Mirosława Zima

On the local spectral radius in partially ordered Banach spaces

Czechoslovak Mathematical Journal, Vol. 49 (1999), No. 4, 835–841

Persistent URL: `http://dml.cz/dmlcz/127533`

Czechoslovak Mathematical Journal, 49 (124) (1999), 835–841

ON THE LOCAL SPECTRAL RADIUS IN PARTIALLY ORDERED BANACH SPACES

Mirosława Zima, Rzeszów

(Received January 21, 1997)

Introduction

Let X denote a Banach space, let $A\colon X \to X$ be a linear bounded operator. By $r(A)$ and $r(A,x)$, where $x \in X$, we will denote the spectral radius of the operator A and the local spectral radius of A at x, respectively. Let us recall that

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$$

and

$$r(A,x) = \limsup_{n\to\infty} \|A^n x\|^{1/n}.$$

Some theorems on the properties of $r(A)$ and $r(A,x)$ can be found for instance in the papers [5, 8, 10]. Particularly, in the paper [5] Daneš proved that if linear bounded operators A and B are commutative then for every $x \in X$

$$r(A+B,x) \leqslant r(A,x) + r(B)$$

and

$$r(AB,x) \leqslant r(A,x)r(B).$$

One can easily show that if the assumption of commutativity is not satisfied then the above inequalities need not hold. The natural question arises wether it can be weakened. For results of this type, concerning the spectral radius, we refer the reader to [6, 13]. In the present paper we give a few lemmas on estimations of $r(A+B,x)$ and $r(AB,x)$, in which the commutativity is replaced by other conditions. The applications of these lemmas to the differential- functional equations of neutral type are also given.

1. Lemmas on the local spectral radius

In what follows, by K we will denote a cone in the Banach space X. Recall that a subset K of X is called a cone if K is convex, closed, $K \neq \{\theta\}$ and if $x \in K$ and $-x \in K$ then $x = \theta$.

Lemma 1. *Assume that K is a normal cone in the Banach space X, i.e. there exists $M > 0$ such that for all $x, y \in X$ if $\theta \prec x \prec y$ then $\|x\| \leqslant M\|y\|$, where $\prec$ denotes the partial order relation given by K (see* [9]*). Moreover, let $A, B\colon X \to X$ be linear bounded and positive operators. If $ABx \prec BAx$ for every $x \in K$ then:*

1) $r(AB, x) \leqslant r(A, x)r(B)$,
2) $r(A + B, x) \leqslant r(A, x) + r(B)$,

for each $x \in K$.

Proof. 1) Since A and B are positive, it follows that they are increasing and $A(K)$, $B(K) \subset K$. Hence for every $x \in K$ we obtain

$$(AB)^2x = AB(ABx) \prec AB(BAx) \prec (BA)^2x \prec B^2A^2x.$$

So $(AB)^2x \prec B^2A^2x$. In the same manner we can prove that for every $n \in \mathbb{N}$ and $x \in K$,

$$(AB)^nx \prec B^nA^nx.$$

Hence

$$\|(AB)^nx\| \leqslant M\|B^n\|\|A^nx\|,$$

which gives $r(AB, x) \leqslant r(A, x)r(B)$.

2) It is easy to check that for every $n \in \mathbb{N}$ and $x \in K$ we have

$$(A + B)^nx \prec \sum_{i=0}^{n} \binom{n}{i} B^{n-i}A^ix.$$

Consequently,

$$\|(A + B)^nx\| \leqslant M \sum_{i=0}^{n} \binom{n}{i} \|B^{n-i}\|\|A^ix\|.$$

The next step of the proof is a modification of those given in [5] or [13], so it can be omitted. □

Remark. If we assume additionally that K is a generating cone, that is $X = K - K$ (see [9]) then in view of Lemma 2 [5] we have

$$r(AB) = \max\{r(AB, x)\colon\ x \in K\}$$

and

$$r(A+B) = \max\{r(A+B,x)\colon\ x \in K\}.$$

Thus we obtain the following corollary for the spectral radius.

Corollary. *If the assumptions of Lemma 1 are satisfied and the cone K is generating then $r(AB) \leqslant r(A)r(B)$ and $r(A+B) \leqslant r(A) + r(B)$.*

In a similar way we can prove the following two lemmas which can be useful in applications.

