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ON NONOSCILLATION OF CANONICAL OR NONCANONICAL DISCONJUGATE FUNCTIONAL EQUATIONS

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Abstract. Qualitative comparison of the nonoscillatory behavior of the equations

$$L_n y(t) + H(t, y(t)) = 0$$

and

$$L_n y(t) + H(t, y(g(t))) = 0$$

is sought by way of finding different nonoscillation criteria for the above equations. L_n is a disconjugate operator of the form

$$L_n = \frac{1}{p_n(t)} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{p_{n-1}(t)} \frac{\mathrm{d}}{\mathrm{d}t} \dots \frac{\mathrm{d}}{\mathrm{d}t} \frac{\cdot}{p_0(t)}.$$

Both canonical and noncanonical forms of L_n have been studied.

Keywords: canonical, noncanonical, oscillatory, nonoscillatory, principal system

MSC 2000: 34K15

1. Introduction

Our main purpose in this work is to study the nonoscillation phenomenon in the disconjugate functional equations

$$L_n y(t) + H(t, y(t)) = 0 \tag{1}$$

and

$$L_n y(t) + H(t, y(g(t))) = 0 \tag{2}$$

where $n \geqslant 2$ and L_n denotes the disconjugate differential operator

$$L_n = \frac{1}{p_n(t)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{p_{n-1}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\dots\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{p_1(t)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\cdot}{p_0(t)}.$$

It will be shown in this work that the retardation $g(t)$ puts an entirely different nonoscillatory structure on equation (2) as compared to equation (1). This difference in the nonoscillatory behavior of (1) and (2), that we point out in this work, seems to be new as borne out by our literature search. Our main theorem essentially gives a set of conditions subject to which every nontrivial solution of equation (1) is nonoscillatory. However the very same conditions in the presence of retardation $g(t)$, allow equation (2) to have oscillatory solutions.

It should be noted that in this setting it is not so much the *difference* in the behavior of (1) and (2) caused by $g(t)$ that we are commenting on, but rather the *quality* of the difference caused by $g(t)$. It is well known, for example (Travis [14] and this author [11]) that the equation

$$y''(t) + \frac{\sin t}{2-\sin t}y(t) = 0 \tag{3}$$

is oscillatory, whereas the equation

$$y''(t) + \frac{\sin t}{2-\sin t}y(t-\pi) = 0 \tag{4}$$

has $y(t) = 2 + \sin t$ as a nonoscillatory solution.

In what follows, we shall assume that p_i, $g\colon [a,\infty) \to \mathbb{R}$ and $H\colon [a,\infty)\times\mathbb{R}\to\mathbb{R}$ are continuous, $p_i(t) > 0$, $0 \leqslant i \leqslant n$, $g(t) \leqslant t$ and $g(t) \to \infty$. The constant $a > 0$ is fixed. We also assume that p_i, $0 \leqslant i \leqslant n-1$ are continuously differentiable and g is differentiable on (a,∞).

$$L_0x(t) = \frac{x(t)}{p_0(t)},\ L_ix(t) = \frac{1}{p_i(t)}\frac{\mathrm{d}}{\mathrm{d}t}L_{i-1}(x(t)), \tag{5}$$

$1 \leqslant i \leqslant n$. The domain of L_n is defined to be the set of all functions $y\colon [T_y,\infty)\to\mathbb{R}$ such that $L_iy(t)$, $0 \leqslant i \leqslant n$, exist and are continuous on $[T_y,\infty)$. By a proper solution of (1) or (2) we mean a function y in the domain of L_n which satisfies the corresponding equation for all sufficiently large t and $\sup\{|y(t)|\colon t \geqslant T\} > 0$ for every $T \geqslant T_y$. A proper solution of (1) or (2) is called *oscillatory* if it has arbitrarily large zeros; otherwise the solution is called nonoscillatory. The term "solution" in the foregoing analysis only applies to a proper solution of the equation under consideration. We assume that such solutions exist.

