colon\ \psi(t) - \varepsilon(t) < V(t,x) < \psi(t),\ a < t < t_1,\ |x - x_0| < b\}, \tag{8}$$

and

$$\mathrm{D}^+ V_f(t,x) \geqslant g(t, V(t,x)) \quad \text{on } \Omega_\varphi \qquad \text{if } \Omega_\varphi \neq \emptyset, \tag{9}$$

$$\mathrm{D}_+ V_f(t,x) \leqslant h(t, V(t,x)) \quad \text{on } \Omega_\psi \qquad \text{if } \Omega_\psi \neq \emptyset. \tag{10}$$

Then the equation (1) *has at least two different solutions* $x(t)$ *on* $(a, t_1]$ *such that*

$$\lim_{t \to a} \frac{V(t, x(t))}{B(t)} = 0. \tag{11}$$

Proof. Choose $x_1, x_2 \in \{x\colon\ |x - x_0| < b\}$, $x_1 \neq x_2$ such that

$$\psi(t_1) < V(t_1, x_j) < \varphi(t_1) \qquad (j = 1, 2). \tag{12}$$

Such a choice is possible in view of (5) and the continuity of V. Consider solutions $x_j(t)$ of

$$x' = f(t,x), \qquad x_j(t_1) = x_j \tag{13_j}$$

for $j = 1, 2$. Put

$$x(t) = x_j(t), \qquad m(t) = V(t, x_j(t))$$

for $j \in \{1, 2\}$. In view of (12) we have

$$\psi(t_1) < m(t_1) < \varphi(t_1). \tag{14}$$

We shall show that the set of $t \in (a, t_1)$ for which the solution $x(t)$ satisfies $(t, x(t)) \in \Omega_\varphi$ is empty. Suppose on the contrary that there is a $\tau \in (a, t_1)$ such that $(\tau, x(\tau)) \in \Omega_\varphi$. With respect to (6), (14) and the continuity, we can assume that

$|x(t) - x_0| < b$ for $t \in [\tau, t_1]$. In view of (14) there exists an interval $I = (t_2, t_3)$ such that $\tau < t_2 < t_3 < t_1$,

$$m(t_3) = \varphi(t_3) \tag{15}$$

and

$$\varphi(s) < m(s) < \varphi(s) + \varepsilon(s) \qquad \text{for} \quad s \in I. \tag{16}$$

Clearly $(s, x(s)) \in \Omega_\varphi$ for $s \in I$.

For $s \in I$ and for $h > 0$ small enough we get

$$\begin{aligned} m(s+h) - m(s) &= V(s+h, x(s+h)) - V(s, x(s)) \\ &= V(s+h, x(s) + hf(s, x(s)) + hR(h)) - V(s, x(s)), \end{aligned} \tag{17}$$

where

$$\lim_{h \to 0+} |R(h)| = 0. \tag{18}$$

As V satisfies locally the Lipschitz condition, we have

$$|m(s+h) - m(s) - V(s+h, x(s) + hf(s, x(s))) + V(s, x(s))| \leqslant Lh|R(h)| \tag{19}$$

for $h > 0$ sufficiently small and for some $L > 0$. The conditions (18), (19) together with the definition of $\mathrm{D}^+ V_f$ yield

$$\mathrm{D}^+ m(s) = \limsup_{h \to 0+} \frac{m(s+h) - m(s)}{h} = \mathrm{D}^+ V_f(s, x(s)). \tag{20}$$

By use of (9) and (20) we obtain

$$\mathrm{D}^+[m(s) - \varphi(s)] = \mathrm{D}^+ m(s) - \varphi'(s) \geqslant g(s, m(s)) - \varphi'(s), \qquad s \in I.$$

The nondecreasing character of $g(s, \cdot)$ implies

$$\mathrm{D}^+[m(s) - \varphi(s)] \geqslant g(s, \varphi(s)) - \varphi'(s) = 0, \qquad s \in I.$$

Thus the function $m(s) - \varphi(s)$ is nondecreasing in I and we get a contradiction with (15) and (16). Hence the set of all $t \in (a, t_1)$ for which $(t, x(t)) \in \Omega_\varphi$ is empty. By virtue of (14) and the continuity we get $m(t) \leqslant \varphi(t)$ for all $t \in (a, t_1]$ for which the solution $x(t)$ exists.