Lemma 2. *Assume that K is a normal cone, the operators $A, B\colon\ X \to X$ are linear bounded and positive and there exists $c > 0$ such that $ABx \prec cBAx$ for every $x \in K$. Then $r(AB, x) \leqslant cr(BA, x)$, $x \in K$.*

Lemma 3. *Suppose that K is a normal cone, the operators $A, B\colon\ X \to X$ are linear bounded and positive and there exists $c > 1$ such that $A^nBx \prec cBA^nx$ for every $n \in \mathbb{N}$ and $x \in K$. Then $r(AB, x) \leqslant cr(A, x)r(B)$ and $r(A+B, x) \leqslant c[r(A, x) + r(B)]$, $x \in K$.*

The proofs are straightforward.

2. Applications of Lemma 1

Now we give two examples of application of Lemma 1 to the problems of neutral type for differential-functional equations. In what follows we will need the fixed point theorem from the paper [12].

Let $(X, \|\cdot\|, \prec, m)$ be a Banach space with a binary relation $\prec$ and a mapping $m\colon\ X \to X$. Assume that:

(i) the relation $\prec$ is transitive,
(ii) the norm $\|\cdot\|$ is monotonic,
(iii) $\theta \prec m(x)$ and $\|m(x)\| = \|x\|$ for every $x \in X$.

Proposition [12]. *In the Banach space considered above, let the operators $\mathcal{A}\colon\ X \to X$, $A\colon\ X \to X$ be given with the following properties:*

(iv) *A is a linear bounded and positive operator with $r(A) < 1$,*
(v) *$m(\mathcal{A}x - \mathcal{A}y) \prec Am(x-y)$ for all $x, y \in X$.*

Then the equation $\mathcal{A}x = x$ has a unique solution in X.

Similar theorems can be found in [11].

Consider the following initial value problem of neutral type

$$x'(t) = f(t, x(h(t)), x'(H(t))),\ t \in [0,T], \tag{1}$$

$$x(0) = 0. \tag{2}$$

By a solution of the problem (1)–(2) we mean a function which is absolutely continuous on $[0,T]$ (AC for short), satisfies the equation (1) a.e. on $[0,T]$ and the initial condition (2). Similar problems have been considered for example in the papers [1], [2] under the assumption of continuity of the function f, and in [7] with f satisfying a sort of Carathéodory type conditions. Suppose that:

1° $h\colon [0,T] \to [0,T]$ is a continuous function,

2° $H\colon [0,T] \to [0,T]$ is a monotonic and AC-function such that $0 < \varepsilon \leqslant |H'(t)| \leqslant 1$ a.e. on $[0,T]$ and $H^{-1}([0,h(H(t))]) \subset [0,h(t)]$,

3° $(t,x,y) \to f(t,x,y)$ is a real function defined on the set $[0,T] \times \mathbb{R}^2$, Lebesgue measurable with respect to t for all $(x,y) \in \mathbb{R}^2$ and satisfying the Lipschitz condition

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leqslant L_1|x_1 - x_2| + L_2|y_1 - y_2|$$

for all $(t,x_1,y_1), (t,x_2,y_2) \in [0,T] \times \mathbb{R}^2$, where $L_1, L_2 > 0$,

4° $r(A_1) + L_2\overline{M} < 1$, where $\overline{M} = (\min\limits_{[0,T]} |H'(t)|)^{-1}$ and $r(A_1)$ denotes the spectral radius of the operator $(A_1x)(t) = L_1 \int_0^{h(t)} x(s)\,\mathrm{d}s,\ t \in [0,T]$, in the space $L^1[0,T]$,

5° there exists a function $g\colon [0,T] \to \mathbb{R}_+$ Lebesgue integrable on $[0,T]$ and such that $|f(t,0,0)| \leqslant g(t)$ a.e. on $[0,T]$.

Theorem 1. *Under the assumptions* 1°–5° *the problem* (1)–(2) *has a unique solution defined on* $[0,T]$.

Proof. To prove the theorem we apply Proposition. In our case $X = L^1[0,T]$, $m(x)(t) = |x(t)|$. We say that $x \prec y$ if and only if $x(t) \leqslant y(t)$ a.e. on $[0,T]$. It is easy to verify that the problem (1)–(2) is equivalent to the integral-functional equation

$$z(t) = f\left(t, \int_0^{h(t)} z(s)\,\mathrm{d}s, z(H(t))\right),\ t \in [0,T],$$

where $x(t) = \int_0^t z(s)\,\mathrm{d}s$.

