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OSCILLATIONS OF CERTAIN FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Sufficient conditions are presented for all bounded solutions of the linear system of delay differential equations

$$(-1)^{m+1}\frac{d^m y_i(t)}{dt^m}+\sum_{j=1}^{n} q_{ij}y_j(t-h_{jj})=0,\quad m\geqslant 1,\ i=1,2,\dots,n,$$

to be oscillatory, where $q_{ij}\varepsilon(-\infty,\infty)$, $h_{jj}\in(0,\infty)$, $i,j=1,2,\dots,n$. Also, we study the oscillatory behavior of all bounded solutions of the linear system of neutral differential equations

$$(-1)^{m+1}\frac{d^m}{dt^m}(y_i(t)+cy_i(t-g))+\sum_{j=1}^{n} q_{ij}y_j(t-h)=0,$$

where c, g and h are real constants and $i=1,2,\dots,n$.

1. Introduction

Consider the system of delay differential equations

$$\text{(E)}\qquad (-1)^{m+1}y_i^{(m)}(t)+\sum_{j=1}^{n} q_{ij}y_j(t-h_{jj})=0,\quad m\geqslant 1,\ i=1,2,\dots,n,$$

where $q_{ij}\in(-\infty,\infty)$, $h_{jj}\in(0,\infty)$, $i,j=1,2,\dots,n$.

We say that a solution $y(t)=\big[y_1(t),\dots,y_n(t)\big]^T$ of (E) oscillates if for some $i\in\{1,2,\dots,n\}$, $y_i(t)$ has arbitrarily large zeros. A solution $y(t)$ of (E) is said to be nonoscillatory if there exists a $t_0\geqslant 0$ such that for each $i=1,2,\dots,n$, $y_i(t)\neq 0$ for $t\geqslant t_0$.

The oscillatory behavior of scalar delay differential equations and/or linear systems of delay differential equations has been the subject of numerous investigations. For a recent survey of results, we refer to the book of Györi and Ladas [3], and for references concerning the oscillation of systems, the reader is referred to [1].

Recently, Gopalsamy [1] and Gopalsamy and Ladas [2] discussed (E) when $m = 1$ and derived some sufficient conditions for the oscillation of (E).

The purpose of this paper is to establish some sufficient conditions for the oscillation of (E), $m \geqslant 1$ and to investigate the oscillatory behavior of the neutral system

$$(\mathrm{E_N}, \mathrm{a}) \qquad (-1)^{m+1}\big(y_i(t) + cy_i(t - ag)\big)^{(m)} + \sum_{j=1}^{n} q_{ij} y_j(t - h) = 0,$$

$i = 1, 2, \ldots, n$, where $a = \pm 1$, c, g and h are real constants.

We note that the results of this paper are extensions of those in [1,2] to higher order systems of differential equations.

2. Main results

The following result concerns the oscillatory behavior of all bounded solutions of (E).

Theorem 1. *Let $q_{ij} \in (-\infty, \infty)$, $h_{jj} \in (0, \infty)$, $i, j = 1, 2, \ldots, n$. If every bounded solution of the equation*

$$(\mathrm{E}^*) \qquad (-1)^{m+1} z^{(m)}(t) + qz(t - h) = 0$$

oscillates, where

$$(1) \qquad q = \min_{1 \leqslant i \leqslant n} \Big\{ q_{ii} - \sum_{\substack{j=1 \\ j \neq 1}}^{n} |q_{ji}| \Big\} > 0 \text{ and } h = \min_{1 \leqslant i \leqslant n} \{h_{ii}\},$$

then every bounded solution of (E) *oscillates.*

P r o o f. Suppose that (E) has a nonoscillatory, bounded and eventually positive solution $y(t) = \big[y_1(t), \ldots, y_n(t)\big]^T$. There exists a $t_0 \geqslant 0$ such that $y_i(t) > 0$ for $t \geqslant t_0$, $i = 1, 2, \ldots, n$. If we let

$$w(t) = \sum_{j=1}^{n} y_j(t),$$

then

$$(-1)^{m+1}w^{(m)}(t) = -\sum_{i=1}^{n} q_{ii}y_i(t-h_{ii}) - \sum_{i=n}^{n}\sum_{\substack{j=1\\ j\neq 1}}^{n} q_{ij}y_j(t-h_{jj})$$

$$\leqslant -\sum_{i=1}^{n} q_{ii}y_i(t-h_{ii}) + \sum_{i=n}^{n}\sum_{\substack{j=1\\ j\neq 1}}^{n} |q_{ji}|y_i(t-h_{ii}).$$