Let

$$i_k \in \{1, \ldots, n-1\},\ 1 \leqslant k \leqslant n-1, \text{ and } t, s \in [a, \infty).$$

We define

$$\begin{cases} I_0 \equiv 1 \\ I_k(t, s; p_{i_k}, \ldots, p_{i_1}) = \int_s^t p_{i_k}(r) I_{k-1}(r, s; p_{i_{k-1}}, \ldots, p_{i_1})\, \mathrm{d}r. \end{cases} \tag{6}$$

It is readily verified that

$$I_k(t, s; p_{i_k}, \ldots, p_{i_1}) = \int_s^t p_{i_1}(r) I_{k-1}(t, r; p_{i_k}, \ldots, p_{i_2})\, \mathrm{d}r. \tag{7}$$

For convenience we let for $0 \leqslant i \leqslant n-1$

$$J_i(t, s) = p_0(t) I_i(t, s; p_1, \ldots, p_i),\ J_i(t) = J_i(t, a), \tag{8}$$

$$K_i(t, s) = p_n(t) I_i(t, s; p_{n-1}, \ldots, p_{n-i}),\ K_i(t) = K_i(t, a). \tag{9}$$

Our literature search reveals that no nonoscillation results seem to be known which guarantee that all solutions of (1) or (2) be nonoscillatory. Theorem 2 of Dahiya and this author [1] gives a nonoscillation criterion for a second order homogeneous equation. Onose [3] obtains oscillatory criteria for differential equations of arbitrary order. The work of Staikos and Philos [13] discusses nonoscillation of bounded solutions of (2) when it is advanced, i.e. $g(t) \geqslant t$. The results obtained by Dzurina and Ohriska [2] assume that all $p_i(t)$, $1 \leqslant i \leqslant n-1$ be the same and hence restricted for equation (2) and different from ours. The works of Philos and Staikos [5], Sficas and Stavroulakis [6] and this author [8, 11, 12] allude to the asymptotic nature of oscillatory and nonoscillatory solutions when (1) and (2) are forced equations. Our work in [7] mostly deals with the asymptotic limits of the oscillatory solutions of equation (2). There are quite a few criteria known (see Philos [4]) ensuring oscillations of all solutions of (1) and (2), but the literature is scanty at best about the results providing complete nonoscillation of (1) and (2).

In section 4, we extend these results to fourth order elliptic equations.

2. Main results

We first consider the case where

$$\int_a^\infty p_i(t)\,\mathrm{d}t = \infty, \quad 1 \leqslant i \leqslant n-1. \tag{10}$$

The differential operator L_n in (1) and (2) is said to be in canonical form if condition (10) is satisfied. It is shown by Trench [15] that any differential operator of the form of L_n can be represented in canonical form in an essentially unique manner.

The next lemma is crucial in our main theorem. It is Theorem 3 in [7].

Lemma 1. *Suppose that* (10) *holds,* $g(t) \leqslant t$ *and there exists a number* $\gamma \in (0,1]$ *such that*

$$|H(t,x)| \leqslant q(t)|x|^\gamma \quad \text{for} \quad (t,x) \in [a,\infty) \times \mathbb{R} \tag{11}$$

where $q\colon\ [a,\infty) \to [0,\infty)$ *is continuous. Further suppose that*

$$\int^\infty [J_{n-1}(g(t))]^\gamma K_{n-1}(t) q(t)\,\mathrm{d}t < \infty. \tag{12}$$

Then every oscillatory solution $y(t)$ *of* (2) *(and hence* (1)*) satisfies*

$$\operatorname*{Lim}_{t\to\infty} \left[\frac{y(t)}{p_0(t)}\right] = 0. \tag{13}$$

