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STABILITY OF GLOBAL SOLUTIONS TO ONE-PHASE STEFAN PROBLEM FOR A SEMILINEAR PARABOLIC EQUATION

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0. Introduction

We consider the following one-phase Stefan problem $SP := SP(u_0, \ell_0)$ for a semilinear parabolic equation in the one-dimensional space: Find a curve (a free boundary) $x = \ell(t) > 0$ on $[0, T]$, $0 < T < \infty$, and a function $u = u(t, x)$ on $Q(T) := (0, T) \times (0, \infty)$ satisfying

$$(0.1) \qquad u_t = u_{xx} + u^{1+\alpha} \quad \text{in } Q_\ell(T) := \{(t, x);\ 0 < t < T,\ 0 < x < \ell(t)\},$$

$$(0.2) \qquad u(0, x) = u_0(x) \quad \text{for } 0 \leqslant x \leqslant \ell_0,$$

$$(0.3) \qquad u_x(t, 0) = 0 \quad \text{for } 0 < t < T,$$

$$(0.4) \qquad u(t, x) = 0 \quad \text{for } 0 < t < T \text{ and } x \geqslant \ell(t),$$

$$(0.5) \qquad \ell'(t) = -u_x(t, \ell(t)) \quad \text{for } 0 < t < T,$$

$$(0.6) \qquad \ell(0) = \ell_0,$$

where α and ℓ_0 are given positive constants and u_0 is a given initial function on $[0, \ell_0]$.

The local existence and the uniqueness for solutions to the above problem SP were already investigated by Fasano-Primicerio [7] and Aiki-Kenmochi [1, 5, 8]. Since there are blow-up solutions of the usual initial boundary value problem for the semilinear equation (0.1) in a bounded domain, by using comparison principle it is clear that SP has blow-up solutions for a large initial data. In previous works [2, 3, 6] we showed some theorems and numerical experiments concerned with the behavior of free boundaries of blow-up solutions to one-phase Stefan problems with homogeneous Neumann and Dirichlet boundary conditions. On global existence (see Theorem 1.2) we obtained in [4] a solution to the problem SP on $[0, \infty)$, an exponential decay of

$|u|_{L^\infty(0,\ell(t))}$ and boundedness of the free boundary ℓ for a small initial function u_0 in the case $\alpha > 1$.

The purpose of the present paper is to establish stability of a global solution to the problem SP in the following sense: Let $\alpha > 1$ and let $\{u, \ell\}$ be a solution to SP on $[0, \infty)$ such that there are positive constants L, M and μ such that

$$\ell(t) \leqslant L \text{ for } t \geqslant 0 \text{ and } |u(t,x)| \leqslant M \exp(-\mu t) \text{ for } t \geqslant 0 \text{ and } x \geqslant 0.$$

Then there exists a positive constant δ such that if $|u_0 - \hat{u}_0|_{L^p(0,\infty)} < \delta$, where $p > 1$ is a suitable constant, the problem $SP(\hat{u}_0, \hat{\ell}_0)$ has a solution $\{\hat{u}, \hat{\ell}\}$ on $[0, \infty)$ such that the free boundary $\{\hat{\ell}(t)\}$ is bounded and $|\hat{u}(t)|_{L^\infty(0,\hat{\ell}(t))}$ decays in exponential order. We note that the global existence and stability concerned with the problem SP are not proved, theoretically, for $0 < \alpha \leqslant 1$.

1. The main result

We give a precise definition of a solution to SP.

Definition 1.1. We say that a pair $\{u, \ell\}$ is a solution of $SP(u_0, \ell_0)$ on $[0, T]$, $0 < T < \infty$, if the following conditions are fulfilled:

(S1) $u \in W^{1,2}(0, T; L^2(0, \ell(t))) \cap L^\infty(0, T; W^{1,2}(0, \ell(t)))$, and $\ell \in W^{1,2}(0, T)$ with $0 < \ell$ on $[0, T]$.

(S2) (0.1) holds in the sense of $\mathcal{D}'(Q_\ell(T))$ and (0.2) $\sim$ (0.6) are satisfied.

Also, we say that a couple $\{u, \ell\}$ is a solution of SP on an interval $[0, T')$, $0 < T' \leqslant \infty$, if it is a solution of SP on $[0, T]$ in the above sense for any $0 < T < T'$.

We introduce the following space in order to describe the class of initial functions which satisfy the compatibility condition:

$$V = \{(z, s);\ s > 0 \text{ and } z \in W^{1,2}(0, \infty) \text{ with } z \geqslant 0 \text{ on } [0, s] \text{ and } z(y) = 0 \text{ for } y \geqslant s\}.$$

First, we recall the theorem concerned with local existence, uniqueness, comparison, continuation and regularity of solutions to SP.

Theorem 1.1. (cf. [1, Theorems 1.1 and 5.1] and [7, Theorem 1]) *Let $\alpha > 0$ and $(u_0, \ell_0) \in V$.*

(i) *Then there is a positive number T_0 such that the problem SP has one and only one solution $\{u, \ell\}$ on $[0, T_0]$.*

(ii) *We assume that $(\hat{u}_0, \hat{\ell}_0) \in V$, $\ell_0 \leqslant \hat{\ell}_0$, $u_0 \leqslant \hat{u}_0$ on $[0, \infty)$ and $u_0 \not\equiv \hat{u}_0$. Let $\{u, \ell\}$ or $\{\hat{u}, \hat{\ell}\}$ be a solution to $SP(u_0, \ell_0)$ or $SP(\hat{u}_0, \hat{\ell}_0)$, respectively, on $[0, T]$,*

$0 < T < \infty$. *Then we have*

$$\ell \leqslant \hat{\ell} \text{ on } [0,T] \text{ and } u < \hat{u} \text{ on } Q(T).$$

(iii) *If* $u_0 \in C^1([0,\ell_0])$ *and* $u_{0x}(0) = 0$, *then the solution* $\{u,\ell\}$ *to* $SP(u_0,\ell_0)$ *on* $[0,T]$ *satisfies that* u_x *is continuous on* $\overline{Q_\ell(T)}$, u_t *and* u_{xx} *are continuous on* $Q_\ell(T)$ *and* $\ell \in C^1([0,T])$.

(iv) *Let* $\{u,\ell\}$ *be a solution to* $SP(u_0,\ell_0)$ *on* $[0,T')$, $0 < T' < \infty$, *and let* M *be any positive number. If* $|u(t,x)| \leqslant M$ *for* $(t,x) \in Q(T')$, *then the solution is extended in time beyond* T'.

Remark 1.1. By Definition 1.1 and Theorem 1.1 (iii), for a solution $\{u,\ell\}$ to SP on $[0,T]$, u_x is continuous on the set $\{(t,x);\ 0 \leqslant x \leqslant \ell(t), 0 < t \leqslant T\}$, u_t and u_{xx} are continuous on $Q_\ell(T)$ and $\ell \in C^1([0,T])$. Hence, applying the strong maximum principle to SP we get the assertion (ii) in Theorem 1.1.

Throughout this paper, given the problem SP, we say that $[0,T)$, $0 < T \leqslant +\infty$, is the maximal interval of existence of the solution if the problem has a solution on the time-interval $[0,T']$ for every T' with $0 < T' < T$ and the solution can not be extended in time beyond T. Also, for simplicity we put

$$E(z,s) = \int_0^s z(x)\,\mathrm{d}x + s \quad \text{ for } (z,s) \in V$$

and

$$V(M,L) = \{(z,s) \in V;\ s \leqslant L \text{ and } z(x) \leqslant M \text{ for } 0 < x < s\},$$

where M and L are positive numbers.

Now, we give a theorem concerned with the global existence of solutions to SP.