It follows from the above inequality that

$$(-1)^{m+1}w^{(m)}(t) + \sum_{i=1}^{n}\Bigg[q_{ii} - \sum_{\substack{j=1\\ j\neq 1}}^{n}|q_{ji}|\Bigg]y_i(t-h_{ii}) \leqslant 0 \quad t \geqslant t_0,$$

or

(2) $$(-1)^{m+1}w^{(m)}(t) + q\sum_{i=1}^{n} y_i(t-h_{ii}) \leqslant 0, \quad t \geqslant t_0.$$

From the boundedness, nonoscillation and eventual positivity of $y_1(t),\dots,y_n(t)$, we see that $w(t)$ is bounded and eventually positive. From the fact that $(-1)^{m+1}w^{(m)}(t) \leqslant 0$ eventually and by the well-known Kiguradze's Lemma [4], the function $w(t)$ is eventually decreasing and satisfies

(3) $$(-1)^k w^{(k)}(t) > 0 \text{ eventually}, \quad k = 0,1,\dots,m.$$

Thus we conclude that $y_i(t)$ converges as $t \to \infty$, $i = 1,2,\dots,n$. We let

$$\lim_{t\to\infty} y_i(t) = b_i \geqslant 0, \quad i = 1,2,\dots,n.$$

We claim that $b_i = 0$, $i = 1,2,\dots,n$; suppose this is not the case. Then there exists a $t_1 > t_0 + h^*$, $h^* = \max\limits_{1\leqslant i\leqslant n}\{h_{ii}\}$, such that

$$y_i(t-h_{ii}) > \frac{1}{2}b_i \quad \text{for } t \geqslant t_1 + h^*.$$

We have from (1) that

$$(-1)^{m+1}w^{(m)}(t) + \tfrac{1}{2}q\sum_{i=1}^{n} b_i \leqslant 0,$$

or

$$(-1)^{m+1}w^{(m)}(t) \leqslant -\tfrac{1}{2}q\sum_{i=1}^{n} b_i \quad \text{for } t \geqslant t_1 + h^*,$$

which leads to

$$(-1)^{m+1}w^{(m-1)}(t) \leqslant -\tfrac{1}{2}q\sum_{i=1}^{n} b_i(t-t_1-h^*)(-1)^{m+1}w^{(m-1)}(t_1+h^*),$$

implying that $(-1)^{m+1}w^{(m-1)}(t)$ can become negative for all sufficiently large t; but this is impossible by the well-known lemma of Kiguradze [4]. Thus, we have $\sum\limits_{i=1}^{n} b_i = 0$, and hence $b_i = 0$, $i = 1,2,\ldots,n$; thus

$$\lim_{t\to\infty} y_i(t) = 0, \quad i = 1,2,\ldots,n,$$

and so $\lim\limits_{t\to\infty} w(t) = 0$.

Integrating (2) m-times from t to u, $u \geqslant t \geqslant t_2 + h$ for some $t_2 \geqslant t_1$, using (3) and letting $u \to \infty$, we obtain

$$\begin{aligned}
w(t) &\geqslant \int_t^\infty \frac{(s-t)^{m-1}}{(m-1)!} q\Big[\sum_{i=1}^{n} y_i(s-h_{ii})\Big]\,\mathrm{d}s = \sum_{i=1}^{n} q\int_t^\infty \frac{(s-t)^{m-1}}{(m-1)!} y_i(s-h_{ii})\,\mathrm{d}s\\
&= \sum_{i=1}^{n} q\int_{t-h_{ii}}^\infty \frac{(v+h_{ii}-t)^{m-1}}{(m-1)!} y_i(v)\,\mathrm{d}v \geqslant \sum_{i=1}^{n} q\int_{t-h}^\infty \frac{(v+h-t)^{m-1}}{(m-1)!} y_i(v)\,\mathrm{d}v\\
&= q\int_{t-h}^\infty \frac{(v+h-t)^{m-1}}{(m-1)!} \sum_{i=1}^{n} y_i(v)\,\mathrm{d}v = q\int_{t-h}^\infty \frac{(v+h-t)^{m-1}}{(m-1)!} w(v)\,\mathrm{d}v.
\end{aligned}$$

Now, we let

$$Z(t) = \int_{t-h}^\infty \frac{(v+h-t)^{m-1}}{(m-1)!} w(v)\,\mathrm{d}v,$$

and derive that

$$(-1)^{m+1}Z^{(m)}(t) = w(t) \leqslant -qZ(t-h) \quad \text{for } t \geqslant t_2 + h. \tag{4}$$

It follows from (4) that the function Z is a bounded and eventually positive solution of

$$(-1)^{m+1}Z^{(m)}(t) + qZ(t-h) \leqslant 0 \quad \text{for } t \geqslant t_2 + h.$$

However, by Corollary 1 of Philos [6], equation (E*) has a bounded and eventually positive solution, a contradiction. This completes the proof. □

Remark. From the results in [5], we see that all bounded solutions of (E*) are oscillatory if the following condition holds:

$$q^{1/m}(h/m)e > 1. \tag{5}$$

Now, we obtain the following oscillation criterion for all bounded solutions of (E).

