Kamil John; Dirk Werner

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M-IDEALS OF COMPACT OPERATORS INTO ℓ_p

Kamil John*, Praha and Dirk Werner, Berlin

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Abstract. We show for $2 \leqslant p < \infty$ and subspaces X of quotients of L_p with a 1-unconditional finite-dimensional Schauder decomposition that $K(X, \ell_p)$ is an M-ideal in $L(X, \ell_p)$.

1. Introduction

A closed subspace J of a Banach space X is called an M-ideal if the dual space X^* decomposes into an ℓ_1-direct sum $X^* = J^\perp \oplus_1 V$, where $J^\perp = \{x^* \in X^*\colon\ x^*\big|_J = 0\}$ is the annihilator of J and V is some closed subspace of X^*. This notion is due to Alfsen and Effros [1], and it is studied in detail in [4].

It has long been known that the space of compact operators $K(\ell_p)$ is an M-ideal in the space of bounded operators $L(\ell_p)$ for $1 < p < \infty$ whereas this property fails for $L_p = L_p[0,1]$ unless $p = 2$; cf. Section VI.4 in [4]. More recently, it was shown in [6] that $K(L_p, \ell_p)$ is an M-ideal if $1 < p \leqslant 2$, and it is not an M-ideal if $p > 2$.

In this paper we wish to examine the M-ideal character of $K(X, \ell_p)$ for subspaces X of quotients of L_p and $2 \leqslant p < \infty$. Our idea is to exploit the fact that those X have Rademacher cotype p with constant 1. This leads to the result mentioned in the abstract.

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2. Results

Here is our main result.

Theorem 2.1. *Let $1 < p < \infty$ and suppose that the Banach space X admits a sequence of operators $K_n \in K(X)$ satisfying*
(a) $K_n x \to x$ *for all* $x \in X$,
(b) $K_n^* x^* \to x^*$ *for all* $x^* \in X^*$,
(c) $\|Id_X - 2K_n\| \to 1$.
Then $K(X, \ell_p)$ is an M-ideal in $L(X, \ell_p)$ if

$$\limsup_n (\|x\|^p + \|x_n\|^p)^{1/p} \leqslant \limsup_n \left(\frac{\|x + x_n\|^p + \|x - x_n\|^p}{2} \right)^{1/p} \tag{2.1}$$

for all $x, x_n \in X$ such that $x_n \to 0$ weakly.

Proof. Let $T\colon X \to \ell_p$ be a contraction. We shall show that T has property (M), i.e.,

$$\limsup_n \|y + Tx_n\| \leqslant \limsup_n \|x + x_n\|$$

whenever $x \in X$, $y \in \ell_p$, $\|y\| \leqslant \|x\|$, and $x_n \to 0$ weakly in X. This implies our claim by [6, Th. 6.3].

In fact, we have

$$\begin{aligned} \limsup_n \|y + Tx_n\| &= \limsup_n \big(\|y\|^p + \|Tx_n\|^p\big)^{1/p} \\ &\leqslant \limsup_n \big(\|x\|^p + \|x_n\|^p\big)^{1/p} \\ &\leqslant \limsup_n \left(\frac{\|x + x_n\|^p + \|x - x_n\|^p}{2} \right)^{1/p}; \end{aligned}$$

so it is enough to show that

$$\limsup_n \|x + x_n\| = \limsup_n \|x - x_n\|. \tag{2.2}$$

Let $\varepsilon > 0$. Pick $m \in \mathbb{N}$ so that

$$\|K_m x - x\| \leqslant \varepsilon, \qquad \|Id - 2K_m\| \leqslant 1 + \varepsilon.$$

Then pick $n_0 \in \mathbb{N}$ so that

$$\|K_m x_n\| \leqslant \varepsilon \qquad \forall n_0;$$

this is possible since $x_n \to 0$ weakly and K_m is compact. We now have for $n \geqslant n_0$

$$
\begin{aligned}
(1+\varepsilon)\|x_n + x\| &\geqslant \|(Id - 2K_m)(x_n + x)\| \\
&= \|x_n - x - 2K_m x_n + 2x - 2K_m x\| \\
&\geqslant \|x_n - x\| - 2\varepsilon - 2\varepsilon
\end{aligned}
$$

so that

$$\limsup_n \|x_n + x\| \geqslant \limsup_n \|x_n - x\|,$$

and by symmetry equality holds. $\square$

We note that (2.1) is not a necessary condition, for essentially trivial reasons: e.g., if $p < 2$ and $X = \ell_2$, then every operator from X to ℓ_p is compact and, therefore, $K(X, \ell_p)$ is an M-ideal, but (2.1) fails.