Consider the operator

$$(\mathcal{A}z)(t) = f\left(t, \int_0^{h(t)} z(s)\,\mathrm{d}s, z(H(t))\right),\ t \in [0,T],$$

where $z \in L^1[0,T]$. Obviously, in view of 1°–3° and 5°, the operator $\mathcal{A}$ maps $L^1[0,T]$ into itself. Moreover, by 3°, we have for all $z, w \in L^1[0,T]$

$$|(\mathcal{A}z)(t) - (\mathcal{A}w)(t)| \leqslant L_1 \int_0^{h(t)} |z(s) - w(s)| \,\mathrm{d}s + L_2 |z(H(t)) - w(H(t))|.$$

Hence

$$|(\mathcal{A}z)(t) - (\mathcal{A}w)(t)| \leqslant (A_1 + A_2)(|z - w|)(t), \tag{3}$$

where

$$(A_1 z)(t) = L_1 \int_0^{h(t)} z(s) \,\mathrm{d}s, \ \ (A_2 z)(t) = L_2 z(H(t)).$$

In view of 1° and 2°, $A_1 + A_2$ is a linear bounded and positive operator mapping $L^1[0,T]$ into itself. The inequality (3) means that the assumption (v) is satisfied. It remains to prove that $r(A_1 + A_2) < 1$. To show this we apply Lemma 1. Let $K = \{x \in L^1[0,T] \colon x(t) \geqslant 0 \text{ a.e. on } [0,T]\}$. Obviously, the cone K is normal. Notice that the relation given by K is the same as the one defined at the beginning of our proof. By 2° and the theorem on integration by substitution for the Lebesgue integral we obtain for $z \in K$

$$(A_2 A_1 z)(t) = L_1 L_2 \int_0^{h(H(t))} z(s) \,\mathrm{d}s \leqslant L_1 L_2 \int_0^{h(t)} z(H(s)) \,\mathrm{d}s = (A_1 A_2 z)(t)$$

a.e. on $[0,T]$, which means that $A_2 A_1 z \prec A_1 A_2 z$ for every $z \in K$. Therefore in virtue of Lemma 1

$$r(A_1 + A_2, z) \leqslant r(A_1) + r(A_2, z), \ \ z \in K.$$

Since $r(A_2, z) \leqslant L_2 \overline{M}$ for every $z \in K$ and K is a generating cone, we have

$$r(A_1 + A_2) \leqslant r(A_1) + L_2 \overline{M}.$$

By 4° we obtain $r(A_1 + A_2) < 1$, which completes the proof. □

Next consider the Darboux problem of neutral type

$$z_{xy} = f(x, y, z(h(x,y)), z_{xy}(H(x,y))), \ \ (x,y) \in I^2, \tag{4}$$

$$z(x,0) = 0, \ \ x \in I, \ \ z(0,y) = 0, \ \ y \in I, \tag{5}$$

where $I = [0,T]$.

By the solution of (4)–(5) we mean a function $z\colon I^2 \to \mathbb{R}$ such that $z(x,y)$ is an AC-function with respect to x and y, z_x is an AC-function with respect to y for a.e. $x \in I$, z_y is an AC-function with respect to x for a.e. $y \in I$, $z_{xy} = f(x,y,z(h(x,y)), z_{xy}(H(x,y)))$ a.e. on I^2, $z(x,0) = 0$ for $x \in I$ and $z(0,y) = 0$ for $y \in I$ (see [2]). The following result for (4)–(5) may be proved in much the same way as Theorem 1.

Theorem 2. *Assume that:*

6° $h\colon I^2 \to I^2$ *is a continuous function,*

7° $U, V \subset \mathbb{R}^2$ *are any open sets such that* $I^2 \subset U$, $I^2 \subset V$, $H\colon U \to V$ *is a diffeomorphism with the property* $H(I^2) \subset I^2$,

8° $|H'(x,y)| \leqslant 1$ *a.e. on* I^2, *where* $H'(x,y)$ *denotes the Jacobian of* H, *and* $D(h(H(x,y)) \subset H(D(h(x,y)))$ *for* $(x,y) \in I^2$, *where* $D(x,y) = \{(t,s) \in I^2\colon\ 0 \leqslant t \leqslant x,\ 0 \leqslant s \leqslant y\}$,