Theorem 1.2. (cf. [4, Theorem 1.2]) *Let* $\alpha > 1$, $(u_0,\ell_0) \in V$. *Then for any positive number* M *there exists a positive number* $\delta_0 = \delta(M,\alpha) \in (0,1]$ *such that if* $\ell_0 \leqslant M$, $\int_0^{\ell_0} u_{0x}^2\,\mathrm{d}x \leqslant M$ *and* $\int_0^{\ell_0} u_0^2\,\mathrm{d}x \leqslant \delta_0$, *then the problem* $SP(u_0,\ell_0)$ *has a solution* $\{u,\ell\}$ *on* $[0,\infty)$ *satisfying*

$$\begin{aligned}
&E(u(t),\ell(t)) \leqslant \left\{C + E\Big(u\Big(\frac{1}{2}\Big),\ell\Big(\frac{1}{2}\Big)\Big)^{\beta}\right\}^{\frac{1}{\beta}} \quad \text{for } t \geqslant \frac{1}{2},\\
&\frac{\mathrm{d}}{\mathrm{d}t}|u_x(t)|^2_{L^2(0,\ell(t))} \leqslant 0 \quad \text{for a.e. } t > 0,\\
&|u(t)|_{L^\infty(0,\ell(t))} \leqslant \sqrt{2}\exp(-\mu t) \quad \text{for } t > 0,
\end{aligned}$$

where $C = C(\alpha)$, $\beta = \beta(\alpha)$ *and* $\mu = \mu(\alpha,\ell_0,|u_0|_{L^\infty(0,\ell_0)})$ *are some positive constants.*

For brevity we introduce the following set $G := G(u_0, \ell_0; M, L, \mu)$ for $(u_0, \ell_0) \in V$ and positive numbers M, L and μ:

$$
\begin{aligned}
G(u_0, \ell_0; M, L, \mu) = \{\{u, \ell\};\ & \{u, \ell\} \text{ is a solution to } SP(u_0, \ell_0) \text{ on } [0, \infty) \text{ satisfying} \\
& |u_x(t)|_{L^2(0,\ell(t))} \leqslant M,\ |u(t)|_{L^\infty(0,\ell(t))} \leqslant M \exp(-\mu t) \\
& \text{and } \ell(t) \leqslant L \text{ for } t \geqslant 0\}.
\end{aligned}
$$

The next theorem is our main result on the stability of global solutions to SP.

Theorem 1.3. *Let $\alpha > 1$, $(u_0, \ell_0) \in V$, let M, L and μ be positive numbers and $\{u, \ell\} \in G(u_0, \ell_0; M, L, \mu)$. Then there is a positive number $p_1 > 0$ depending only on α possessing the following property:*

For any positive number $\tilde{M}$ there exists a positive constant δ such that for any $(\hat{u}_0, \hat{\ell}_0) \in V(\tilde{M}, \tilde{M})$ with $|u_0 - \hat{u}_0|_{L^{p_1}(0,\infty)} < \delta$ and $|\ell_0 - \hat{\ell}_0| < \delta$ the problem $SP(\hat{u}_0, \hat{\ell}_0)$ has a solution $\{\hat{u}, \hat{\ell}\}$ on $[0, \infty)$ satisfying

$$\hat{\ell}(t) \leqslant \hat{L} \text{ and } |\hat{u}(t)|_{L^\infty(0,\hat{\ell}(t))} \leqslant \hat{M} \exp(-\hat{\mu} t) \text{ for } t \geqslant 0,$$

where $\hat{M}$, $\hat{L}$ and $\hat{\mu}$ are positive constants depending on α, M, L, μ, $\tilde{M}$ and δ.

We will prove Theorem 1.3 in the following way. First, we give some useful inequalities in Sobolev spaces and an ordinary differential inequality in Section 2. Secondly, some properties of a global solution belonging to the set $G(u_0, \ell_0; M, L, \mu)$ are shown (see Section 3). Next, we obtain the following decay for $v := \hat{u} - u$ under the condition $\ell_0 \leqslant \hat{\ell}_0$ and $u_0 \leqslant \hat{u}_0$:

$$|v(t)|_{L^{p_1}(0,\infty)} \leqslant c(1+t)^{-\beta} \quad \text{for } t \geqslant 0,$$

where c and β are positive constants. Finally, we give the complete proof of Theorem 1.3 by applying Theorem 1.2.

At the end of this section we introduce some notation. In order to avoid surplus confusion for notation we write the set of positive constants, α, M, L, μ, $\tilde{M}$ and $\tilde{L}$ as (D). Since $\alpha > 1$ we can take numbers satisfying

$$
\tag{1.1}
\begin{cases}
p_1 > \max\left\{2 + \alpha, \dfrac{1+\alpha}{\alpha - 1}\right\} \text{ and } \dfrac{p_1}{1+\alpha} + \dfrac{1}{2} < \dfrac{1}{r_0} < \dfrac{p_1}{2}, \\
\left(\dfrac{1}{r_0} - \dfrac{1}{2}\right)\dfrac{1+\alpha}{p_1} = 1 + \beta_0, \\
p_0 = \dfrac{p_1 r_0}{2}.
\end{cases}
$$

Clearly, we obtain that $1 < p_0 < p_1$ and $0 < r_0 < 2$. These numbers play an important role in our proof.

2. Auxiliary lemmas

At the beginning of this section we list some useful inequalities in Sobolev spaces (cf. O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva [10, Chap. 2, Theorem 2.2]): Let d be any positive number. Then

$$\int_0^d u^{p+\alpha}\,\mathrm{d}x \leqslant \Big(\frac{q+2}{2}\Big)^{\frac{2(q-r)}{r+2}} |(u^{\frac{p}{2}})_x|_{L^2(0,d)}^{\frac{2(q-r)}{r+2}} \Big(\int_0^d u^{\frac{pr}{2}}\,\mathrm{d}x\Big)^{\frac{q+2}{r+2}} \tag{2.1}$$
$$\text{for } u \in W^{1,2}(0,d) \text{ with } u(d)=0,$$

where $p \geqslant 2$, $\alpha \geqslant 0$, $q = 2(p+\alpha)/p$ and $r \in (0,q)$;

$$|u|_{L^2(0,d)} \leqslant \frac{d}{\sqrt{2}}|u_x|_{L^2(0,d)} \qquad \text{for } u \in W^{1,2}(0,d) \text{ with } u(d)=0; \tag{2.2}$$

$$|u|_{L^\infty(0,d)} \leqslant \left(\frac{q+2}{2}\right)^{\frac{2}{q+2}} |u_x|_{L^2(0,d)}^{\frac{2}{q+2}} |u|_{L^q(0,d)}^{\frac{q}{q+2}} \tag{2.3}$$
$$\text{for } u \in W^{1,2}(0,d) \text{ with } u(d)=0,$$

where $q \geqslant 1$.

The first lemma is concerned with an ordinary differential inequality.

Lemma 2.1. *Let a, b and μ be positive numbers, $0 < r < 2$ and let z be a non-negative absolutely continuous function on $[0,T]$, $0 < T < \infty$, satisfying*

$$\frac{\mathrm{d}}{\mathrm{d}t}z(t) + az^{\frac{2+r}{2-r}}(t) \leqslant b\exp(-\mu t) \qquad \textit{for a.e. } t \in [0,T].$$

Then there is a positive constant $N_0 = N_0(a,b,r,\mu)$ such that

$$z(t) \leqslant N_0(1+z(0))(1+a\beta t)^{-\frac{1}{\beta}} \qquad \textit{for any } t \in [0,T], \tag{2.4}$$

where $\beta = \dfrac{2r}{2-r}$.

