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LINEAR EXTENSIONS OF ORDERINGS

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Dedicated to Professor Josef Novák on the occasion of his 95th birthday.

Abstract. A construction is given which makes it possible to find all linear extensions of a given ordered set and, conversely, to find all orderings on a given set with a prescribed linear extension. Further, dense subsets of ordered sets are studied and a procedure is presented which extends a linear extension constructed on a dense subset to the whole set.

Keywords: ordered set, linear extension, natural representation, lexicographic sum, dense subset

MSC 2000: 06A06

0. Introduction

In 1930, E. Szpilrajn published his famous theorem on the existence of a linear extension of any ordered set [25]. Further development showed that this theorem was one of the most significant results in the theory of ordered sets. We can mention that the theory of order dimension, intensively studied recently (see, e.g., [3] or [6]), has its background in Szpilrajn's theorem. Many authors later presented other constructions of a linear extension (e.g. [5], [8], [21]). Another problem is a characterization of those ordered sets which have a linear extension with a given property. So, it is well known that a well founded ordered set has a linear extension which is a well ordering [11], [16]. M. Pouzet and I. Rival characterized ordered sets having a complete linear extension [17]. A. Rutkowski gave some sufficient conditions for an ordered set to have a linear extension which is isomorphic to the set of rational numbers [18]. A good survey on these problems is given in [16].

In [12] a construction is presented which provides a well ordered extension of any well founded ordered set, and any well ordered extension can be constructed in such a way. In this paper the same problem is solved for any linear extension. More exactly, a construction is given, which makes it possible

(1) for a given ordered set, to find all its linear extensions, and

(2) for a given linearly ordered set, to find all orderings on its carrier having a linear extension identical with the given linear ordering.

Our construction is, however, limited to finding a linear ordering on the carrier with a certain special property and we have not a complete description of all linear orderings with this property (see Problem 1.6.). Nevertheless, we hope that this construction offers a possibility of looking into the structure of all linear extensions.

In Sections 2. and 3. we show that it suffices to construct a linear extension of a dense subset of a given ordered set. The concept of a dense subset was introduced by F. Hausdorff for linearly ordered sets [4]; see also D. Kurepa [7], W. Sierpiński [22] etc. A generalization to ordered sets can be found in J. Schmidt [20]; other generalizations are presented in [13], [14], [15]. In this paper we define a dense subset of an ordered set in such a way that it has the Hausdorff property in subsets that are linearly ordered. Furthermore, we present a construction that offers a possibility to extend a linear extension defined on a dense subset to the whole set. This construction is universal in the sense that any linear extension can be found in this way.

The fundamental notions used here can be found in [1] or [24]. By a homomorphism of an ordered set into another one we mean an isotone mapping. If f is a bijection, then f^{-1} denotes its inverse. If f, g are binary relations, then $g \circ f$ is their composite, i.e. $(x, y) \in g \circ f$ means the existence of an element z such that $(x, z) \in f$, $(z, y) \in g$. If S is a set, then $|S|$ denotes its cardinality. For an equivalence relation e on a set S we denote by S/e the corresponding decomposition of S; elements of S/e are called blocks.

Both authors wish to express their gratitude to Professor Josef Novák; his results on linearly ordered continua (e.g. [10]) inspired them to study ordered sets.

1. Linear extensions

Let S be a set. A function on S whose values are only 0 and 1 will be called a *01-function*. Let F be a set of 01-functions on S. We define a binary relation N on F in the following way: For any $f, g \in F$ we put $(f, g) \in N$ if $f(t) \leqslant g(t)$ for any $t \in S$. It is easy to see that N is an ordering relation on F; it will be referred to as the *natural ordering* on F.

Let (S,R) be an ordered set. We define a 01-function $f[x]$ on S for any $x \in S$ as follows:

$$f[x](t) = \begin{cases} 0 \text{ if } (x,t) \in R, \\ 1 \text{ if } (x,t) \notin R, \end{cases}$$

and put $F = \{f[x];\ x \in S\}$.

The following simple assertion is known and its proof is trivial.

1.1. Lemma. *The mapping $f\colon\ S \to F$ is an isomorphism of (S,R) onto (F,N).*

Let (S,R) be an ordered set, F the set of 01-functions $f[x]$ on S constructed above for any $x \in S$ and N the natural ordering on F. Then (F,N) will be said to be the *natural representative* of (S,R) and f will be referred to as the *natural representation* of (S,R). Clearly, $R = f^{-1} \circ N \circ f$.

If (S,R) is an ordered set and M an ordering relation on S such that $R \subseteq M$, then M is said to be an *extension* of R. If, moreover, M is a linear ordering on S then it is called a *linear extension* of R.

Let (S,L) be a linearly ordered set, F a set of 01-functions on S. For $f,g \in F$ put $(f,g) \in A(L)$ if either $f = g$ or there exists $s \in S$ such that $f(s) < g(s)$ and $f(t) \leqslant g(t)$ for any $t \in S$ with $(t,s) \in L$. Then $A(L)$ is an ordering relation on F (see [19]; the relation $A(L)$ is denoted by $\mathcal{WR}_2$ there); it will be referred to as the *alphabetical ordering* on F with respect to L. Trivially, $A(L)$ is an extension of the natural ordering N on F.