Similarly we can prove that $m(t) \geqslant \psi(t)$ for all $t \in (a, t_1]$ for which the solution $x(t)$ exists. Therefore

$$\psi(t) \leqslant m(t) \leqslant \varphi(t) \tag{21}$$

for all $t \in (a, t_1]$ for which $x(t)$ is defined. In view of (6) the solution $x(t)$ is defined for all $t \in (a, t_1]$ and the inequality (21) holds for $t \in (a, t_1]$. On account of the hypothesis (i) we have proved that

$$\lim_{t \to a} \frac{V(t, x_j(t))}{B(t)} = 0$$

for j=1,2. □

Remark 1. 1. Suppose additionally

$$|V(t, x)| \geqslant \Phi(t)\,\Psi(|x - z(t)|) \qquad \text{for} \quad a < t \leqslant t_1, \ |x - x_0| < b \tag{22}$$

where $\Phi \in C[(a, t_1], \mathbb{R}^+]$, $\Psi \in C[[0, 2b), \mathbb{R}^+]$, $z \in C[(a, t_1], \mathbb{R}^n]$ are such that

$$\liminf_{t \to a} \frac{\Phi(t)}{B(t)} > 0, \qquad \Psi(0) = 0, \qquad \Psi(u) > 0 \quad \text{for} \quad u \in (0, 2b) \tag{23}$$

and

$$\lim_{t \to a} z(t) = x_0, \qquad |z(t) - x_0| < b \quad \text{for} \quad t \in (a, t_1]. \tag{24}$$

Then Theorem 1 ensures that the initial value problem (2) has at least two different solutions $x(t)$ on $(a, t_1]$ which satisfy the condition (11). Moreover, if $a > -\infty$, $\lim_{t \to a} \varphi(t) = \lim_{t \to a} \psi(t) = 0$ and $V \in C[\overline{R}_a, \mathbb{R}]$, $\overline{R}_a$ denoting the closure of R_a, then the condition (22) may be replaced by

$$V(a, x) = 0 \Leftrightarrow x = x_0.$$

2. Let the condition (5) in Theorem 1 be satisfied with $y_0 \in \mathbb{R}^n$, $|y_0 - x_0| < \frac{1}{2}b$. If $a > -\infty$, $|f(t, x)| \leqslant M$ for $(t, x) \in R_a$, and $t_1 \in (a, A)$ is such that $(t_1 - a)M \leqslant \frac{1}{2}b$, then the solutions $x_j(t)$ of (13_j) are defined for $t \in (a, t_1]$ and satisfy[1] $|x_j(t) - x_0| < b$; hence the condition (6) may be omitted in this case.

Remark 2. Theorem 1 together with Remark 1 generalize the results of [2].

[1] $|x_j(t) - x_0| \leqslant |x_j(t) - x_j| + |x_j - x_0| \leqslant \big|x_j - x_0\big| + \big|\int_{t_1}^{t} f(s, x(s))\,\mathrm{d}s\big| \leqslant |x_j - x_0| + M(t_1 - t) \leqslant |x_j - x_0| + M(t_1 - a) < \frac{1}{2}b + \frac{1}{2}b = b$

Corollary 1. *Let $t_1 \in (a, A)$. Assume that*

(i) *there exists a function $q \in C[(a, t_1] \times \mathbb{R}^+, \mathbb{R}]$ nondecreasing in the second variable and such that a certain solution $\varphi(t)$, $t \in (a, t_1]$ of*

$$u' = q(t, u)$$

satisfies conditions

$$\varphi(t_1) > 0, \qquad \lim_{t \to a} \frac{\varphi(t)}{B(t)} = 0,$$

where $B \in C[(a, t_1], \mathbb{R}]$ is positive;

(ii) *$V \in C[R_a, \mathbb{R}^+]$ is such that*

$$V(t_1, y_0) < \varphi(t_1) \qquad \text{for some } \ y_0 \in \mathbb{R}^n, \ |y_0 - x_0| < b, \tag{25}$$

$$V(t, x) > \varphi(t) \qquad \text{for } \ a < t < t_1, \ |x - x_0| = b, \tag{26}$$

$$V(t, x) \geqslant \Phi(t)\,\Psi(|x - z(t)|) \qquad \text{for } \ a < t \leqslant t_1, \ |x - x_0| < b, \tag{27}$$

where $\Phi \in C[(a, t_1], \mathbb{R}^+]$, $\Psi \in C[[0, 2b), \mathbb{R}^+]$, $z \in C[(a, t_1], \mathbb{R}^n]$ satisfy (23), (24);

(iii) *there exists a positive function $\varepsilon \in C[(a, t_1), \mathbb{R}^+]$ such that $V(t, x)$ satisfies locally the Lipschitz condition with respect to x for $(t, x) \in \Omega_\varphi$ and*

$$\mathrm{D}^+ V_f(t, x) \geqslant q(t, V(t, x)) \qquad \text{on } \ \Omega_\varphi \tag{28}$$

holds, Ω_φ being defined by (7).