P r o o f. Let N_1 be any positive number and

$$\psi(t) = N_1(1+a\beta t)^{-1/\beta} \qquad \text{for } t \in [0,T].$$

By elementary calculation we obtain that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(z(t)-\psi(t)) + a\frac{2+r}{2-r}\psi^{\frac{2r}{2-r}}(t)(z(t)-\psi(t))\\ &\leqslant b\exp(-\mu t) - a\big(N_1^{\frac{2+r}{2-r}} - N_1\big)(1+a\beta t)^{-\frac{r+2}{2r}} \qquad \text{for a.e. } t \in [0,T].\end{aligned}$$

Hence, we take a positive number $N_0 \geqslant 1$ such that

$$\left(\frac{b}{a}\right)^{\frac{2r}{2+r}}\left(1+\frac{a\beta(2+r)}{2r\mu\exp(1)}\right) \leqslant (N_0^{\frac{2+r}{2-r}} - N_0)^{\frac{2r}{2+r}},$$

and put $N_1 = N_0(1+z(0))$.

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(z(t)-\psi(t)) + a\frac{2+r}{2-r}\psi^{\frac{2r}{2-r}}(t)(z(t)-\psi(t)) \leqslant 0 \qquad \text{for a.e. } t \in [0,T].$$

Using Gronwall's argument we see that

$$\begin{aligned} z(t)-\psi(t) &\leqslant (z(0)-\psi(0))\exp\left\{-a\int_0^t \frac{2-r}{2+r}\psi^{\frac{2r}{2-r}}(\tau)\,\mathrm{d}\tau\right\} \\ &\leqslant z(0) - N_1(1+z(0)) \leqslant 0 \quad \text{ for any } t\in[0,T]. \end{aligned}$$

Thus, we get (2.4). □

Lemma 2.2. *Let $p > 1$ and $d > 0$. We suppose that $u \in W^{2,2}(0,d)$ with $u_x(0) = 0$, $u(d) = 0$ and $u > 0$ on $(0,d)$. Then $(u^{p/2})_x \in L^2(0,d)$.*

Proof. It is sufficient to show that there is a function $f \in L^2(0,d)$ such that

$$-\int_0^d u^{p/2}\eta_x\,\mathrm{d}x = \int_0^d f\eta\,\mathrm{d}x \quad \text{ for any } \eta \in C_0^\infty([0,d]). \tag{2.5}$$

Let $\eta \in C_0^\infty([0,d])$. Then there is a positive number ε such that $\operatorname{supp}(\eta) \subset [\varepsilon, d-\varepsilon]$ so that $u \geqslant \delta > 0$ on $[\varepsilon, d-\varepsilon]$ for some positive number δ. Clearly, we have

$$-\int_0^d u^{p/2}\eta_x\,\mathrm{d}x = \int_\varepsilon^{d-\varepsilon}(u^{p/2})_x\eta\,\mathrm{d}x = \frac{p}{2}\int_\varepsilon^{d-\varepsilon} u_x u^{\frac{p}{2}-1}\eta\,\mathrm{d}x.$$

Hence,

$$\left|-\int_0^d u^{p/2}\eta_x\,\mathrm{d}x\right| \leqslant \frac{p}{2}\left(\int_\varepsilon^{d-\varepsilon}|u_x|^2|u|^{p-2}\,\mathrm{d}x\right)^{1/2}\left(\int_0^d \eta^2\,\mathrm{d}x\right)^{1/2}.$$

Here we note that

$$\begin{aligned} \int_\varepsilon^{d-\varepsilon}|u_x|^2|u|^{p-2}\,\mathrm{d}x &= \int_\varepsilon^{d-\varepsilon} u_x\left(\frac{1}{p-1}u^{p-1}\right)_x\mathrm{d}x \\ &= -\frac{1}{p-1}\int_\varepsilon^{d-\varepsilon}u_{xx}u^{p-1}\,\mathrm{d}x + \frac{1}{p-1}\{u_x(d-\varepsilon)u^{p-1}(d-\varepsilon) - u_x(\varepsilon)u^{p-1}(\varepsilon)\}. \end{aligned} \tag{2.6}$$

Letting $\varepsilon \downarrow 0$ in (2.6), in virtue of continuity of u_x on $[0,d]$ we obtain that

$$\lim_{\varepsilon\downarrow 0}\int_\varepsilon^{d-\varepsilon}|u_x|^2|u|^{p-2}\,\mathrm{d}x = -\frac{1}{p-1}\int_0^d u_{xx}u^{p-1}\,\mathrm{d}x,$$

that is,

$$\left|-\int_0^d u^{p/2}\eta_x\,\mathrm{d}x\right| \leqslant C|\eta|_{L^2(0,d)} \qquad \text{for any } \eta\in C_0^\infty([0,d]),$$

where C is a positive constant.

Immediately, we conclude that there is a function $f\in L^2(0,d)$ satisfying (2.5). $\square$

3. Properties of a global solution

In this section we show some estimates for a global solution to SP. First, we recall some useful equations for a solution to SP.

Lemma 3.1. (cf. [9, Lemma 5.1] and [4, Lemma 2.1]) *Let $(u_0,\ell_0)\in V$ and let $\{u,\ell\}$ be a solution to $SP(u_0,\ell_0)$ on $[0,T]$, $0<T<\infty$.*

(1) *We have*

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u(t),\ell(t)) = \int_0^{\ell(t)}u^{1+\alpha}(t,x)\,\mathrm{d}x \qquad \text{for a.e. } t\in[0,T]. \tag{3.1}$$

(2) *For a.e. $t\in[0,T]$ we have*

$$|u_t(t)|^2_{L^2(0,\ell(t))}+\frac{1}{2}|\ell'(t)|^3+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_x(t)|^2_{L^2(0,\ell(t))} = \frac{1}{2+\alpha}\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|^{2+\alpha}_{L^{2+\alpha}(0,\ell(t))}. \tag{3.2}$$

The following lemma guarantees a decay for u_x.

Lemma 3.2. *Let M, L and μ be positive numbers, $(u_0,\ell_0)\in V$ and $\{u,\ell\}\in G(u_0,\ell_0;M,L,\mu)$. Then there are positive constants L_1 and μ_1 such that*

$$|u_x(t)|_{L^2(0,\ell(t))}\leqslant L_1\exp(-\mu_1 t) \qquad \text{for } t>0.$$

P r o o f. By the argument in the proof of [9, Lemma 5.2] we have

$$\int_0^{\ell(t)}u_t(t)u_{xx}(t)\,\mathrm{d}x = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\ell(t)}|u_x(t)|^2\,\mathrm{d}x-\frac{1}{2}|\ell'(t)|^3 \qquad \text{for } t\geqslant 0. \tag{3.3}$$

Also, from (0.1) we see that

$$
\begin{aligned}
(3.4)\qquad \int_0^{\ell(t)} u_t(t)u_{xx}(t)\,\mathrm{d}x &= \int_0^{\ell(t)} (u_{xx}(t)+u^{1+\alpha}(t))u_{xx}(t)\,\mathrm{d}x \\
&= \int_0^{\ell(t)} (u_{xx})^2(t)\,\mathrm{d}x - (1+\alpha)\int_0^{\ell(t)} u^{\alpha}(t)(u_x)^2(t)\,\mathrm{d}x \qquad \text{for } t>0.
\end{aligned}
$$

It follows from (3.3), (3.4) and (2.2) that

$$
\begin{aligned}
(3.5)\qquad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\ell(t)} |u_x(t)|^2\,\mathrm{d}x + |\ell'(t)|^3 + \frac{1}{4L^2}\int_0^{\ell(t)} |u_x(t)|^2\,\mathrm{d}x \\
&\leqslant \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\ell(t)} |u_x(t)|^2\,\mathrm{d}x + |\ell'(t)|^3 + \int_0^{\ell(t)} |u_{xx}(t)|^2\,\mathrm{d}x \\
&\leqslant (1+\alpha)M^{\alpha}\exp(-\alpha\mu t)\int_0^{\ell(t)} |u_x(t)|^2\,\mathrm{d}x \qquad \text{for } t>0.
\end{aligned}
$$