As the proof shows, one can as well consider all the Banach spaces sharing the property

$$\limsup_n \|y + y_n\| \leqslant \limsup_n \big(\|y\|^p + \|y_n\|^p\big)^{1/p}$$

whenever $y_n \to 0$ weakly, e.g., ℓ_q or the Lorentz spaces $d(w, q)$ for $p \leqslant q < \infty$. So our theorem is closely related to [10, Th. 3] and [11, Prop. 4.2]. Actually, we needed assumptions (a)–(c) only to ensure (2.2), a condition that could be called property (wM) in accordance with Lima's property (wM^*) [7].

Now we wish to give more concrete examples where Theorem 2.1 applies. There is a natural class of Banach spaces in which inequality (2.1) is valid. Recall that a Banach space X has Rademacher type p with constant C if for all finite families $\{x_1, \ldots, x_n\} \subset X$, with $r_1, r_2, \ldots$ denoting the Rademacher functions,

$$\left(\int_0^1 \left\|\sum_{k=1}^n r_k(t) x_k\right\|^p \mathrm{d}t\right)^{1/p} \leqslant C\left(\sum_{k=1}^n \|x_k\|^p\right)^{1/p};$$

it has Rademacher cotype p with constant C if

$$\left(\sum_{k=1}^n \|x_k\|^p\right)^{1/p} \leqslant C\left(\int_0^1 \left\|\sum_{k=1}^n r_k(t) x_k\right\|^p \mathrm{d}t\right)^{1/p}$$

instead. Thus we see that the inequality (2.1) is always satisfied when X has Rademacher cotype p with constant 1, which is the case if X is a subspace of a quotient of L_p for $2 \leqslant p < \infty$. As for assumptions (a)–(c) from Theorem 2.1, these conditions are obviously fulfilled if X has a shrinking 1-unconditional finite-dimensional Schauder decomposition or merely the shrinking unconditional metric compact approximation property of [2] and [3]. Let us mention that the "shrinking" character of these properties holds, by a well-known convex combinations argument (cf. [4, Lemma VI.4.9]), for reflexive spaces automatically. These observations yield the next corollary.

Corollary 2.2. *Let X be a subspace of a quotient of L_p, $2 \leqslant p < \infty$, and let X have a 1-unconditional finite-dimensional Schauder decomposition or merely the unconditional metric compact approximation property. Then $K(X, \ell_p)$ is an M-ideal in $L(X, \ell_p)$.*

More explicitly, we note that for instance ℓ_p, $\ell_p \oplus_p \ell_r$ and $\ell_p(\ell_r)$, where $2 \leqslant r \leqslant p < \infty$, satisfy these assumptions; but for these spaces the result of Corollary 2.2 has already been known from [11] or [4, p. 327]. Yet there are other examples. In fact, Li [8] has exhibited spaces of Λ-spectral functions $L^p_\Lambda(\mathbb{T})$ for certain $\Lambda \subset \mathbb{Z}$ that enjoy the unconditional metric compact approximation property. Moreover, since for $2 \leqslant q \leqslant p < \infty$ the space L_q is isometric to a quotient of L_p, one can substitute q for p in the above list of examples.

Another way to see that (2.1) holds for L_p, $2 \leqslant p < \infty$, is to observe that (2.1) follows immediately from Clarkson's inequality in L_p, that is

$$\|f\|^p + \|g\|^p \leqslant \frac{\|f+g\|^p + \|f-g\|^p}{2}$$

for $p \geqslant 2$. Now, Clarkson's inequalities are valid in the Schatten classes as well [9]. Therefore we obtain a noncommutative version of the previous corollary. (Actually, this argument is not that different, because the Clarkson inequality entails the desired cotype property.)