Then the problem (2) has at least two different solutions $x(t)$ on $(a, t_1]$ such that (11) is valid.

P r o o f. Let $t^* \in (a, t_1)$ be fixed. Put

$$g(t, u) = \begin{cases} q(t, u) & \text{for } \ (t, u) \in (a, t_1] \times \mathbb{R}^+, \\ q(t, 0) & \text{for } \ (t, u) \in (a, t_1] \times \mathbb{R}^-. \end{cases}$$

Setting $h(t, u) = \sqrt[3]{u}$ for $(t, u) \in (a, t_1] \times \mathbb{R}$,

$$\psi(t) = \begin{cases} 0 & \text{for } \ t \in (a, t^*), \\ -\frac{2\sqrt{2}}{3\sqrt{3}}(t - t^*)^{\frac{3}{2}} & \text{for } \ t \in [t^*, t_1], \end{cases}$$

we can easily see that the assumptions of Theorem 1 are satisfied with $\Omega_\psi = \emptyset$. In view of Remark 1 we get the desired statement. $\square$

As a consequence we obtain the following revised and generalized form of NOWAK's Nonuniqueness Theorem [5]:

Corollary 2. *Let* $t_1 \in (a, A)$ *and let* $F \in C[R_a, \mathbb{R}^n]$ *be such that the equation*

$$z' = F(t, z) \tag{29}$$

has a solution $z(t)$ *defined on* $(a, t_1]$ *and satisfying* (24). *Suppose that the hypothesis* (i) *of Corollary* 1 *holds true, while the hypotheses* (ii), (iii) *are replaced by*

(ii′) $v \in C[(a, A) \times \mathbb{R}^n, \mathbb{R}^+]$ *is such that*

$$v(t_1, y_0 - z(t_1)) < \varphi(t_1) \qquad \text{for some } y_0 \in \mathbb{R}^n,\ |y_0 - x_0| < b, \tag{30}$$

$$v(t, x - z(t)) > \varphi(t) \qquad \text{for } a < t < t_1,\ |x - x_0| = b, \tag{31}$$

$$v(t, x - z(t)) \geqslant \Phi(t)\,\Psi(|x - z(t)|) \qquad \text{for } a < t \leqslant t_1,\ |x - x_0| < b, \tag{32}$$

where $\Phi \in C[(a, t_1], \mathbb{R}^+]$, $\Psi \in C[[0, 2b), \mathbb{R}^+]$ *satisfy* (23);

(iii′) $v(t, x)$ *satisfies locally the Lipschitz condition with respect to* x *and*

$$\mathrm{D}^+ v_{fF}(t, x - z(t)) \geqslant q(t, v(t, x - z(t))) \qquad \text{on } \Omega,$$

where $\Omega = \{(t, x)\colon\ \varphi(t) < v(t, x - z(t)),\ a < t < t_1,\ |x - x_0| < b\}$.

Then there exist at least two different solutions $x(t)$ *of* (2) *on* $(a, t_1]$ *such that*

$$\lim_{t \to a} \frac{v(t, x(t) - z(t))}{B(t)} = 0.$$

P r o o f. Put $V(t, x) = v(t, x - z(t))$. Then

$$\begin{aligned}
\mathrm{D}^+ V_f(t, x) &= \limsup_{h \to 0+} \frac{V(t + h, x + hf(t, x)) - V(t, x)}{h} \\
&= \limsup_{h \to 0+} \frac{v(t + h, x - z(t + h) + hf(t, x)) - v(t, x - z(t))}{h} \\
&= \limsup_{h \to 0+} \frac{v(t + h, x + hf(t, x) - z(t) - hF(t, z(t)) - hR(h)) - v(t, x - z(t))}{h},
\end{aligned}$$

where $\lim\limits_{h \to 0+} |R(h)| = 0$. Since $v(t, x)$ satisfies locally the Lipschitz condition, we have

$$\mathrm{D}^+ V_f(t, x) = \mathrm{D}^+ v_{fF}(t, x - z(t)) \geqslant q(t, v(t, x - z(t))).$$