Here we can take a positive number t_0 such that $(1+\alpha)M^{\alpha}\exp(-\alpha\mu t)\leqslant \frac{1}{8L^2}$ for $t\geqslant t_0$. Consequently, for $t\geqslant t_0$ we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\ell(t)} |u_x(t)|^2\,\mathrm{d}x + \frac{1}{4L^2}\int_0^{\ell(t)} |u_x(t)|^2\,\mathrm{d}x \leqslant 0,
$$

and hence

$$
\int_0^{\ell(t)} |u_x(t)|^2\,\mathrm{d}x \leqslant \exp\Big\{-\frac{1}{4L^2}(t-t_0)\Big\}\int_0^{\ell(t)} |u_x(t_0)|^2\,\mathrm{d}x.
$$

On the other hand, (3.5) implies

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\ell(t)} |u_x(t)|^2\,\mathrm{d}x \leqslant 2(1+\alpha)M^{\alpha}\int_0^{\ell(t)} |u_x(t)|^2\,\mathrm{d}x.
$$

By Gronwall's inequality, we have

$$
\int_0^{\ell(t)} |u_x(t)|^2\,\mathrm{d}x \leqslant \exp\Big\{2(1+\alpha)M^{\alpha}t_0+\frac{t_0}{4L^2}\Big\}\exp\Big\{-\frac{t}{4L^2}\Big\}|u_{0x}|^2_{L^2(0,\ell_0)} \quad \text{for } t\in[0,t_0].
$$

Therefore, putting

$$
L_1 = \exp\Big\{2(1+\alpha)M^{\alpha}t_0+\frac{t_0}{4L^2}\Big\}|u_{0x}|^2_{L^2(0,\ell_0)} \text{ and } \mu_1 = \frac{1}{4L^2},
$$

we get the assertion of the lemma. □

The following lemma shows the decay of ℓ', which is a key for the proof of Theorem 1.3.

Lemma 3.3. *We suppose that the same assumptions as in Lemma* 3.2 *hold and* $1 < q < 4$. *Then for some positive number* $\mu_0 = \mu_0(\mu, q)$, *we have*

$$\int_0^\infty |\ell'(t)|^q \exp(\mu_0 t)\,\mathrm{d}t < \infty.$$

Clearly, the above fact implies that

$$\int_0^\infty |\ell'(t)|^q\,\mathrm{d}t < \infty.$$

Proof. Let L_1 and μ_1 be positive constants defined in Lemma 3.2. According to (2.3) and Lemma 3.2, we see that for any $t > 0$

$$\begin{aligned} |\ell'(t)|^q &= |u_x(t, \ell(t)-)|^q \\ &\leqslant \sqrt{2L_1}\exp\Bigl(-\frac{q}{2}\mu_1 t\Bigr)|u_{xx}(t)|^{q/2}_{L^2(0,\ell(t))}, \end{aligned}$$

and hence

$$|\ell'(t)|^q \exp\Bigl(\frac{\mu_1 q}{4}t\Bigr) \leqslant C|u_{xx}(t)|^2_{L^2(0,\ell(t))} + C\exp\Bigl(-\frac{\mu_1 q}{4-q}t\Bigr), \tag{3.6}$$

where C is a suitable positive constant.

By using (3.5) and Lemma 3.2 again, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^{\ell(\tau)} |u_x(\tau)|^2\,\mathrm{d}x + \int_0^{\ell(\tau)} |u_{xx}(\tau)|^2\,\mathrm{d}x \\ &\leqslant (1+\alpha)M^\alpha \exp(-\alpha\mu\tau)\int_0^{\ell(\tau)} |u_x(\tau)|^2\,\mathrm{d}x \\ &\leqslant (1+\alpha)M^\alpha L_1^2 \exp\{-(\alpha\mu + 2\mu_1)\tau\} \quad \text{ for } \tau > 0. \end{aligned}$$

Integrating this inequality over $[0, t]$, $0 < t < \infty$, we obtain that

$$\begin{aligned} &\int_0^{\ell(t)} |u_x(t)|^2\,\mathrm{d}x + \int_0^t \int_0^{\ell(\tau)} |u_{xx}(\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau \\ &\leqslant (1+\alpha)M^\alpha L_1^2 \int_0^t \exp\{-(\alpha\mu + 2\mu_1)\tau\}\,\mathrm{d}\tau + \int_0^{\ell_0} |u_{0x}|^2\,\mathrm{d}x \quad \text{ for } t \geqslant 0. \end{aligned}$$

Adding to (3.6), we conclude that $\int_0^\infty |\ell'(t)|^q \exp(\mu_0 t)\,\mathrm{d}t < \infty$ where $\mu_0 = \frac{\mu_1 q}{4}$. $\square$

4. Energy inequalities

The purpose of this section is to establish the following lemmas concerned with global estimates for the difference $\hat{u} - u$ of solutions to SP.

Lemma 4.1. *Let $(u_0, \ell_0), (\hat{u}_0, \hat{\ell}_0) \in V$, let M, L and μ be positive numbers, $\{u, \ell\} \in G(u_0, \ell_0; M, L, \mu)$, and let $\{\hat{u}, \hat{\ell}\}$ be a solution to $SP(\hat{u}_0, \hat{\ell}_0)$ on $[0,T]$, $0 < T < \infty$. Moreover, we suppose that $\ell_0 \leqslant \hat{\ell}_0$, $u_0 \leqslant \hat{u}_0$ on $[0,\infty)$ and $\hat{u}_0 \not\equiv u_0$. Then putting $v = \hat{u} - u$ we obtain that for $t \in (0,T]$ and $p \in [p_0, p_1]$ (see (1.1))*

$$
\begin{aligned}
(4.1)\qquad & \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\hat{\ell}(t)} v^p(t)\,\mathrm{d}x \\
& \leqslant \left\{ -C_1 + C_2 \hat{\ell}(t)^{2-\frac{1+\alpha}{p_1}} \left(\int_0^{\hat{\ell(t)}} v^{p_1}(t)\,\mathrm{d}x \right)^{\frac{\alpha}{p_1}} \right\} |(v^{\frac{p}{2}})_x(t)|^2_{L^2(0,\hat{\ell}(t))} \\
& \quad + C_2 \exp(-\alpha\mu t) \int_0^{\hat{\ell}(t)} v^p(t)\,\mathrm{d}x + C_2 \ell'(t)^{\frac{2p}{p+1}} \left(\int_0^{\hat{\ell}(t)} v^p(t)\,\mathrm{d}x \right)^{\frac{p-1}{p+1}},
\end{aligned}
$$

where C_1 and C_2 are positive constants depending on α, p_0, p_1 and M.

Proof. For simplicity, we put $H(t) = L^2(0, \hat{\ell}(t))$.