Theorem 1. *Suppose* $g(t) \equiv t$ *and conditions of Lemma* 1 *hold. Further suppose that* $p_0(t) \leqslant M_0$ *for* $t \in [a,\infty)$ *for some constant* $M_0 > 0$. *Let* $\gamma \equiv 1$. *Then all solutions of* (1) *are nonoscillatory.*

Proof. Suppose to the contrary that (1) has an oscillatory solution $y(t)$. By Lemma 1, $y(t) \to 0$ as $t \to \infty$. Let $T > a$ be large enough so that condition (12) holds for $t \geqslant T$. From (12) it follows that T can be chosen large enough so that

$$M_0 \int_T^\infty K_{n-1}(t,T) q(t)\,\mathrm{d}t < \frac{1}{2}. \tag{14}$$

Let $t_2 > t_1 > T$ be such that $L_0 y(t_1) = L_0 y(t_2) = 0$, $L_0 y(t) \neq 0$, $t \in (t_1,t_2)$. Let $0 < M = \operatorname{Max}|L_0 y(t)|$ for $t_1 \leqslant t \leqslant t_2$. There exists $T_1 > t_2$ such that $L_0 y(T_1) = 0$ and

$$\sup\{|L_0 y(t)|\colon\ t \geqslant T_1\} < M. \tag{15}$$

Since $L_i(y(t))$ is oscillatory for $i = 0, 1, 2, \ldots, n-1$, let e_i, $i = 1, 2, \ldots, n-1$ be such that

$$e_1 < e_2 < e_3 < \ldots < e_{n-1}, \quad e_1 > T_1 \tag{16}$$

and

$$L_i(y(e_i)) = 0, \quad i = 1, 2, \ldots, n-1. \tag{17}$$

Let

$$M_1 = |L_0 y(T_2)| = \mathrm{Max}\{|L_0 y(t)|\colon\ t \in [t_1, T_1]\}. \tag{18}$$

It is clear from (15) and (18) that $M_1 \geqslant M$ and $T_2 \in [t_1, T_1]$. Thus

$$\sup\{|L_0 y(t)|\colon\ t \geqslant t_1\} \leqslant M_1 = |L_0 y(T_2)| \tag{19}$$

where

$$t_1 < T_2 \leqslant T_1. \tag{20}$$

On repeated integration we obtain from (1)

$$\begin{aligned} \pm(L_0 y(t))' = p_1(t) \int_t^{e_1} p_2(x_2) \int_{x_2}^{e_2} p_3(x_3) \ldots p_{n-1}(x_{n-1}) \\ \times \int_{p_{n-1}}^{e_{n-1}} p_n(x) H(x, y(x)) \, \mathrm{d}x \, \mathrm{d}x_{n-1} \ldots \mathrm{d}x_2. \end{aligned} \tag{21}$$

Integrating (21) between t and T_2 we have

$$\begin{aligned} \pm M_1 = \int_{t_1}^{T_2} p_1(x_1) \int_{x_1}^{e_1} p_x(x_2) \int_{x_2}^{e_2} \ldots p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{e_{n-1}} p_n(x) H(x, y(x)) \, \mathrm{d}x \\ \times \, \mathrm{d}x_{n-1} \, \mathrm{d}x_{n-2} \ldots \mathrm{d}x_2 \, \mathrm{d}x_1. \end{aligned} \tag{22}$$

From condition (11) we have

$$\begin{aligned} M_1 \leqslant \int_{t_1}^{T_2} p_1(x_1) \int_{x_1}^{e_1} p_2(x_2) \int_{x_2}^{e_2} \ldots p_{n-1}(x_{n-1}) \\ \times \int_{x_{n-1}}^{e_{n-1}} p_n(x) q(x) |y(x)| \, \mathrm{d}x \, \mathrm{d}x_{n-1} \ldots \mathrm{d}x_1, \end{aligned} \tag{23}$$

