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ON GLOBAL TRANSFORMATIONS OF ORDINARY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

Václav Tryhuk, Brno

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Abstract. The paper describes the general form of an ordinary differential equation of the second order which allows a nontrivial global transformation consisting of the change of the independent variable and of a nonvanishing factor. A result given by J. Aczél is generalized. A functional equation of the form

$$f(t, vy, wy + uvz) = f(x, y, z)u^2v + g(t, x, u, v, w)vz + h(t, x, u, v, w)y + 2uwz$$

is solved on $\mathbb{R}$ for $y \neq 0$, $v \neq 0$.

Keywords: ordinary differential equations, linear differential equations, global transformations, functional equations

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1. Introduction

Pointwise transformations and canonical forms of ordinary linear differential equations have been studied by many authors. The theory of global transformations, converting any homogeneous linear differential equation of the n-th order into another equation of the same kind and order on the whole interval of their definition, was developed in the monograph of F. Neuman [6] (see historical remarks, definitions, results and applications).

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The most general form of such global transformations for ordinary homogeneous linear differential equations of the n-th order ($n \geqslant 2$) is

$$z(t) = L(t)y(\varphi(t)) \tag{1}$$

where φ is a bijection of an interval J onto an interval I ($J \subseteq \mathbb{R}$, $I \subseteq \mathbb{R}$) and $L(t)$ is a nonvanishing function on J, i.e. this global transformation consists of a change of the independent variables and of a nonvanishing factor L. The most general form of the pointwise transformation of the class C^n was derived by P. Stäckel [8]. If we suppose that (1), with continuous functions L and φ, transforms the set of all solutions of a linear homogeneous differential equation of the n-th order onto the set of all solutions of an equation of the same type then differentiability of the n-th order of the functions L and φ follows (see [7]). An interesting problem is solved by J. Aczél [1] by means of functional equations, eliminating regularity conditions from [5]. In [1] the ordinary differential equation of the second order in the explicit form

$$y''(x) = f(x, y(x), y'(x)) \tag{2}$$

is considered together with solutions $y(x)$ and $y(h(x))$ of the equation (2), where h satisfies a differential equation

$$h''(x) = g(x, h(x), h'(x)).$$

We can also formulate Aczél's problem by using transformation (1) with the factor $L \equiv 1$ under the conditions $\varphi(I) = I$, $\varphi''(x) = g(x, \varphi(x), \varphi'(x))$, $x \in I$, such that (1) converts any equation $y''(x) = f(x, y(x), y'(x))$, $x \in I$, into itself, i.e. by using a nontrivial stationary transformation.

In this paper we derive, similarly to [1], the most general form of the ordinary differential equation (2) which allows (1) and we generalize Aczél's result. We prove that the most general differential equation of the second order with the above property, defined for $y \in \mathbb{R}$, is the linear differential equation.

2. Notation, basic definitions

Denote by (f) and (f^*) the ordinary differential equations of the second order $y''(x) = f(x, y(x), y'(x))$ and $z''(t) = f^*(t, z(t), z'(t))$, defined on I and J, respectively.

Definition. We say that (f) is globally transformable into (f^*) if there exist two functions φ, L such that

– the function L is of the class $C^2(J)$ and is nonvanishing on J,

– the function φ is a C^2 diffeomorphism of the interval J onto the interval I,

and the function

$$z(t) = L(t)y(\varphi(t)),\ t \in J \tag{3}$$

is a solution of the equation (f) whenever y is a solution of the equation (f^*).

We say that (3) is a *stationary transformation* if it globally transforms an equation (f) into itself on I, i.e. if $L \in C^2(I)$, $L(x) \neq 0$ on I, φ is a C^2 diffeomorphism of I onto $I = \varphi(I)$ and the function $z(x) = L(x)y(\varphi(x))$ is a solution of $z''(x) = f(x, z(x), z'(x))$ whenever y is a solution of $y''(x) = f(x, y(x), y'(x))$, $x \in I$. Hence, if (3) is a stationary transformation of the equation (f) with a solution $y(x)$ then $L(x)y(\varphi(x))$ is also a solution of the same differential equation.