Corollary 2.3. *Let X be a subspace of a quotient of the Schatten class c_p, $2 \leqslant p < \infty$, and let X have a 1-unconditional finite-dimensional Schauder decomposition or merely the unconditional metric compact approximation property. Then $K(X, \ell_p)$ is an M-ideal in $L(X, \ell_p)$.*

There is a dual version of Theorem 2.1 which we state for completeness.

Theorem 2.4. *Let $1 < p < \infty$ and $1/p + 1/p' = 1$. Suppose that the Banach space Y admits a sequence of operators $K_n \in K(Y)$ satisfying*

(a) $K_n y \to y$ *for all* $y \in Y$,
(b) $K_n^* y^* \to y^*$ *for all* $y^* \in Y^*$,
(c) $\|Id_Y - 2K_n\| \to 1$.

Then $K(\ell_p, Y)$ is an M-ideal in $L(\ell_p, Y)$ if

$$\limsup_n (\|y^*\|^{p'} + \|y_n^*\|^{p'})^{1/p'} \leqslant \limsup_n \left(\frac{\|y^* + y_n^*\|^{p'} + \|y^* - y_n^*\|^{p'}}{2} \right)^{1/p'} \tag{2.3}$$

for all $y^*, y_n^* \in Y^*$ such that $y_n^* \to 0$ weak*.

The proof of Theorem 2.4 can be accomplished along the same lines as above using property (M^*) of a contraction (cf. [6, p. 171] instead.

Again, inequality (2.3) is always satisfied when Y^* has Rademacher cotype p' with constant 1, which is the case if Y has Rademacher type p with constant 1. The latter holds if Y is a subspace of a quotient of L_p or c_p for $1 < p \leqslant 2$.

3. Concluding remarks

The conditions (2.1) and (2.3) can be understood as averaging conditions. In an earlier draft of this manuscript we used these conditions to establish what we call p-averaged versions of the properties (M) and (M^*) of contractions T, that is

$$\limsup_n \|y + Tx_n\| \leqslant \begin{cases} \limsup_n \left(\frac{\|x + x_n\|^p + \|x - x_n\|^p}{2} \right)^{1/p} & \text{for } p < \infty \\ \limsup_n \max(\|x + x_n\|, \|x - x_n\|) & \text{for } p = \infty \end{cases}$$

whenever $x \in X$, $y \in Y$ with $\|y\| \leqslant \|x\|$ and $x_n \to 0$ weakly in X; respectively,

$$\limsup_n \|x^* + T^* y_n^*\| \leqslant \begin{cases} \limsup_n \left(\frac{\|y^* + y_n^*\|^p + \|y^* - y_n^*\|^p}{2} \right)^{1/p} & \text{for } p < \infty \\ \limsup_n \max(\|y^* + y_n^*\|, \|y^* - y_n^*\|) & \text{for } p = \infty \end{cases}$$

for all $x^* \in X^*$, $y^* \in Y^*$ such that $\|x^*\| \leqslant \|y^*\|$ and for all weak* null sequences $(y_n^*) \subset Y^*$. (As a matter of fact, (2.3) implies the p'-averaged property (M^*) for a contraction $T\colon \ell_p \to Y$.) Using techniques from [6] (which in turn depend on those from [5]) one can prove the following results.

Proposition 3.1. *Let $1 \leqslant p \leqslant \infty$ and suppose that the Banach space X admits a sequence of operators $K_n \in K(X)$ satisfying*

(a) $K_n x \to x$ *for all $x \in X$,*
(b) $K_n^* x^* \to x^*$ *for all $x^* \in X^*$,*
(c) $\|Id_X - 2K_n\| \to 1$.

Let Y be a Banach space. Then $K(X, Y)$ is an M-ideal in $L(X, Y)$ if and only if every contraction $T\colon X \to Y$ has p-averaged (M).

Proposition 3.2. *Let $1 \leqslant p \leqslant \infty$ and suppose that the Banach space Y admits a sequence of operators $K_n \in K(Y)$ satisfying*
(a) $K_n y \to y$ *for all* $y \in Y$,
(b) $K_n^* y^* \to y^*$ *for all* $y^* \in Y^*$,
(c) $\|Id_Y - 2K_n\| \to 1$.
Let X be a Banach space. Then $K(X,Y)$ is an M-ideal in $L(X,Y)$ if and only if every contraction $T\colon X \to Y$ has p-averaged (M^*).