Thus the hypotheses of Corollary 1 are fulfilled. □

Taking into account Remark 1, we easily get a generalization of SAMIMI's Nonuniqueness Theorem [7] (see also [1], page 201):

Corollary 3. *Let the assumptions of Corollary 2 be fulfilled with the exception that $a > -\infty$, $v \in C[[a, A) \times \mathbb{R}^n, \mathbb{R}^+]$, the condition (30) is satisfied for some $y_0 \in \mathbb{R}^n$, $|y_0 - x_0| < \frac{1}{2}b$, the condition (31) is omitted and (32) is replaced by $v(a, x) = 0 \Leftrightarrow x = 0$. If, moreover, $\lim_{t\to a} \varphi(t) = 0$, $|f(t, x)| \leqslant M$ for $(t, x) \in R_a$, and the number t_1 is such that $(t_1 - a)M \leqslant \frac{1}{2}b$, then the problem (2) has at least two different solutions $x(t)$ on $(a, t_1]$ such that*

$$\lim_{t\to a} \frac{v(t, x(t) - z(t))}{B(t)} = 0.$$

Since $|x - z| < 2b$ for $|x - x_0| \leqslant b$, $|z - x_0| < b$, it is obvious that the function $v(t, x)$ can be considered for $a < t < A$, $|x| < 2b$ instead of $(t, x) \in (a, A) \times \mathbb{R}^n$. Supposing $a > -\infty$, $B(t) \equiv 1$, we obtain the following generalization of the revised STETTNER's Nonuniqueness Theorem (see [8] and [6]):

Corollary 4. *Let $a > -\infty$, $0 < \delta < A - a$ and let $F \in C[R_a, \mathbb{R}^n]$ be such that the equation (29) has a solution $z(t)$ defined on $(a, a + \delta)$ and satisfying (24). Suppose there is an $M > 0$ such that $|f(t, x)| \leqslant M$ for $(t, x) \in R_a$ and assume that*

(i) *the function $q \in C[(a, A) \times \mathbb{R}^+, \mathbb{R}]$ is nondecreasing in the second variable and has the following property: the equation*

$$u' = q(t, u)$$

possesses a positive solution $\varphi(t)$ such that $\lim_{t\to a} \varphi(t) = 0$;

(ii) *v is continuous for $a \leqslant t < A$, $|x| < 2b$ with values in $\mathbb{R}^+$ and satisfying locally the Lipschitz condition with respect to x for $a < t < A$, $0 < |x| < 2b$ and such that*

$$v(t, x) = 0 \Leftrightarrow x = 0 \qquad \text{for } a \leqslant t < A; \tag{33}$$

(iii) *for $a < t < A$, $|x - x_0| < b$, $|y - x_0| < b$, $x \neq y$ the inequality*

$$\mathrm{D}^+ v_{fF}(t, x - y) \geqslant q(t, v(t, x - y)) \tag{34}$$

holds.

Then the initial value problem (2) is nonunique.

Proof. Choose $t_1 \in (a, a + \delta)$ such that $(t_1 - a)M < \frac{1}{2}b$, the solution $\varphi(t)$ is defined in $(a, t_1]$, and $|z(t) - x_0| < \frac{1}{2}b$ holds for $t \in (a, t_1]$. Put $B(t) \equiv 1$. From (33) it follows that the condition (30) of Corollary 2 is fulfilled with $y_0 \in \mathbb{R}^n$, $|y_0 - x_0| < \frac{1}{2}b$. In view of (34) we have

$$\mathrm{D}^+ v_{fF}(t, x - z(t)) \geqslant q(t, v(t, x - z(t)))$$

on $\Omega = \{(t,x)\colon\ \varphi(t) < v(t, x - z(t)),\ a < t < t_1,\ |x - x_0| < b\}$. With respect to Remark 1 we can omit the relations (31), (32) and Corollary 2 yields the desired result. □

Remark 3. If $f \in C[\overline{R}_a, \mathbb{R}^n]$, $F(t,z) = f(t,z)$ for $(t,z) \in \overline{R}_a$ in Corollary 4, we need not assume the existence of the solution $z(t)$ of (29) which satisfies (24).