9° $(x,y,w,z) \to f(x,y,w,z)$ *is a real function defined on the set* $I^2 \times \mathbb{R}^2$, *Lebesgue measurable with respect to* (x,y) *for every* $(w,z) \in \mathbb{R}^2$ *and satisfying the Lipschitz condition*

$$|f(x,y,w_1,z_1) - f(x,y,w_2,z_2)| \leqslant L_1|w_1 - w_2| + L_2|z_1 - z_2|$$

for all $(x,y,w_1,z_1), (x,y,w_2,z_2) \in I^2 \times \mathbb{R}^2$, *where* $L_1, L_2 > 0$,

10° $r(A_1) + L_2\tilde{M} < 1$, *where* $\tilde{M} = (\min_{I^2} |H'(x,y)|)^{-1}$ *and* $r(A_1)$ *denotes the spectral radius of the operator* $(A_1 z)(x,y) = \int_0^{h(x,y)} z(t,s)\,\mathrm{d}t\,\mathrm{d}s$ *in the space* $L^1(I^2)$,

11° *there exists a function* $g\colon I^2 \to \mathbb{R}_+$ *which is Lebesgue integrable and* $|f(x,y,0,0)| \leqslant g(x,y)$ *a.e. on* I^2.

Then the problem (4)–(5) *has a unique solution defined on* I^2.

Remark. The problem (4)–(5) was considered for example in the papers [3, 4]. Particularly, in [4] we applied the Banach fixed point theorem to prove the existence and uniqueness of a solution of (4)–(5) in the space $L^1(I^2)$. To obtain the uniqueness we assumed that $L_1T^2 + L_2\tilde{M} < 1$, which is in general a more restrictive condition than 10°. For example, if $h(x,y) \leqslant (x,y)$ a.e. on I^2, that is $h_1(x,y) \leqslant x$ and $h_2(x,y) \leqslant y$ a.e. on I^2, where $h(x,y) = (h_1(x,y), h_2(x,y))$, we have $r(A_1) = 0$ and the assumption 10° becomes $L_2\tilde{M} < 1$.

References

[1] *A. Augustynowicz, M. Kwapisz*: On a numerical-analytic method of solving of boundary value problem for functional differential equation of neutral type. Math. Nachr. *145* (1990), 255–269.
[2] *J. Banaś*: Applications of measures of noncompactness to various problems. Folia Scientiarum Universitatis Technicae Resoviensis *34* (1987).
[3] *D. Bugajewski*: On some applications of theorems on the spectral radius to differential equations. J. Anal. Appl. *16* (1997), 479–484.
[4] *D. Bugajewski, M. Zima*: On the Darboux problem of neutral type. Bull. Austral. Math. Soc. *54* (1996), 451–458.
[5] *J. Daneš*: On local spectral radius. Čas. pěst. mat. *112* (1987), 177–187.
[6] *A. R. Esayan*: On the estimation of the spectral radius of the sum of positive semicommutative operators (in Russian). Sib. Mat. Zhur. *7*, 460–464.
[7] *L. Faina*: Existence and continuous dependence for a class of neutral functional differential equations. Ann. Polon. Math. *64* (1996), 215–226.
[8] *K.-H. Förster, B. Nagy*: On the local spectral radius of a nonnegative element with respect to an irreducible operator. Acta Sci. Math. *55* (1991), 155–166.
[9] *M. A. Krasnoselski et al.*: Approximate solutions of operator equations. Noordhoff, Groningen, 1972.
[10] *V. Müller*: Local spectral radius formula for operators in Banach spaces. Czechoslovak Math. J. *38* (1988), 726–729.
[11] *P. P. Zabrejko*: The contraction mapping principle in K-metric and locally convex spaces (in Russian). Dokl. Akad. Nauk BSSR *34* (1990), 1065–1068.
[12] *M. Zima*: A certain fixed point theorem and its applications to integral-functional equations. Bull. Austral. Math. Soc. *46* (1992), 179–186.
[13] *M. Zima*: A theorem on the spectral radius of the sum of two operators and its applications. Bull. Austral. Math. Soc. *48* (1993), 427–434.

Author's address: Department of Mathematics, Pedagogical University of Rzeszów, Rejtana 16 A, 35-310 Rzeszów, Poland.