from which we get

$$M_1 \leqslant \int_{t_1}^{e_{n-1}} p_1(x_1) \int_{x_1}^{e_{n-1}} p_2(x_2) \int_{x_2}^{e_{n-1}} \dots p_{n-1}(x_{n-1}) \times \int_{x_{n-1}}^{e_{n-1}} p_n(x)q(x)|y(x)| \,\mathrm{d}x \,\mathrm{d}x_{n-1} \dots \mathrm{d}x_1.$$

$$M_1 \leqslant \int_{t_1}^{e_{n-1}} p_n(x) I_{n-1}(x, t_1; p_{n-1}, \dots, p_1) q(x)|y(x)| \,\mathrm{d}x$$

which in view of the definition of K_{n-1} in (9) yields

$$M_1 \leqslant \int_{t_1}^{e_{n-1}} k_{n-1}(x, t_1) q(x)|y(x)| \,\mathrm{d}x. \tag{25}$$

Inequality (25), in view of Lemma 1 and (14) where $|L_0 y(t)| \leqslant M_1$ and $p_0(t) \leqslant M_0$ for $t \geqslant t_1 \geqslant T$, yields

$$1 \leqslant M_0 \int_{t_1}^{\infty} K_{n-1}(x, t_1) g(x) \,\mathrm{d}x < \frac{1}{2}. \tag{26}$$

This contradiction completes the proof. □

Example 1. Consider the equation

$$(\mathrm{e}^t y(t))''' + \mathrm{e}^{(-t-3\pi/2)} y\Big(\frac{t-3\pi}{2}\Big) = 0, \quad t \geqslant \frac{3\pi}{2}. \tag{27}$$

In can be easily verified that all conditions of Theorem 1 are satisfied. Hence all solutions of the equation

$$(\mathrm{e}^t y(t))''' + \mathrm{e}^{(-t-3\pi/2)} y(t) = 0 \tag{28}$$

are nonoscillatory. The conclusion, however, need not be true for the retarded equation (27) since the proof of Theorem 1 fails when $g(t)$ is present. Since this difference in the behavior of (27) and (28) is being caused by the delay term, we need to restrict it in order to obtain nonoscillation conditions for equation (2). Our next theorem obtains these conditions.

Theorem 2. *Suppose*

$$\gamma \equiv 1, \quad p_0(t) \equiv M_0$$

for some constant $M_0 > 0$ *for all* $t \geqslant a$. *Suppose moreover that condition* (11) *of Lemma* 1 *holds. Let*

$$\ell_0 \leqslant g'(t) \leqslant Q_0 \quad \text{where} \quad Q_0 > \ell_0 > 0 \quad \text{for } t \geqslant a.$$

Define

$$P_i(t) = \frac{p_i(t)}{g'(t)}, \quad 1 \leqslant i \leqslant n. \tag{29}$$

Further suppose that condition (12) *of Lemma* 1 *is replaced by*

$$\int^{\infty} J_{n-1}(t) K_{n-1}(g^{-1}(t)) q(g^{-1}(t))\,\mathrm{d}t < \infty. \tag{30}$$

Then all solutions of equation (2) *are nonoscillatory.*

Proof. Let g^{-1} be the inverse function of g. The substitution $u = g(t)$ transforms equation (2) into

$$\Delta_n y(u) + H(g^{-1}(u), y(u)) = 0 \tag{31}$$

where

$$\begin{cases} \Delta_0 y(u) = \frac{y(g^{-1}(u))}{M_0} \\ \Delta_1 y(u) = \frac{1}{P_1(g^{-1}(u))}(\Delta_0 y(u))' \\ \vdots \\ \Delta_i y(u) = \frac{1}{P_i(g^{-1}(u))}(\Delta_{i-1} y(u))' \end{cases} \tag{32}$$

for $1 \leqslant i \leqslant n$. Suppose to the contrary that $y(u)$ is an oscillatory solution of (31). Replacing e_i with appropriate u_i, $1 \leqslant i \leqslant n-1$, and following the proof of Theorem 1 identically with p_i replaced by P_i, $1 \leqslant i \leqslant n$, we reach inequality (26) which in the present setting states