Observation 1. *Every homogeneous linear differential equation of the second order is a particular case of the equation (f) and for two globally transformable linear differential equations $y''(x) = p(x)y'(x) + q(x)y(x)$, $x \in I$; $z''(t) = P(t)z'(t) + Q(t)z(t)$, $t \in J$ there always exist relations*

$$\begin{aligned}
\varphi''(t) &= P(t)\varphi'(t) - p(\varphi(t))\varphi'(t)^2 - 2\frac{L'(t)}{L(t)} = g(t, \varphi(t), \varphi'(t), L(t), L'(t)),\\
L''(t) &= P(t)L'(t) + (Q(t) - q(\varphi(t))\varphi'(t)^2)L(t) = h(t, \varphi(t), \varphi'(t), L(t), L'(t))
\end{aligned}$$

between the functions φ, L and the coefficients of the differential equations.

Assumption. We suppose that there exist two differential equations such that φ, L are related by

$$\varphi''(t) = g(t, \varphi(t), \varphi'(t), L(t), L'(t)), \quad L''(t) = h(t, \varphi(t), \varphi'(t), L(t), L'(t)),\ t \in J.$$

3. Global transformations

Lemma 1. *The transformation* (3) *is a stationary transformation of the equation (f) if and only if $I = \varphi(I)$ and the real functions f, g, h satisfy the functional equation in several variables*

$$f(t, vy, wy + uvz) = f(x, y, z)u^2v + g(t, x, u, v, w)vz + h(t, x, u, v, w)y + 2uwz \tag{4}$$

for all $t, x \in I \subseteq \mathbb{R}$ and $u, v, w, z \in \mathbb{R}$; $y \neq 0$, $v \neq 0$.

P r o o f. The transformation (3) is a stationary transformation of the equation (f) if and only if $y''(x) = f(x, y(x), y'(x))$ is globally transformable into the equation $z''(t) = f(t, z(t), z'(t))$ by means of

$$z(t) = L(t)y(\varphi(t)),\ t \in I = \varphi(I).$$

Then

$$\begin{aligned} z'(t) &= L'(t)y(\varphi(t)) + L(t)\dot{y}(\varphi(t))\varphi'(t)\ (^{\cdot} = \mathrm{d}/\mathrm{d}\varphi), \\ z'' &= L''y(\varphi) + 2L'\dot{y}(\varphi)\varphi' + L\ddot{y}(\varphi)\varphi'^2 + L\dot{y}(\varphi)\varphi'' \end{aligned}$$

are true, where $\varphi'' = g(t, \varphi, \varphi', L, L')$, $L'' = h(t, \varphi, \varphi', L, L')$ on I, according to Assumption.

The functions $z(t)$, $y(\varphi(t))$ satisfy simultaneously the differential equations

$$z'' = f(t, z, z'), \quad \ddot{y}(\varphi) = f(\varphi, y(\varphi), \dot{y}(\varphi)),$$

because (3) is a stationary transformation.

Using the above relations we obtain

$$\begin{aligned} (5) \quad (z'' =)\ & h(t, \varphi, \varphi', L, L')y(\varphi) + 2L'\dot{y}(\varphi)\varphi' + f(\varphi, y(\varphi), \dot{y}(\varphi))L\varphi'^2 \\ & + g(t, \varphi, \varphi', L, L')L\dot{y}(\varphi) = f(t, Ly(\varphi), L'y(\varphi) + L\dot{y}(\varphi)\varphi')\ (= f(t, z, z')), \end{aligned}$$

which is for $x = \varphi(t)$, $u = \varphi'(t)$, $v = L(t)$, $w = L'(t)$, $y = y(\varphi(t))$, $z = \dot{y}(\varphi(t))$ equivalent to the functional equation (4). Here $t, x \in I \subseteq \mathbb{R}$; $u, v, w, z \in \mathbb{R}$; $v \neq 0$, in accordance with the definition of the global transformation.

Of course, if (4) is satisfied, then (5) holds for $x = \varphi(t)$, $u = \varphi'(t)$, $v = L(t)$, $w = L'(t)$, $y = y(\varphi(t))$, $z = \dot{y}(\varphi(t))$, where $t \in I \subseteq \mathbb{R}$ and $\varphi(I) = I$. $\square$

Theorem 1. *The general continuous solution of the functional equation* (4) *is given by*

$$\begin{aligned} f(x, y, z) &= k\frac{z^2}{y} + p(x)z + q(x)y, \\ g(t, x, u, v, w) &= 2(k-1)\frac{uw}{v} + p(t)u - p(x)u^2, \qquad (6) \\ h(t, x, u, v, w) &= k\frac{w^2}{v} + p(t)w + (q(t) - q(x)u^2)v \end{aligned}$$

where p, q are arbitrary functions and k is a real constant.