In the following Corollary 5 we will suppose that the norm $|\cdot|$ is Euclidean. We denote this norm by $\|\cdot\|$, and the scalar product in $\mathbb{R}^n$ by $\cdot$. Put $\hat{R}_a = \{(t,x) \in \mathbb{R}^{n+1}\colon\ a < t < A,\ \|x - x_0\| \leqslant b\}$.

Corollary 5. *Let $F \in C[\hat{R}_a, \mathbb{R}^n]$ be such that the equation* (29) *has a solution $z(t)$ defined on (a, A) and satisfying* (24)*. Assume $f \in C[\hat{R}_a, \mathbb{R}^n]$ and*

(i) *there exists a function $q \in C[(a,A) \times \mathbb{R}^+, \mathbb{R}]$ nondecreasing in the second variable and such that a certain solution $\varphi(t)$, $t \in (a,A)$ of*

$$u' = q(t,u)$$

satisfies conditions

$$\lim_{t\to a} \varphi(t) = 0, \qquad \lim_{t\to a} \frac{\varphi(t)}{B(t)} = 0, \qquad \varphi(t) > 0 \quad \text{for} \ \ t \in (a,A),$$

where $B \in C[(a,A), \mathbb{R}]$ is positive;

(ii) *there exists a positive function $\varepsilon \in C[(a,A), \mathbb{R}^+]$ such that the inequality*

$$(f(t,x) - F(t,z(t))) \cdot (x - z(t)) \geqslant \|x - z(t)\|\, q(t, \|x - z(t)\|) \tag{35}$$

holds on $\hat{\Omega} = \{(t,x)\colon\ \varphi(t) < \|x - z(t)\| < \varphi(t) + \varepsilon(t),\ a < t < A,\ \|x - x_0\| < b\}$.

Then, for any $t_1 \in (a,A)$ sufficiently close to a, the problem (2) *has at least two different solutions $x(t)$ on $(a, t_1]$ such that*

$$\lim_{t\to a} \frac{\|x(t) - z(t)\|}{B(t)} = 0. \tag{36}$$

Proof. From (i) it follows that $\lim_{t\to a} \varphi(t) = 0$. There exists a $t_2 \in (a,A)$ such that $\|z(t) - x_0\| < \frac{1}{2}b$ and $\varphi(t) \leqslant \frac{1}{2}b$ for $t \in (a, t_2]$. Choose $t_1 \in (a, t_2]$ arbitrary. Define

$$V(t,x) = \|x - z(t)\| \qquad \text{for} \ \ (t,x) \in \hat{R}_a.$$

Since

$$\begin{aligned}\mathrm{D}^+V_f(t,x) &= \frac{1}{\|x-z(t)\|}\,(f(t,x)-z'(t))\cdot(x-z(t))\\ &= \frac{1}{\|x-z(t)\|}\,(f(t,x)-F(t,z(t)))\cdot(x-z(t))\end{aligned}$$

is true for $a<t<t_1$, $\|x-x_0\|<b$, $x\neq z(t)$, we get

$$\mathrm{D}^+V_f(t,x)\geqslant q(t,\|x-z(t)\|)=q(t,V(t,x))\qquad \text{for}\quad (t,x)\in\hat{\Omega},\ \ t<t_1,$$

in view of (35). Moreover, we have

$$V(t,x)=\|x-z(t)\|\geqslant\|x-x_0\|-\|z(t)-x_0\|>\frac{b}{2}\geqslant\varphi(t)$$

for $t\in(a,t_2]$, $\|x-x_0\|=b$. Corollary 1 and Remark 1, where $\Phi(t)\equiv1$, $\Psi(u)\equiv u$, imply that (2) has at least two different solutions on $(a,t_1]$ such that (36) holds. □

Remark 4. Similarly as in Corollary 4 we can modify Corollary 5 in such a way that (35) takes the form

$$(35') \qquad (f(t,x)-F(t,y))\cdot(x-y)\geqslant\|x-y\|\,q(t,\|x-y\|)$$

for $a<t<A$, $\|x-x_0\|<b$, $\|y-y_0\|<b$, $x\neq y$. Thus we can obtain a vector variant of the results of V. Lakshmikantham [3] (see also [1], page 99, or [4], page 55) and M. Samimi [7] (see also [1], page 101) for scalar differential equations.

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