First, by Theorem 1.1 (ii) we have $v = \hat{u} - u > 0$ on $Q_{\hat{\ell}}(T)$. The difference of (0.1) and (0.1) taken $u = \hat{u}$ is multiplied by v^{p-1} for $p \in [p_0, p_1]$. Then Lemma 2.2 is applied to get

$$
\begin{aligned}
(4.2)\quad & \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\hat{\ell}(t)} v^p(t)\,\mathrm{d}x \\
& = p\int_0^{\hat{\ell}(t)} v_t(t) v^{p-1}(t)\,\mathrm{d}x \\
& = p\int_0^{\hat{\ell}(t)} (\hat{u}_{xx}(t) + \hat{u}^{1+\alpha}(t)) v^{p-1}(t)\,\mathrm{d}x - p\int_0^{\ell(t)} (u_{xx}(t) + u^{1+\alpha}(t)) v^{p-1}(t)\,\mathrm{d}x \\
& = -p(p-1)\int_0^{\hat{\ell}(t)} (v_x)^2(t) v^{p-2}(t)\,\mathrm{d}x + p\ell'(t) v^{p-1}(t, \ell(t)) \\
& \quad + p\int_0^{\hat{\ell}(t)} (\hat{u}^{1+\alpha}(t) - u^{1+\alpha}(t)) v^{p-1}(t)\,\mathrm{d}x \\
& \leqslant -\frac{4(p-1)}{p}\int_0^{\hat{\ell}(t)} ((v^{p/2})_x(t))^2\,\mathrm{d}x + p\ell'(t) v^{p-1}(t, \ell(t)) \\
& \quad + p2^\alpha(1+\alpha)\int_0^{\hat{\ell}(t)} (v^{p+\alpha}(t) + u^\alpha(t) v^p(t))\,\mathrm{d}x \qquad \text{for } t \in (0,T].
\end{aligned}
$$

We note that

$$\hat{u}^{1+\alpha} - u^{1+\alpha} \leqslant 2^\alpha(1+\alpha)(v^{1+\alpha} + vu^\alpha). \tag{4.3}$$

From (2.1) and Hölder's inequality it follows that

$$\begin{aligned} \int_0^{\hat{\ell}(t)} v^{p+\alpha}(t)\,\mathrm{d}x &\leqslant \Big(\frac{\alpha}{p}+2\Big)^2 |(v^{p/2})_x(t)|^2_{H(t)} \Big(\int_0^{\hat{\ell}(t)} v^{\alpha/2}(t)\,\mathrm{d}x\Big)^2 \\ &\leqslant \Big(\frac{\alpha}{p}+2\Big)^2 |(v^{p/2})_x(t)|^2_{H(t)} \hat{\ell}(t)^{2-\frac{\alpha}{p_1}} \Big(\int_0^{\hat{\ell}(t)} v^{p_1}(t)\,\mathrm{d}x\Big)^{\frac{\alpha}{p_1}} \quad \text{for } t \in (0,T]. \end{aligned} \tag{4.4}$$

Also, according to (2.3) we have

$$\begin{aligned} & p\ell'(t)v^{p-1}(t,\ell(t)) \\ &\leqslant p\ell'(t)|v^{p/2}(t)|^{\frac{2(p-1)}{p}}_{L^\infty(0,\hat{\ell}(t))} \\ &\leqslant 2^{\frac{p-1}{p}} p\ell'(t)|(v^{p/2})_x(t)|^{\frac{p-1}{p}}_{H(t)}|(v^{p/2})(t)|^{\frac{p-1}{p}}_{H(t)} \\ &\leqslant \frac{2(p-1)}{p}|(v^{p/2})_x(t)|^2_{H(t)} + C_p|\ell'(t)|^{\frac{2p}{p+1}} \Big(\int_0^{\hat{\ell}(t)} v^p(t)\,\mathrm{d}x\Big)^{\frac{p-1}{p+1}} \quad \text{for } t \in (0,T], \end{aligned} \tag{4.5}$$

where C_p is a positive constant depending only on p.

It follows from (4.2) $\sim$ (4.5) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\hat{\ell}(t)} v^p(t)\mathrm{d}x \leqslant& \left\{-C_1 + C_2\hat{\ell}(t)^{2-\frac{\alpha}{p_1}} \Big(\int_0^{\hat{\ell}(t)} v^{p_1}(t)\,\mathrm{d}x\Big)^{\frac{\alpha}{p_1}}\right\} |(v^{p/2})_x(t)|^2_{H(t)} \\ &+ C_2\exp(-\alpha\mu t)\int_0^{\hat{\ell}(t)} v^p(t)\mathrm{d}x + C_2|\ell'(t)|^{\frac{2p}{p+1}} \Big(\int_0^{\hat{\ell}(t)} v^p(t)\,\mathrm{d}x\Big)^{\frac{p-1}{p+1}} \end{aligned}$$

for $t \in (0,T]$ where

$$C_1 = \frac{2(p_0-1)}{p_1} \text{ and } C_2 = 2^\alpha p_1\Big(2+\frac{\alpha}{p_0}\Big)^2(1+\alpha) + 2^\alpha p_1(1+\alpha)M + \max_{p_0\leqslant p\leqslant p_1} C_p.$$

This is the conclusion of the lemma. □

Lemma 4.2. *Let B_1 and B_2 be positive numbers. We suppose that the same assumptions as in Lemma 4.1 hold. Moreover, we suppose that for $p \in [p_0, p_1]$ and $t \in (0,T]$, $0 < T < \infty$,*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\hat{\ell}(t)} v^p(t)\,\mathrm{d}x \leqslant& -B_0|(v^{\frac{p}{2}})_x(t)|^2_{L^2(0,\hat{\ell}(t))} + B_1\exp(-\alpha\mu t)\int_0^{\hat{\ell}(t)} v^p(t)\,\mathrm{d}x \\ &+ B_1\ell'(t)^{\frac{2p}{p+1}}\Big(\int_0^{\hat{\ell}(t)} v^p(t)\,\mathrm{d}x\Big)^{\frac{p-1}{p+1}}. \end{aligned} \tag{4.6}$$

Then there is a positive constant C_3 depending on α, p_0, p_1, μ, M and L which satisfies

$$(4.7)\quad \int_0^{\hat{\ell}(t)} v^p(t)\,\mathrm{d}x \leqslant C_3\Big(\int_0^{\hat{\ell}_0} v^p(0)\,\mathrm{d}x + 1\Big) \quad \textit{for } p \in [p_0, p_1] \textit{ and } t \in [0, T].$$

P r o o f. For simplicity, we put

$$F_p(t) = \int_0^{\hat{\ell}(t)} v^p(t)\,\mathrm{d}x \quad \text{for } p \in [p_0, p_1] \text{ and } t \in [0, T].$$

Obviously, we obtain

$$\ell'(t)^{\frac{2p}{p+1}} F_p(t)^{\frac{p-1}{p+1}} \leqslant \frac{2}{p+1}\ell'(t)^{\frac{2p}{p+1}} + \frac{p-1}{p+1}\ell'(t)^{\frac{2p}{p+1}} F_p(t) \quad \text{for } t \in [0, T].$$

Hence, (4.6) implies that for $p \in [p_0, p_1]$

$$\frac{\mathrm{d}}{\mathrm{d}t}F_p(t) \leqslant C_4 \exp(-\alpha\mu t)F_p(t) + C_4|\ell'(t)|^{\frac{2p}{p+1}} + C_4|\ell'(t)|^{\frac{2p}{p+1}} F_p(t) \quad \text{for } t \in (0, T],$$

where $C_4 = B_1 + B_1\big(\frac{1}{p_0+1} + \frac{p_1-1}{p_0+1}\big)$.

Since $1 < \frac{2p}{p+1} < 4$, hence by applying Lemma 3.3 and Gronwall's argument to the above inequality we get

$$F_p(t) \leqslant (F_p(0) + C_4\int_0^\infty |\ell'|^{\frac{2p}{p+1}}\,\mathrm{d}t)\exp\Big(\int_0^\infty J_p(t)\,\mathrm{d}t\Big) \quad \text{for } t \in [0, T],$$

where $J_p(t) = C_4\exp(-\alpha\mu t) + C_4|\ell'(t)|^{\frac{2p}{p+1}}$.

Thus, the lemma has been proved. $\square$

Lemma 4.3. *Let $\tilde{M}$ and $\tilde{L}$ be positive numbers. Then under the same conditions as in Lemma 4.2, there are positive constants C_5 and C_6 depending only on* (D) *such that*

$$(4.8)\quad \int_0^{\hat{\ell}(t)} v^{p_1}(t)\,\mathrm{d}x \leqslant C_5\Big(1 + \int_0^{\hat{\ell}_0} v^{p_1}(0)\,\mathrm{d}x\Big)(1 + C_6 t)^{-\frac{2-r_0}{2r_0}} \quad \textit{for } t \in [0, T],$$

where r_0 is a positive constant defined by (1.1).