$$1 \leqslant \int_{u_1}^{\infty} K_{n-1}(g^{-1}(u), g^{-1}(U)) q(g^{-1}(u)) du < \frac{1}{2} \tag{33}$$

where $u_i = g(e_i)$, $T = g(U)$ for some $U > a$, and $|\Delta_0 y(u)| \leqslant M_1$ for $t \geqslant u_1$. Since $M_1 > 0$, a contradiction readily follows in view of condition (30). The proof is complete. □

Example 2. Consider the equation

$$y''(t) + \mathrm{e}^{-t}(y(t-\pi)) = 0, \quad t > \pi. \tag{34}$$

It is easily verified that all conditions of Theorem 2 are satisfied. Hence all solutions of this equation are nonoscillatory.

3. Noncanonical L_n

Suppose now that the operator L_n in (1) and (2) is not in canonical form. Thus condition (10) no longer holds. According to Trench [15], a different set of functions $\widetilde{p_i}(t)$ obtained from $p_i(t)$, $1 \leqslant i \leqslant n-1$ can be derived uniquely so that

$$\int_a^\infty \widetilde{p_i}(t)\,\mathrm{d}t = \infty, \quad 1 \leqslant i \leqslant n-1. \tag{35}$$

The functions

$$\widetilde{p_i}(t), \quad 1 \leqslant i \leqslant n-1,$$

are determined up to positive multiplicative constants with product 1.

Even though the actual derivation of the functions

$$\widetilde{p_i}(t), \quad 1 \leqslant i \leqslant n-1,$$

is tedious and difficult to obtain, we shall in this section obtain analogues of Theorem 1 and Theorem 2. To this end we need the concept of a *principal system* associated with the operator L_n.

By a principal system for L_n is meant a set of n solutions $x_1(t)$, $x_2(t), \ldots, x_n(t)$ of $L_n x(t) = 0$ which are eventually positive and satisfy

$$\operatorname*{Lim}_{t\to\infty} \frac{x_i(t)}{x_j(t)} = 0 \quad \text{for} \quad 1 \leqslant i < j \leqslant n.$$

In case L_n is in canonical form, the set of functions

$$\{J_0(t), J_1(t), \ldots, J_{n-1}(t)\} \tag{36}$$

defined earlier is a principal system for L_n, and the set of functions

$$\{K_0(t), K_1(t), \ldots, K_{n-1}(t)\} \tag{37}$$

defined by (9) is a principal system for the operator

$$M_n = \frac{1}{p_0(t)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{p_1(t)}\frac{\mathrm{d}}{\mathrm{d}t}\ldots\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{p_{n-1}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\cdot}{p_n(t)} \tag{38}$$

which is also in canonical form. For a general operator L_n a principal system can be easily obtained by direct integration of the equation

$$L_n x(t) = 0. \tag{39}$$

A basic property of principal systems is that if

$$\{x_1(t), x_2(t), \ldots, x_n(t)\} \quad \text{and} \quad \{\widetilde{x}_1(t), \widetilde{x}_2(t), \ldots, \widetilde{x}_n(t)\}$$

are any two principal systems for the same operator L_n, then the limits

$$\operatorname*{Lim}_{t\to\infty} \frac{\widetilde{x}_i(t)}{x_i(t)} > 0, \quad 1 \leqslant i \leqslant n, \tag{40}$$

exist and are finite (Trench [15]). Theorem 3 below is an analogue of Theorem 1.