It is well known (cf. [4, Th. I.2.2]) that a closed subspace J of a Banach space X is an M-ideal in X if and only if the following 3-ball property holds: For all $y_1, y_2, y_3 \in B_J$, all $x \in B_X$ and all $\varepsilon > 0$ there is $y \in J$ such that $\|x + y_i - y\| \leqslant 1 + \varepsilon$ for $i = 1, 2, 3$. (Here B_X denotes the closed unit ball of X.) Upon replacing the number 3 by some $n \in \mathbb{N}$ we obtain the n-ball property, which is equivalent to the 3-ball property provided $n \geqslant 3$. One may "average" this condition as well and obtain the following characterisation of M-ideals by means of an averaged 3-ball property.

Proposition 3.3. *A closed subspace J of a Banach space X is an M-ideal in X if and only if*

(A) *For all $y_1, y_2, y_3 \in B_J, x \in B_X$ and $\varepsilon > 0$ there is $y \in J$ such that*
$$\|x + y_i - y\| + \|x - y_i - y\| \leqslant 2(1 + \varepsilon) \quad \text{for } i = 1, 2, 3$$

holds.

Proof. Evidently the 6-ball property implies (A). Conversely, suppose (A). In order to show that J is an M-ideal in X we will verify the ordinary 3-ball property (see above). Now an inspection of the proof of [4, Theorem I.2.2] shows that one may additionally assume that $\operatorname{dist}(x, J) \geqslant 1 - \varepsilon$, in which case (A) implies that

$$\|x + y_i - y\| \leqslant 2(1 + \varepsilon) - \|x - y_i - y\| \leqslant 1 + 3\varepsilon, \qquad i = 1, 2, 3,$$

and we are done. □

References

[1] *E. M. Alfsen and E. G. Effros*: Structure in real Banach spaces. Parts I and II. Ann. of Math. *96* (1972), 98–173.
[2] *P. G. Casazza and N. J. Kalton*: Notes on approximation properties in separable Banach spaces. Geometry of Banach Spaces, Proc. Conf. Strobl 1989 (P. F. X. Müller and W. Schachermayer, eds.). London Mathematical Society Lecture Note Series 158, Cambridge University Press, 1990, pp. 49–63.
[3] *G. Godefroy, N. J. Kalton, and P. D. Saphar*: Unconditional ideals in Banach spaces. Studia Math. *104* (1993), 13–59.

[4] *P. Harmand, D. Werner, and W. Werner*: M-Ideals in Banach Spaces and Banach Algebras. Lecture Notes in Math. 1547. Springer, Berlin-Heidelberg-New York, 1993.
[5] *N. J. Kalton*: M-ideals of compact operators. Illinois J. Math. *37* (1993), 147–169.
[6] *N. J. Kalton and D. Werner*: Property (M), M-ideals and almost isometric structure of Banach spaces. J. Reine Angew. Math. *461* (1995), 137–178.
[7] *A. Lima*: Property (wM^*) and the unconditional metric compact approximation property. Studia Math. *113* (1995), 249–263.
[8] *D. Li*: Complex unconditional metric approximation property for $C_\Lambda(\mathbf{T})$ spaces. Preprint (1995).
[9] *Ch. A. McCarthy*: c_p. Israel J. Math. *5* (1967), 249–271.
[10] *E. Oja*: Dual de l'espace des opérateurs linéaires continus. C. R. Acad. Sc. Paris, Sér. A *309* (1989), 983–986.
[11] *D. Werner*: New classes of Banach spaces which are M-ideals in their biduals. Math. Proc. Cambridge Phil. Soc. *111* (1992), 337–354.

Authors' addresses: K. John, Mathematical Institute, Czech Academy of Sciences, Žitná 25, CZ-115 67 Prague 1, Czech Republic, e-mail: kjohn@math.cas.cz; D. Werner, I. Mathematisches Institut, Freie Universität Berlin, Arnimallee 2–6, D-14 195 Berlin, Germany, e-mail: werner@math.fu-berlin.de.