Corollary 1. *Let* q_{ij}, h_{jj}, $i, j = 1, 2, \ldots, n$, q *and* h *be defined as in* (1). *If condition* (5) *is satisfied, then all bounded solutions of* (E) *are oscillatory.*

The following example is illustrative:

Example 1. Consider the system of equations

$$(\mathrm{E}_1) \qquad \begin{cases} (-1)^{m+1} y_1^{(m)}(t) + 2y_1(t - \frac{1}{2}m) - y_2(t - m) = 0, \\ (-1)^{m+1} y_2^{(m)}(t) - y_1(t - \frac{1}{2}m) + 2y_2(t - m) = 0. \end{cases}$$

All conditions of Corollary 1 are satisfied and hence all bounded solutions of (E_1) are oscillatory.

Next, we consider $(\mathrm{E_N}, \mathrm{a})$ and obtain the following results:

Theorem 2. *Let* $q_{ij} \in (-\infty, \infty)$, g, $h \in (0, \infty)$ *and* $c \in (0, 1)$, $i, j = 1, 2, \ldots, n$. *If every bounded solution of the equation*

$$(\mathrm{E}_1^*) \qquad (-1)^{m+1} v^{(m)}(t) + q(1 - c)v(t - h) = 0$$

is oscillatory, where q *is defined as in* (1), *then every bounded solution of* $(\mathrm{E_N}, -1)$ *is oscillatory.*

Theorem 3. *Let* $q_{ij} \in (-\infty, \infty)$, g, $h \in (0, \infty)$ *and* $c \in (1, \infty)$, $i, j = 1, 2, \ldots, n$. *If* $h > g$ *and every bounded solution of the equation*

$$(\mathrm{E}_2^*) \qquad (-1)^{m+1} u^{(m)}(t) + q\big((c - 1)/c^2\big)u\big(t - (h - g)\big) = 0$$

is oscillatory, where q *is defined as in* (1), *then every bounded solution of* $(\mathrm{E_N}, 1)$ *is oscillatory.*

Theorem 4. *Let* $q_{ij} \in (-\infty, \infty)$, g, $h \in (0, \infty)$ *and* $-c = c^* \in (0, 1]$, $i, j = 1, 2, \ldots, n$. *If every bounded solution of the equation*

$$(\mathrm{E}_3^*) \qquad (-1)^{m+1} w^{(m)}(t) + qw(t - h) = 0$$

is oscillatory, where q *is defined as in* (1), *then every bounded solution of* $(\mathrm{E_N}, 1)$ *is oscillatory.*

Proof of Theorems 2–4. Let $y(t) = \big[y_1(t), \ldots, y_n(t)\big]^T$ be a nonoscillatory bounded and eventually positive solution of $(\mathrm{E_N}, \mathrm{a})$. There exists a $t_0 \geqslant 0$ such that $y_i(t) > 0$ for $t > t_0$, $i = 1, 2, \ldots, n$. We let

$$(6) \qquad z(t) = \sum_{i=1}^{n} y_i(t) + c \sum_{i=1}^{n} y_i(t - ag)$$

and

$$w(t)=\sum_{i=1}^{n} y_i(t). \tag{7}$$

Then

$$(-1)^{m+1} z^{(m)}(t)+\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij} y_j(t-h)=0.$$

As in the proof of Theorem 1, we see that

$$(-1)^{m+1} z^{(m)}(t)+q w(t-h) \leqslant 0 \quad \text{for } t \geqslant t_0, \tag{8}$$

where q is defined as in (1). Clearly z and w are positive and bounded functions on $[t_0,\infty)$ provided $c>0$ and hence

$$(-1)^{m+1} z^{(m)}(t) \leqslant 0 \quad \text{for } t \geqslant t_0. \tag{9}$$

As in the proof of Theorem 1, one can easily derive that $\lim\limits_{t\to\infty} z(t)=0$ and $z(t)$ satisfies

$$(-1)^{k} z^{(k)}(t)>0 \quad \text{eventually, } k=0,1,\ldots,m. \tag{10}$$

Using this fact, we see form (6) that if $a=-1$ and $c\in(0,1)$, then

$$z(t)=w(t)+cw(t+g) \tag{11}$$

or

$$w(t)=z(t)-cw(t+g)=z(t)-cz(t+g)+cw(t+2g)\geqslant(1-c)z(t)$$

for all large t, and that if $a=1$ and $c\in(1,\infty)$, then

$$\begin{aligned} w(t)&=(1/c)[z(t+g)-w(t+g)]\\ &=(1/c)z(t+g)-(1/c^2)[z(t+2g)-w(t+2g)]\\ &\geqslant\big((c-1)/c^2\big)z(t+g) \end{aligned} \tag{12}$$