Theorem 3. *Suppose $g(t) \equiv t$ and there exists a continuous function $q\colon [a,\infty) \to [0,\infty)$ such that*

$$|H(t,x)| \leqslant q(t)|x| \tag{41}$$

for $(t,x) \in [a,\infty) \times \mathbb{R}$. Further suppose that there exists a constant $M_0 > 0$ such that $p_0(t) \leqslant M_0$ for $t \geqslant a$. Let $\{X_1(t), X_2(t), \ldots, X_n(t)\}$ be a principal system for L_n and let $\{Y_1(t), Y_2(t), \ldots, Y_n(t)\}$ be a principal system for M_n defined by (38). *Suppose that*

$$\int^\infty [X_n(t)] Y_n(t) q(t)\,\mathrm{d}t < \infty. \tag{42}$$

Then all solutions of (1) *are nonoscillatory.*

Proof. Let $\widetilde{p}_i(t)$, $1 \leqslant i \leqslant n-1$ be the functions obtained from $p_i s$ in the sense of Trench [15] to give the canonical representation of L_n. Thus condition (35) holds. Let

$$\{\widetilde{X}_1(t), \widetilde{X}_2(t), \ldots, \widetilde{X}_n(t)\} \quad \text{be the set} \quad \{\widetilde{J}_0(t), \widetilde{J}_1(t), \ldots, \widetilde{J}_{n-1}(t)\}.$$

In this notation, condition (12) of Lemma 1 simply states

$$\int^\infty [\widetilde{X}_n(t)] Y_n(t) q(t)\,\mathrm{d}t < \infty. \tag{43}$$

All conditions of Theorem 1 are satisfied and the proof is complete. □

Example 3. Consider the equation

$$(t^4 x''(t))'' + \frac{1}{t^4} x(t) = 0, \quad t \geqslant 1. \tag{44}$$

If we set $L_4 x(t) = (t^4 x''(t))''$ then $L_4 \equiv M_4$. On repeated integration of $L_4 x(t) = 0$ we obtain $\{t^{-2}, t^{-1}, 1, t\}$ as a principal system for $L_4 \equiv M_4$. We notice that all conditions of Theorem 3 are satisfied. Hence all solutions of equation (44) are nonoscillatory.

Our next theorem is an analogue of Theorem 2. Its proof is similar to the proof of Theorem 3.

Theorem 4. *Suppose $p_0(t) \equiv M_0$ for some constant $M_0 > 0$ for all $t \geqslant a$. Further suppose that $g(t) \leqslant t$ and there exists a continuous function $q(t)\colon\ [a,\infty) \to [0,\infty)$ such that* (41) *holds. Let $g'(t)$ be bounded and bounded away from zero for $t \geqslant a$. Suppose $P_i(t)$, $1 \leqslant i \leqslant n$ are defined by* (29). *Let $\{X_1(t), X_2(t), \ldots, X_n(t)\}$ and $\{Y_1(t), Y_2(t), \ldots, Y_n(t)\}$ be principal systems for L_n and M_n respectively. Suppose that*

$$\int^{\infty} X_n(t) Y_n(g^{-1}(t)) q(g^{-1}(t))\,\mathrm{d}t < \infty. \tag{45}$$

Then all solutions of (2) *are nonoscillatory.*

Example 4. In a manner of Example 3 all solutions of the retarded equation

$$(t^4 x''(t))'' + \frac{1}{t^4} x(\mathrm{e}^{\frac{-\pi}{2} t}) = 0 \tag{46}$$

are nonoscillatory. Since $L_4 \equiv M_4$, it is easily seen that all conditions of Theorem 4 are satisfied. The principal system for $L_4 \equiv M_4$ is the same as in Example 2.

4. Extension to elliptic equations

In this section, we apply the preceding results to the fourth order elliptic equation

$$\Delta^2 u(|s|) + q(|s|) u(|s|) = 0 \tag{47}$$

in an exterior domain

$$\Omega_T = \{s = (s_1, s_2, s_3) \in \mathbb{R}^3 \colon\ |s| \geqslant T\} \tag{48}$$

where T and $q(t)$ are the same as before and

$$|s| = \left(\sum_{i-1}^{3} s_i^2\right)^{\frac{1}{2}}. \tag{49}$$

The operator Δ is the three dimensional Laplacian operator.