P r o o f. For brevity, we put $F(t)=\int_0^{\hat{\ell}(t)} v^{p_1}(t)\,\mathrm{d}x$ for $t\in[0,T]$ and note that $\int_0^{\hat{\ell}_0} v^{p_0}(0)\,\mathrm{d}x\leqslant \tilde{M}^{p_0}\tilde{L}$. According to (2.1) and the previous lemma, we infer that

$$\begin{aligned}F(t)&\leqslant 2^{\frac{2(2-r_0)}{r_0+2}}|(v^{p_1/2})_x(t)|_{H(t)}^{\frac{2(2-r_0)}{r_0+2}}\Bigl(\int_0^{\hat{\ell(t)}} v^{p_0}(t)\,\mathrm{d}x\Bigr)^{\frac{4}{r_0+2}}\\&\leqslant 2^{\frac{2(2-r_0)}{r_0+2}}(C_3(\tilde{M}^{p_0}\tilde{L}+1))^{\frac{4}{r_0+2}}|(v^{p_1/2})_x(t)|_{H(t)}^{\frac{2(2-r_0)}{r_0+2}}\quad\text{for } t\in(0,T],\end{aligned}$$

and hence

$$|(v^{p_1/2})_x(t)|^2_{H(t)}\geqslant\frac{1}{4(C_3(\tilde{M}^{p_0}\tilde{L}+1))^{\frac{4}{2+r_0}}}F(t)^{\frac{2+r_0}{2-r_0}}\quad\text{for } t\in(0,T].$$

We note that

$$\begin{aligned}&\ell'(t)^{\frac{2p_1}{p_1+1}}F(t)^{\frac{p_1-1}{p_1+1}}\\&\leqslant\frac{p_1-1}{p_1+1}\ell'(t)^{\frac{2p_1}{p_1-1}}F(t)\exp(\mu_0 t)+\frac{2}{p_1+1}\exp\Bigl(-\frac{p_1-1}{2}\mu_0 t\Bigr)\quad\text{for } t\in(0,T],\end{aligned}$$

where μ_0 is the positive constant defined in Lemma 3.3, and $1<\frac{2p_1}{p_1-1}<4$.

Therefore, by adding the above inequalities together to (4.6), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F(t)\leqslant&-K_1F(t)^{\frac{2+r_0}{2-r_0}}+K_2|\ell'(t)|^{\frac{2p_1}{p_1-1}}F(t)\exp(\mu_0 t)\\&+K_2\exp(-\mu_2 t)+K_2F(t)\exp(-\mu_3 t)\quad\text{for } t\in(0,T],\end{aligned}$$

where K_1, K_2, μ_2 and μ_3 are suitable positive constants.

For simplicity, we put

$$J(t)=K_2(|\ell'(t)|^{\frac{2p_1}{p_1-1}}\exp(\mu_0 t)+\exp(-\mu_3 t))\text{ and }\Phi(t)=F(t)\exp\Bigl(-\int_0^t J(\tau)\,\mathrm{d}\tau\Bigr).$$

It is clear that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t)+K_1\Phi(t)^{\frac{2+r_0}{2-r_0}}\exp\Bigl(\frac{2r_0}{2-r_0}\int_0^t J(\tau)\,\mathrm{d}\tau\Bigr)\leqslant K_2\exp(-\mu_2 t)\quad\text{for } t\in(0,T]$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t)+K_1\Phi(t)^{\frac{2+r_0}{2-r_0}}\leqslant K_2\exp(-\mu_2 t)\quad\text{for } t\in(0,T].$$

By Lemma 2.1 we obtain that

$$\Phi(t)\leqslant N_0(1+\Phi(0))(1+\beta_1K_1t)^{-\frac{1}{\beta_1}}\quad\text{for } t\in[0,T],$$

where $\beta_1=\frac{2r_0}{2-r_0}$ and $N_0=N_0(K_1,K_2,r_0,\mu_2)>0$.

Consequently, this implies that

$$F(t) \leqslant N_0(1+F(0))(1+\beta_1 K_1 t)^{-\frac{1}{\beta_1}} \exp\Big(\int_0^\infty J(\tau)\,\mathrm{d}\tau\Big) \quad \text{for } t \in (0,T];$$

since $1 < 2p_1/(p_1-1) < 4$, the integration in the above inequality makes sense. We get the assertion of Lemma 4.3. □

At the end of this section, we give a global estimate for $E(\hat{u}(t), \hat{\ell}(t))$.

Lemma 4.4. *We suppose that the same assumptions as in Lemma* 4.3 *hold. Then there exists a positive constant* C_7 *depending only on* (D) *which satisfies*

$$E(\hat{u}(t), \hat{\ell}(t)) \leqslant C_7 \Big\{ E(\hat{u}_0, \hat{\ell}_0) + \Big(\int_0^{\hat{\ell}_0} v^{p_1}(0)\,\mathrm{d}x\Big)^{\frac{p_1\beta_0}{1+\alpha}} + 1 \Big\} \quad \textit{for } t \in [0,T], \tag{4.9}$$

where β_0 *is a positive constant defined by* (1.1).

P r o o f. For simplicity, we use the same notation as in the proof of the previous lemmas and put

$$E(t) = E(u(t), \ell(t)) \text{ and } \hat{E}(t) = E(\hat{u}(t), \hat{\ell}(t)).$$

It follows from (3.1) with help of (4.3) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\hat{E}(t) - E(t)) \\
&\leqslant 2^\alpha(1+\alpha)\Big\{\int_0^{\hat{\ell}(t)} v^{1+\alpha}(t)\,\mathrm{d}x + (\int_0^{\hat{\ell}(t)} v^{1+\alpha}(t)\,\mathrm{d}x)^{\frac{1}{1+\alpha}}(\int_0^{\ell(t)} u^{1+\alpha}(t)\,\mathrm{d}x)^{\frac{\alpha}{1+\alpha}}\Big\} \\
&\leqslant 2^\alpha(1+\alpha)\Big\{\int_0^{\hat{\ell}(t)} v^{1+\alpha}(t)\,\mathrm{d}x + \frac{1}{1+\alpha}\int_0^{\hat{\ell}(t)} v^{1+\alpha}(t)\,\mathrm{d}x + \frac{\alpha}{1+\alpha}\int_0^{\ell(t)} u^{1+\alpha}(t)\,\mathrm{d}x\Big\} \\
&\leqslant 2^\alpha(1+\alpha)\Big\{\frac{\alpha+2}{\alpha+1}\Big(\int_0^{\hat{\ell}(t)} v^{p_1}(t)\,\mathrm{d}x\Big)^{\frac{1+\alpha}{p_1}} \hat{\ell}(t)^{1-\frac{1+\alpha}{p_1}} + \frac{\alpha}{1+\alpha}M^{1+\alpha}\exp(-(1+\alpha)\mu t)\Big\} \\
&\leqslant K_3 F(t)^{\frac{1+\alpha}{p_1}} \hat{E}(t)^{1-\frac{1+\alpha}{p_1}} + K_3\exp(-(1+\alpha)\mu t) \quad \text{for } t \in (0,T],
\end{aligned}$$

where $K_3 = 2^\alpha(1+\alpha)(\frac{\alpha+2}{\alpha+1} + \frac{\alpha}{1+\alpha}M^{1+\alpha})$, and hence

$$\begin{aligned}
&\Big(\frac{\mathrm{d}}{\mathrm{d}t}\hat{E}(t)\Big)\hat{E}(t)^{\frac{1+\alpha}{p_1}-1} \\
&\leqslant K_3 F(t)^{\frac{1+\alpha}{p_1}} + K_3 \hat{E}(t)^{\frac{1+\alpha}{p_1}-1}\exp(-(1+\alpha)\mu t) + \hat{E}(t)^{\frac{1+\alpha}{p_1}-1}\frac{\mathrm{d}}{\mathrm{d}t}E(t) \quad \text{for } t \in (0,T].
\end{aligned}$$