for all large t. Next, we consider the case when $a=1$ and $-c=c^*\in(0,1]$. Clearly $w(t)$ is bounded and eventually positive. Since (9) holds, we see that $z(t)$ is either eventually positive or else eventually negative. If $z(t)<0$ eventually, there is a sequence $\{t_k\}$ such that $\lim\limits_{k\to\infty} t_k=\infty$ and $\lim\limits_{k\to\infty} w(t_k)=\limsup\limits_{t\to\infty} w(t)$. Without loss of generality, we assume that $\{w(t_k-g)\}$ is convergent. Then

$$0>\lim_{k\to\infty} z(t_k)\geqslant\limsup_{t\to\infty} w(t)(1-c^*)\geqslant 0,$$

and hence we conclude that $z(t) < 0$ eventually is impossible. As in the proof of Theorem 1, we see that $\lim_{t\to\infty} z(t) = 0$, (10) holds eventually and

$$z(t) \leqslant w(t) \tag{13}$$

for all large t. Now, we have the following results:

(i) Suppose $a = -1$ and $c \in (0,1)$. From (8) and (11), we obtain

$$(-1)^{m+1} z^{(m)}(t) + q(1-c)z(t-h) \leqslant 0 \tag{14}$$

for all large t.

(ii) Suppose $a = 1$ and $c \in (1,\infty)$. From (8) and (12), we have

$$(-1)^{m+1} z^{(m)}(t) + q\big((c-1)/c^2\big)z\big((t-(h-g)\big) \leqslant 0 \tag{15}$$

for all large t.

(iii) Suppose $a = 1$ and $-c = c^* \in (0,1]$. From (8) and (13), we obtain

$$(-1)^{m+1} z^{(m)}(t) + qz(t-h) \leqslant 0 \tag{16}$$

for all large t.

The rest of the proof is similar to that of Theorem 1 and hence is omitted. $\square$

The following three corollaries are immediate.

Corollary 2. *Let $q_{ij} \in (-\infty,\infty)$, g, $h \in (0,\infty)$ and $c \in (0,1)$. If*

$$\big((1-c)q\big)^{1/m}(h/m)e > 1, \tag{17}$$

where q is defined as in (1), *then every bounded solution of* $(\mathrm{E_N}, 1)$ *is oscillatory.*

Corollary 3. *Let $q_{ij} \in (-\infty,\infty)$, h, $g \in (0,\infty)$ and $c \in (1,\infty)$. If $h > g$ and*

$$\Big(\big((c-1)/c^2\big)q\Big)^{1/m}\big((h-g)/m\big)e > 1, \tag{18}$$

where q is defined as in (1), *then every bounded solution of* $(\mathrm{E_n}, 1)$ *is oscillatory.*

Corollary 4. *Let $q_{ij} \in (-\infty,\infty)$, g, $h \in (0,\infty)$ and $-c = c^* \in (1,\infty]$. If*

$$q^{1/m}(h/m)e > 1, \tag{19}$$

where q is defined as in (1), *then every bounded solution of* $(\mathrm{E_N}, 1)$ *is oscillatory.*

The following example is illustrative:

Example 2. Consider the system of neutral equations

$$
\text{(E,a)} \qquad \begin{cases} (-1)^{m+1}\bigl(y_1(t)+cy_1(t-ag)\bigr)^{(m)}+2y_1(t-m)-y_2(t-m)=0,\\ (-1)^{m+1}\bigl(y_2(t)+cy_2(t-ag)\bigr)^{(m)}-y_1(t-m)+2y_2(t-m)=0, \end{cases}
$$

where $a=\pm 1$, g is a nonnegative real number.

One can easily conclude that the following assertions hold:

(I) If $c\in(0,1)$ and $(1-c)^{1/m}e>1$, then all conditions of Corollary 2 are satisfied and hence all bounded solutions of (E,-1) are oscillatory.

(II) If $c\in[1,\infty)$, $m>g$ and $\bigl((c-1)/c^2\bigr)^{1/m}\bigl((m-g)/m\bigr)e>1$, then all conditions of Corollary 3 are satisfied and hence all solutions of (E,1) are oscillatory.

(III) If $c\in[-1,0)$, then all conditions of Corollary 4 are satisfied, and hence all bounded solutions of (E,1) are oscillatory.

Remark. It would be interesting to obtain criteria similar to those presented here for nonautonomous systems (with variable coefficients and variable delays) and also to nonlinear systems.

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