Proof. Inserting into (4) $w = 0$, $v = 1$, $x = x_0 \in I$ (arbitrary fixed), we obtain

$$f(t, y, uz) = f(x_0, y, z)u^2 + \tilde{g}(t, u)z + \tilde{h}(t, u)y \tag{7}$$

where $\tilde{g}(t, u) = g(t, x_0, u, 1, 0)$, $\tilde{h}(t, u) = h(t, x_0, u, 1, 0)$.

Together with $z = 1$ we have

$$f(t, y, u) = \delta(y)u^2 + \tilde{g}(t, u) + \tilde{h}(t, u)y. \tag{8}$$

But $u = 1, z = U, D(y, U) = f(x_0, y, U), p(t) = \tilde{g}(t, 1), Q(t) = \tilde{h}(t, 1)$ in (7) give

$$f(t, y, U) = D(y, U) + p(t)U + Q(t)y = \delta(y)U^2 + \tilde{g}(t, U) + \tilde{h}(t, U)y. \tag{9}$$

Comparison of the terms depending on y gives

$$D(y, U) = (\tilde{h}(t, U) - Q(t))y + \delta(y)U^2 \tag{10}$$

and further on

$$\tilde{g}(t, u) = p(t)u. \tag{11}$$

According to (10), $\tilde{h}(t, u) - Q(t) = C(u)$ is independent of t, hence

$$\tilde{h}(t, u) = Q(t) + C(u). \tag{12}$$

So (10) gives $D(y, u) = C(u)y + \delta(y)u^2$. Substituting this expression into (9) we obtain

$$f(t, y, u) = \delta(y)u^2 + p(t)u + (C(u) + Q(t))y \tag{13}$$

i.e.

$$f(x_0, y, z) = \delta(y)z^2 + p(x_0)z + (C(z) + Q(x_0))y. \tag{14}$$

We substitute (11), (12), (14) into (7)

$$\begin{aligned}&\delta(y)u^2z^2 + p(t)uz + (C(uz) + Q(t))y \\ &\quad = \delta(y)u^2z^2 + p(x_0)u^2z + (C(z) + Q(x_0))u^2y + p(t)uz + C(u)y + Q(t)y\end{aligned}$$

and compare the terms containing y

$$C(uz) = (C(z) + Q(x_0))u^2 + C(u) = (C(u) + Q(x_0))z^2 + C(z) = C(zu).$$

Thus

$$C(u)(z^2-1) = C(z)(u^2-1) + Q(x_0)(u^2-z^2), \quad z \notin \{-1,1\}$$

or, with $z = z_0 \notin \{-1,1\}$ constant,

$$C(u) = cu^2 + d. \tag{15}$$

So (13) becomes

$$f(t,y,u) = (\delta(y)+cy)u^2 + p(t)u + (Q(t)+d)y = a(y)u^2 + p(t)u + q(t)y. \tag{16}$$

To find the function $h(t,x,u,v,w)$ we put $z=0$, $y=1$ in (4). Then

$$f(t,v,w) = f(x,1,0)u^2v + h(t,x,u,v,w) \tag{17}$$

and (17) combined with (16) gives

$$\begin{aligned} h(t,x,u,v,w) &= f(t,v,w) - f(x,1,0)u^2v \\ &= a(v)w^2 + p(t)w + q(t)v - (a(1)\cdot 0 + p(x)\cdot 0 + q(x))u^2v, \end{aligned}$$

i.e.

$$h(t,x,u,v,w) = a(v)w^2 + p(t)w + (q(t) - q(x)u^2)v. \tag{18}$$

In order to determine the function $g(t,x,u,v,w)$ we put $y=1$, $z=1$ in (4). We obtain

$$f(t,v,w+uv) = f(x,1,1)u^2v + g(t,x,u,v,w)v + h(t,x,u,v,w) + 2uw$$

and

$$\begin{aligned} g(t,x,u,v,w)v &= f(t,v,w+uv) - f(x,1,1)u^2v - h(t,x,u,v,w) - 2uw \\ &= a(v)(2w+uv)uv + p(t)uv - p(x)u^2v - 2uw - a(1)u^2v \end{aligned}$$

using (16), (18). So we have

$$g(t,x,u,v,w) = a(v)(2w+uv)u + p(t)u - p(x)u^2 - a(1)u^2 - 2u\frac{w}{v} \tag{19}$$

because of $v \neq 0$.