Moreover, since $\frac{1+\alpha}{p_1} - 1 < 0$, $(\hat{E}(t))^{\frac{1+\alpha}{p_1}-1} \leqslant (E(t))^{\frac{1+\alpha}{p_1}-1}$ and $\frac{\mathrm{d}}{\mathrm{d}t}E(t) \geqslant 0$, we see that

$$\frac{p_1}{1+\alpha}\frac{\mathrm{d}}{\mathrm{d}\tau}\big(\hat{E}(\tau)^{\frac{1+\alpha}{p_1}}\big) \leqslant K_3 F(\tau)^{\frac{1+\alpha}{p_1}} + K_3 E(0)^{\frac{1+\alpha}{p_1}-1}\exp(-(1+\alpha)\mu\tau) + \frac{p_1}{1+\alpha}\frac{\mathrm{d}}{\mathrm{d}\tau}(E(\tau)^{\frac{1+\alpha}{p_1}}) \qquad \text{for } \tau \in (0,T]. \tag{4.10}$$

Integrating (4.10) over $[0,t]$, $0 \leqslant t \leqslant T$, we conclude that

$$\begin{aligned} &\frac{p_1}{1+\alpha}\Big(\hat{E}(t)^{\frac{1+\alpha}{p_1}} - \hat{E}(0)^{\frac{1+\alpha}{p_1}}\Big) \\ &\quad \leqslant \frac{p_1}{1+\alpha}\Big(E(t)^{\frac{1+\alpha}{p_1}} - E(0)^{\frac{1+\alpha}{p_1}}\Big) + K_3\int_0^t F(\tau)^{\frac{1+\alpha}{p_1}}\,\mathrm{d}\tau \\ &\qquad + K_3 E(0)^{\frac{1+\alpha}{p_1}-1}\int_0^t \exp(-(1+\alpha)\mu\tau)\,\mathrm{d}\tau \quad \text{for } t \in [0,T]. \end{aligned}$$

Hence, it follows from Lemma 4.3 that

$$\begin{aligned} \int_0^t F(\tau)^{\frac{1+\alpha}{p_1}}\,\mathrm{d}\tau &\leqslant \{C_5(1+F(0))\}^{\frac{1+\alpha}{p_1}}\int_0^\infty (1+C_6\tau)^{-\frac{(1+\alpha)(2-r_0)}{2p_1r_0}}\,\mathrm{d}\tau \\ &\leqslant \{C_5(1+F(0))\}^{\frac{1+\alpha}{p_1}}\int_0^\infty (1+C_6\tau)^{-1-\beta_0}\,\mathrm{d}\tau \quad \text{for } t \in [0,T]. \end{aligned}$$

Therefore, it is easy to check that (4.9) holds. □

5. Stability of global solutions

First, we shall prove Theorem 1.3 in case the following condition $(*)$ holds:

$$\ell_0 \leqslant \hat{\ell}_0, u_0 \leqslant \hat{u}_0 \text{ on } [0,\infty) \text{ and } u_0 \not\equiv \hat{u}_0. \tag{$*$}$$

Proof of Theorem 1.3 under the condition $(*)$. Let $\{\hat{u},\hat{\ell}\}$ be a solution of $SP(\hat{u}_0,\hat{\ell}_0)$ on $[0,T_1]$, $0 < T_1 < \infty$, $\delta \in (0,1]$ and $v := \hat{u} - u$. We assume that $\int_0^{\hat{\ell}_0} v^{p_1}(0)\,\mathrm{d}x \leqslant \delta$ and $\hat{\ell}_0 \leqslant \ell_0+\delta$. Since the function $t \to \int_0^{\hat{\ell(t)}} v^{p_1}(t)\,\mathrm{d}x$ is continuous, there is a positive constant $T_2 \leqslant T_1$ such that

$$\int_0^{\hat{\ell}(t)} v^{p_1}(t)\,\mathrm{d}x \leqslant 2\delta \text{ and } \hat{\ell}(t) \leqslant L+2 =: L_2 \quad \text{for } t \in [0,T_2].$$

Lemma 4.1 implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\hat{\ell}(t)} v^p(t)\,\mathrm{d}x \leqslant \{-C_1 + C_2(L_2)^{2-\frac{1+\alpha}{p_1}}(2\delta)^{\frac{\alpha}{p_1}}\}|(v^{\frac{p}{2}})_x(t)|^2_{L^2(0,\hat{\ell}(t))}$$
$$+ C_2\exp(-\alpha\mu t)\int_0^{\hat{\ell}(t)} v^p(t)\,\mathrm{d}x$$
$$+ C_2\ell'(t)^{\frac{2p}{p+1}}\Big(\int_0^{\hat{\ell}(t)} v^p(t)\,\mathrm{d}x\Big)^{\frac{p-1}{p+1}} \quad \text{for } t\in(0,T_2] \text{ and } p\in[p_0,p_1].$$

We choose a positive number δ_1 such that

$$C_2(L_2)^{2-\frac{1+\alpha}{p_1}}(2\delta_1)^{\frac{\alpha}{p_1}} \leqslant \frac{C_1}{2},$$

and clearly, for $\delta \leqslant \delta_1$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\hat{\ell}(t)} v^p(t)\,\mathrm{d}t \tag{5.1}$$
$$\leqslant -\frac{C_1}{2}|(v^{p/2})_x(t)|^2_{L^2(0,\hat{\ell}(t))} + C_2\exp(-\alpha\mu t)\int_0^{\hat{\ell}(t)} v^p(t)\,\mathrm{d}x$$
$$+ C_2\ell'(t)\Big(\int_0^{\hat{\ell}(t)} v^p(t)\,\mathrm{d}x\Big)^{\frac{p-1}{p+1}} \quad \text{for } t\in(0,T_2] \text{ and } p\in[p_0,p_1].$$

By virtue of Lemmas 4.3 and 4.4, for $\delta < \delta_1$ we have

$$\begin{aligned}\int_0^{\hat{\ell}(t)} v^{p_1}(t)\,\mathrm{d}x &\leqslant M_1(1+\delta)(1+M_2t)^{-\frac{2-r_0}{2r_0}} \quad \text{for } t\in[0,T_2],\\ E(\hat{u}(t),\hat{\ell}(t)) &\leqslant M_1(1+\delta^{\frac{p_1\beta_0}{1+\alpha}}) \quad \text{for } t\in[0,T_2],\end{aligned} \tag{5.2}$$

where M_1 and M_2 are positive constants depending only on (D).

It follows from (3.2) that

$$|\hat{u}_x(t)|^2_{L^2(0,\hat{\ell}(t))} \leqslant |\hat{u}_{0x}|^2_{L^2(0,\hat{\ell}_0)} + \frac{2}{2+\alpha}|\hat{u}(t)|^{2+\alpha}_{L^{2+\alpha}(0,\hat{\ell}(t))} \quad \text{for } t\in[0,T_2],$$

and hence with the aid of (5.2) there is a positive constant M_3 depending on (D) such that

$$\left.\begin{aligned}|\hat{u}_x(t)|_{L^2(0,\hat{\ell}(t))} &\leqslant M_3\\ |v(t)|_{L^\infty(0,\hat{\ell}(t))} &\leqslant M_3\end{aligned}\right\} \quad \text{for } t\in[0,T_2]. \tag{5.3}$$

Also, putting $\varphi(t) = E(\hat{u}(t), \hat{\ell}(t)) - E(u(t), \ell(t))$, we have

$$\begin{aligned}
(5.4)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) &= \int_0^{\hat{\ell}(t)} (\hat{u}^{1+\alpha}(t) - u^{1+\alpha}(t))\,\mathrm{d}x \\
&\leqslant 2^\alpha(1+\alpha)\int_0^{\hat{\ell}(t)} (v^{1+\alpha}(t) + v(t)u^\alpha(t))\,\mathrm{d}x \\
&\leqslant 2^\alpha(1+\alpha)(M_3^\alpha + M^\alpha\exp(-\alpha\mu t))\varphi(t) \qquad \text{for } t\in(0,T_2].
\end{aligned}$$

Consequently, by using Gronwall's inequality we infer that

$$\varphi(t) \leqslant \varphi(0)\exp(M_4 t) \qquad \text{for } t\in[0,T_2],$$

where $M_4 = \exp\{2^\alpha(1+\alpha)(M_3^\alpha + M^\alpha)\}$.