We are concerned with spherically symmetric solutions $u = u(|s|)$ of (47) which exist in exterior domain of the type (48) for sufficiently large positive number T. In a manner similar to Lemma 2 of Singh [10], it follows that a function $u(|s|)$ in Ω is a solution of (47) if and only if $u(t)$ is a solution of the differential equation

$$\frac{1}{t}\frac{\mathrm{d}^4}{\mathrm{d}t^4}(tu)+q(t)u(t)=0 \tag{50}$$

on the interval $[T,\infty)$ where $t=|s|$. Equation (50) is equivalent to

$$\frac{\mathrm{d}^4 w}{\mathrm{d}t^4}+q(t)w=0. \tag{51}$$

Since $J_i(t)$ and $K_i(t)$ associated with the operator

$$L_4\equiv\frac{\mathrm{d}^4}{\mathrm{d}t^4} \tag{52}$$

can be taken to be

$$J_i(t)=K_i(t)=t^i \quad 0\leqslant i\leqslant 3, \tag{53}$$

we essentially have the following theorem as an analogue of Theorem 1 for the elliptic equation (47).

Theorem 5. *Suppose* $q\colon [a,\infty)\to(0,\infty)$ *is continuous and*

$$\int^{\infty}|s|^6 q(|s|)\,\mathrm{d}s<\infty. \tag{54}$$

Then all nontrivial solutions of equation (50) *existing in the exterior domain* Ω *are nonoscillatory.*

Proof. The companion differential equation (51) satisfies all conditions of Theorem 1. The condition (12) of Lemma 1 translates to

$$\int^{\infty}t^6 q(t)\,\mathrm{d}t<\infty. \tag{55}$$

Thus equation (51) is nonoscillatory. Since $t=|s|$, condition (54) implies that the elliptic equation (50) is nonoscillatory. This completes the proof. $\square$

Example 5. Consider the equation

$$\Delta^2 u(|s|)+\mathrm{e}^{-|s|}u(|s|)=0. \tag{56}$$

This equation satisfies all conditions of Theorem 5. Hence all spherically symmetric solutions existing in the domain Ω are nonoscillatory.

In a similar way, we have an analogue of Theorem 2 for the corresponding retarded elliptic equation

$$\Delta^2 u(|s|) + q(|s|)u(|g(s)|) = 0 \tag{57}$$

where once again the solutions are being considered in the slightly modified domain Ω, where $g(|s|) > T$.

Theorem 6. *Suppose $\ell_0 \leqslant g'(t) \leqslant Q_0$ where $Q_0 > \ell_0 > 0$ for $t \geqslant a$. Define*

$$P_i(t) = \frac{1}{g'(t)} \quad 1 \leqslant i \leqslant 4. \tag{58}$$

Further suppose that $q(t)\colon [a, \infty) \to (0, \infty)$ is continuous and

$$\int^{\infty} t^3 (g^{-1}(t))^3 q(g^{-1}(t))\, \mathrm{d}t < \infty \tag{59}$$

where $g^{-1}(t)$ is the inverse function of $g(t)$. Then all solutions of the retarded elliptic equation (57) *existing in the domain Ω are nonoscillatory.*

Proof. In a manner of Theorem 5, the companion differential equation

$$\frac{\mathrm{d}^4 w}{\mathrm{d}t^4} + t^2 q(t) w(g(t)) = 0 \tag{60}$$

satisfies the conditions of Theorem 2. The proof is complete. $\square$

Example 6. The retarded elliptic equation

$$\Delta^2 u(|s|) + \mathrm{e}^{-|s|} u(t - \pi) = 0 \tag{61}$$

satisfies the conditions and conclusion of Theorem 6. Hence all nontrivial solution of equation (61) existing in the domain Ω are nonoscillatory.

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