Finally, the equation (4) with the functions (16), (18) and (19) implies that

$$a(vy)(wy+uvz)^2 = a(v)(2w+uv)uvz + a(v)w^2y + a(y)u^2vz^2 - a(1)u^2vz. \tag{20}$$

Consider (20) and $v = w = 1$. Then (20) becomes $a(y)y^2 = a(1)y$. Thus $a(y) = \frac{k}{y}$, $k = a(1)$ and from (16), (18), (19) we obtain the assertion of Theorem 1. □

Theorem 2. *The transformation* (3) *is a stationary transformation of the equation* (f) *if and only if the following equation holds:*

$$f(x,y,y') = k\frac{y'^2}{y} + p(x)y' + q(x)y,\ y = y(x) \tag{21}$$

and

$$\begin{gathered} \varphi'' = 2(k-1)\frac{L'}{L}\varphi' + p(t)\varphi' - p(\varphi)\varphi'^2, \\ \Big(\frac{L'}{L}\Big)' = (k-1)\Big(\frac{L'}{L}\Big)^2 + p(t)\frac{L'}{L} + q(t) - q(\varphi)\varphi'^2, \\ \varphi = \varphi(t),\ L = L(t),\ t \in I,\ \varphi(I) = I \end{gathered} \tag{22}$$

are satisfied, where p, q *are arbitrary functions and* k *is a real constant.*

Proof. The assertion of Theorem 2 follows immediately from Lemma 1 and Theorem 1 using $x = \varphi(t)$, $u = \varphi'(t)$, $v = L(t)$, $w = L'(t)$, $y = y(\varphi(t))$, $z = \dot{y}(\varphi(t))$, $t \in I$. $\square$

Remark 1. If the conditions of Theorem 2 hold but y vanishes at some point of I, then $k\frac{y'^2}{y}$ exists only if $k = 0$. Hence (f) is a linear differential equation.

Theorem 3. *The transformation* $z(t) = L(t)y(\varphi(t))$ *is the most general transformation converting any equation*

$$y'' = k\frac{y'^2}{y} + p(x)y' + q(x)y,\ y = y(x),\ x \in I, \tag{23}$$

into an equation

$$z'' = k\frac{z'^2}{z} + P(t)z' + Q(t)z,\ z = z(t),\ t \in J;\ k \in \mathbb{R}. \tag{24}$$

Moreover, the functions L, φ *satisfy the conditions*

$$\begin{gathered} P(t) = \frac{\varphi''}{\varphi'} + p(\varphi)\varphi' + 2(1-k)\frac{L'}{L}, \\ Q(t) = \frac{L''}{L} + q(\varphi)\varphi'^2 - P(t)\frac{L'}{L} - k\Big(\frac{L'}{L}\Big)^2, \end{gathered} \tag{25}$$

$\varphi = \varphi(t)$, $L = L(t)$ *on* J.

P r o o f. We prove that the transformation $z(t) = L(t)y(\varphi(t))$ converts any equation (23) into an equation (24). We have

$$
\begin{aligned}
y(x) &= y(\varphi(t)) = \frac{z(t)}{L(t)},\\
y' &= \frac{1}{L\varphi'}(z' - z\frac{L'}{L}),\\
y'' &= \frac{1}{L\varphi'^2}\Big(z'' - z'(2\frac{L'}{L} + \frac{\varphi''}{\varphi'}) - z\Big(\frac{L''}{L} - 2\Big(\frac{L'}{L}\Big)^2 - \frac{\varphi''}{\varphi'}\frac{L'}{L}\Big)\Big)
\end{aligned}
$$

and the equation (23) becomes (24), where

$$
\begin{aligned}
P(t) &= \frac{\varphi''}{\varphi'} + p(\varphi)\varphi' + 2(1-k)\frac{L'}{L},\\
Q(t) &= \frac{L''}{L} + q(\varphi)\varphi'^2 - \frac{L'}{L}\Big(\frac{\varphi''}{\varphi'} + 2\frac{L'}{L} + p(\varphi)\varphi' - k\frac{L'}{L}\Big)\\
&= \frac{L''}{L} + q(\varphi)\varphi'^2 - P(t)\frac{L'}{L} - k\Big(\frac{L'}{L}\Big)^2
\end{aligned}
$$

and (25) is satisfied on J.