Moreover, we observe that

$$\begin{aligned}
(5.5)\qquad \int_0^{\ell(t)} v^{p_1}(t)\,\mathrm{d}x &\leqslant M_3^{p_1-1}\varphi(t) \\
&\leqslant M_3^{p_1-1}\varphi(0)\exp(M_4 t) \\
&= M_3^{p_1-1}\Big(\int_0^{\hat{\ell}_0} v(0)\,\mathrm{d}x + \hat{\ell}_0 - \ell_0\Big)\exp(M_4 t) \\
&\leqslant M_3^{p_1-1}(L_2^{1-1/p_1}\delta^{1/p_1} + \delta)\exp(M_4 t) \\
&\leqslant M_5\exp(M_4 t)\delta^{1/p_1} \qquad \text{for } \delta<\delta_1 \text{ and } t\in(0,T_2],
\end{aligned}$$

where $M_5 = M_3^{p_1-1}(L_2^{1-1/p_1}+1)$.

It follows from Theorem 1.1 (iv) that we can extend the solution $\{\hat{u},\hat{\ell}\}$ to $[0,T_3)$ for some $T_3 > T_2$. Now, we take positive numbers $0<\delta_3<\delta_2<\delta_1$ and T_0 such that

$$\begin{aligned}
&C_2 L_2^{2-\frac{1+\alpha}{p_1}}(3\delta_2)^{\frac{\alpha}{p_1}} \leqslant \frac{C_1}{2}, \\
&2M_1(1+M_2T_0)^{-\frac{2-r_0}{2r_0}} \leqslant 2\delta_2, \\
&C_2 L_2^{2-\frac{1+\alpha}{p_1}}\{2M_5\exp(M_4T_0)(2\delta_3)^{1/p_1}\}^{\frac{\alpha}{p_1}} \leqslant \frac{C_1}{2}.
\end{aligned}$$

We suppose that $\int_0^{\hat{\ell}_0} v^{p_1}(0)\,\mathrm{d}x < \delta_3$. Noting that $M_5\exp(M_4T_0)\delta_3^{1/p_1} > \delta_3$, if necessary we choose $M_5>1$ again. Now, if there is a positive number $t_0\in(0,T_0)$ such that

$$M_5\exp(M_4T_0)\delta_3^{\frac{1}{p_1}} \leqslant \int_0^{\hat{\ell}(t_0)} v^{p_1}(t_0)\,\mathrm{d}x < 2M_5\exp(M_4T_0)\delta_3^{\frac{1}{p_1}},$$

then the inequality (5.1) holds for $t \in (0, t_0]$ and $p \in [p_0, p_1]$, and hence by virtue of (5.5) we get the inequality $\int_0^{\hat{\ell}(t_0)} v^{p_1}(t_0)\,\mathrm{d}x < M_5 \exp(M_4 T_0)\delta_3^{\frac{1}{p_1}}$. This is a contradiction.

Therefore, the following inequality holds:

$$\int_0^{\hat{\ell}(t)} v^{p_1}(t)\,\mathrm{d}x \leqslant M_5 \exp(M_4 T_0)\delta_3^{\frac{1}{p_1}} \qquad \text{for } t \in [0, T_0].$$

Similarly, $\{\hat{u}, \hat{\ell}\}$ is the solution on $[0, T_0]$ and in virtue of (5.2) we have

$$\int_0^{\hat{\ell}(T_0)} v^{p_1}(T_0)\,\mathrm{d}x \leqslant 2M_1(1 + M_2 T_0)^{-\frac{2-r_0}{2r_0}} \leqslant 2\delta_2.$$

Furthermore, if there is a positive number $t_1 > T_0$ such that $2\delta_2 < \int_0^{\hat{\ell}(t_1)} v^{p_1}(t_1)\,\mathrm{d}x \leqslant 3\delta_2$, then this is a contradiction to (5.2). Hence, we conclude that

$$\begin{aligned} &\int_0^{\hat{\ell}(t)} v^{p_1}(t)\,\mathrm{d}x \leqslant 2M_1(1 + M_2 t)^{-\frac{2-r_0}{2r_0}} \qquad \text{for } t \geqslant T_0,\\ &|\hat{u}_x(t)|_{L^2(0,\hat{\ell}(t))} \leqslant M_3 \qquad \text{for } t \geqslant 0,\\ &\quad E(\hat{u}(t), \hat{\ell}(t)) \leqslant 2M_1 \qquad \text{for } t \geqslant 0. \end{aligned}$$

Therefore, Theorem 1.2 implies that Theorem 1.3 is valid under the condition $(*)$.

Finally, we give the complete proof of the theorem.

P r o o f of Theorem 1.3. First, we put $X = L^{p_1}(0, \infty)$,

$$u_{01} = \min\{u_0, \hat{u}_0\}, u_{02} = \max\{u_0, \hat{u}_0\}, \ell_{01} = \min\{\ell_0, \hat{\ell}_0\} \text{ and } \ell_{02} = \max\{\ell_0, \hat{\ell}_0\}.$$

Let $\{u_1, \ell_1\}$ and $\{u_2, \ell_2\}$ be solutions to $SP(u_{01}, \ell_{01})$ and $SP(u_{02}, \ell_{02})$ on $[0, T_1]$ and $[0, T_2]$, respectively. Putting $T_3 = \min\{T_1, T_2\}$ it is clear that $\{u_1, \ell_2\} \in G(u_0, \ell_0; M, L, \mu)$ and $|u_{02} - u_0|_X \leqslant |\hat{u}_0 - u_0|_X$, $u_1 \leqslant u$, $\hat{u} \leqslant u_2$ on $Q(T_3)$ and $\ell_1 \leqslant \ell$, $\hat{\ell} \leqslant \ell_2$ on $[0, T_3]$,

$$\begin{aligned} |u(t) - \hat{u}(t)|_X &\leqslant |u_1(t) - u_2(t)|_X \\ &\leqslant |u_2(t) - u(t)|_X + |u(t)|_X + |u_1(t)|_X. \end{aligned}$$

By the above argument there is a positive number δ such that if $|u_0 - u_{02}|_{L^{p_1}(0,\ell_{02})} < \delta$ and $\ell_0 < \ell_{02} < \ell_0 + \delta$, hence $\{u_2, \ell_2\}$ is the global solution to SP and satisfies

$$\begin{aligned} &\ell_2(t) \leqslant 2M_1 \qquad \text{for } t \geqslant 0,\\ &|u(t) - u_2(t)|_X \leqslant 2M_1(1 + M_6 t)^{-\frac{2-r_0}{2r_0}} \qquad \text{for } t \geqslant T_0. \end{aligned}$$

Therefore, if $|\hat{u}_0 - u_0|_{L^{p_1}(0,\hat{\ell}_0)} < \delta$ and $|\hat{\ell}_0 - \ell_0| < \delta$, then $\{\hat{u}, \hat{\ell}\}$ satisfies the required conditions. $\square$

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