According to Remark 1, every linear differential equation is a particular case of the equation (f). The most general pointwise transformation for the linear differential equation (see Stäckel [8], Wilczynski [9], Čadek [3] and [4], Neuman [6]) is of the form $z(t) = L(t)y(\varphi(t))$. This fact implies that (3) is the most general transformation for the equation (23). □

Remark 2. In the case when $L(t) \equiv 1$ $(= v)$ we get $L'(t) \equiv 0 (= w), L'' \equiv 0$ $(= h(t, \varphi, \varphi', 1, 0))$ and the functional equation (4) is of the form

$$f(t, y, uz) = f(x, y, z)u^2 + g(t, x, u)z$$

with the general solution

$$f(x, y, z) = a(y)z^2 + p(x)z, \quad g(t, x, u) = p(t)u - p(x)u^2,$$

where a, p are arbitrary functions (see Aczél [1]). Similarly to Theorem 2 we can prove the following:

The transformation $z(t) = y(\varphi(t))$ is a stationary transformation of the equation (f) if and only if the relations

$$f(x, y, y') = a(y)y'^2 + p(x)y' + q(x)y, \ \ y = y(x)$$

and

$$\varphi'' = p(t)\varphi' - p(\varphi)\varphi'^2, \quad \varphi = \varphi(t),\ t \in I,\ \varphi(I) = I$$

hold, where a, p are arbitrary functions and the function q satisfies the condition $q(t) - q(\varphi)\varphi'^2 = 0$.

This result is not a special case of Theorem 2 because there L can be arbitrary and accordingly $a(y) = \frac{k}{y}$, while here $L(t) \equiv 1$ and a is an arbitrary function.

Example. We investigate stationary transformations of the equation

$$y''(x) = \frac{y'(x)^2}{y(x)} + \frac{1}{x}y'(x) - (x \sin x)y(x),\ x \in I = (0, \infty).$$

Using the conditions of Theorem 2 we have

$$\varphi''(x) = 2(k-1)\frac{L'(x)}{L(x)}\varphi'(x) + \frac{1}{x}\varphi'(x) - \frac{1}{\varphi(x)}\varphi'(x)^2 = (\frac{1}{x} - \frac{\varphi'(x)}{\varphi(x)})\varphi'(x),$$

i.e. $\varphi(x)^2 = a_1 x^2 + b$ and $\varphi(0) = 0$ for $I = \varphi(I)$, thus

$$\varphi(x) = ax, \quad a \in \mathbb{R} - \{0\}.$$

Then

$$\begin{aligned}\left(\frac{L'(x)}{L(x)}\right)' &= (k-1)\left(\frac{L'(x)}{L(x)}\right)^2 + p(x)\frac{L'(x)}{L(x)} + q(x) - q(\varphi(x))\varphi'(x)^2 \\ &= \frac{1}{x}\frac{L'(x)}{L(x)} - x\sin x + a^3 x \sin ax\end{aligned}$$

and solving this differential equation we obtain

$$L(x) = b\frac{\exp\{x\sin x + \cos x\}}{\exp\{ax \sin ax + \cos ax\}}\exp\{dx^2\};\ b, d \in \mathbb{R},\ b \neq 0.$$

The general solution of the given equation is of the form

$$y(x) = K\exp\{Cx^2\}\exp\{x\sin x + \cos x\};\ K, C \in \mathbb{R},\ K \neq 0.$$

Moreover,

$$\begin{aligned}L(x)y(\varphi(x)) &= bK\exp\{(d + Ca^2)x^2\}\exp\{x\sin x + \cos x\} \\ &= K_1\exp\{C_1x^2\}\exp\{x\sin x + \cos x\};\end{aligned}$$

K_1, $C_1 \in \mathbb{R}$, $K_1 \neq 0$ and $L(x)y(\varphi(x))$ is a solution whenever $y(x)$ is a solution.

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Author's address: Department of Mathematics, Faculty of Civil Engineering, Technical University of Brno, Žižkova 17, 602 00 Brno, Czech Republic, e-mail: `mdtry@fce.vutbr.cz`.