1.2. Lemma. *Let (S,R) be an ordered set, (F,N) its natural representative, f its natural representation. If L is an arbitrary linear ordering on S, then $f^{-1} \circ A(L) \circ f$ is an extension of R.*

Proof. As $A(L)$ is an extension of N, $f^{-1} \circ A(L) \circ f$ is an extension of $f^{-1} \circ N \circ f = R$. □

1.3. Theorem. *Let (S,R) be an ordered set, (F,N) its natural representative, f its natural representation. If L is a linear ordering on S, then the following conditions are equivalent:*

(i) *L is a linear extension of R;*

(ii) *$L = f^{-1} \circ A(L) \circ f$;*

(iii) *there exists a linear ordering M on S such that $L = f^{-1} \circ A(M) \circ f$.*

Proof. 1. Let (i) hold. We prove that f is a homomorphism of (S,L) onto $(F, A(L))$. Suppose $x,y \in S$, $(x,y) \in L$. If $x = y$ then $f[x] = f[y]$; thus let $x \neq y$. As L is an extension of R, $(y,x) \in R$ is impossible. Hence $(y,x) \notin R$ which entails $f[y](x) = 1$. Further, by definition, $f[x](x) = 0$. If there exists $t \in S$ with $(t, x) \in L$ and $f[y](t) = 0$, then $(y, t) \in R$. It follows that $(y, t) \in L$ and thus $(y, x) \in L$. This contradicts the assumption $(x, y) \in L$, $x \neq y$. Thus $f[y](t) = 1$ for any $t \in S$ with $(t, x) \in L$, which implies $f[x](t) \leqslant f[y](t)$ for any such t and, therefore, $(f[x], f[y]) \in A(L)$. Thus, f is a bijective homomorphism of the linearly ordered set (S, L) onto the ordered set $(F, A(L))$. Hence $A(L)$ is a linear ordering on F, f is an isomorphism and consequently $L = f^{-1} \circ A(L) \circ f$. Thus (ii) holds.

2. Condition (ii) implies (iii) trivially.

3. Let (iii) hold. By 1.2., $f^{-1} \circ A(M) \circ f = L$ is an extension of R, thus a linear extension, and (i) holds. □

1.4. Lemma. *Let (S, R) be an ordered set, (F, N) its natural representative, f its natural representation. If L is a well ordering on S then $f^{-1} \circ A(L) \circ f$ is a linear extension of R.*

P r o o f. $f^{-1} \circ A(L) \circ f$ is an extension of R by 1.2. As $A(L)$ is trivially a linear ordering on F, $f^{-1} \circ A(L) \circ f$ is a linear ordering on S. □

1.5. Corollary. *Let (S, L) be a linearly ordered set. Let $|S| = \aleph_\nu$ where ν is an ordinal. Then (S, L) is isomorphic to a set of sequences of numbers 0 and 1 of type ω_ν where the sequences are ordered alphabetically ([23], [14], [9]).*

P r o o f. Let (F, N) be the natural representative of (S, L), f its natural representation. Choose any well ordering M on S. By 1.4., $f^{-1} \circ A(M) \circ f$ is a linear extension of L. As L is linear, we have $L = f^{-1} \circ A(M) \circ f$ and f is an isomorphism of (S, L) onto $(F, A(M))$; the elements of F may be regarded as sequences of type ω_ν. □

Let (S, R) be an ordered set, (F, N) its natural representative, f its natural representation. Denote by $E(R)$ the set of all linear orderings M on S such that $A(M)$ is a linear ordering on F (and so, $f^{-1} \circ A(M) \circ f$ is a linear extension of R). Any well ordering on S is an element of $E(R)$ by 1.4. Also, by 1.3, $R \in E(R)$ if R is a linear ordering.

1.6. Problem. Characterize the set $E(R)$.

1.7. Construction of all linear extensions of a given ordering relation. *Let an ordered set (S, R) be given. Construct the natural representative (F, N) and the natural representation f of (S, R). Choose an arbitrary element M of $E(R)$. Then $f^{-1} \circ A(M) \circ f$ is a linear extension of R. Any linear extension of R can be constructed in this way.*

P r o o f. $f^{-1} \circ A(M) \circ f$ is an extension of R by 1.2. As $A(M)$ is linear, $f^{-1} \circ A(M) \circ f$ is linear, too. On the other hand, if L is a linear extension of R, then, by 1.3, $L \in E(R)$ and $L = f^{-1} \circ A(L) \circ f$. □

1.8. Example. Let (S, R) be an ordered set and $x, y \in S$ elements such that $(x, y) \notin R$, $(y, x) \notin R$. Let (F, N) be the natural representative of (S, R), f its natural representation. We have $f[x](x) = 0$, $f[y](x) = 1$. Choose any well ordering M on S such that x is the least element in (S, M). Then $(f[x], f[y]) \in A(M)$. By 1.4, $M \in E(R)$; if we put $L = f^{-1} \circ A(M) \circ f$ then L is a linear extension of R and $(x, y) \in L$. Thus, for any ordered set (S, R) and any elements $x, y \in S$ which are incomparable with respect to R there exists a linear extension L of R such that $(x, y) \in L$ ([25]).

1.9. Construction of all ordering relations with a given linear extension. *Let (S, L) be a linearly ordered set. Choose a set F of 01-functions on S and a linear ordering M on S such that (S, L) is isomorphic to $(F, A(M))$; let f be the corresponding isomorphism. Construct the natural ordering N on F. Then $(S, f^{-1} \circ N \circ f)$ is an ordered set such that L is a linear extension of $f^{-1} \circ N \circ f$. Any ordered set (S, R) such that L is a linear extension of R can be constructed in this way.*

P r o o f. If F, M and f have the desired property, then $L = f^{-1} \circ A(M) \circ f$. As $A(M)$ is an extension of N, $f^{-1} \circ A(M) \circ f = L$ is an extension of $f^{-1} \circ N \circ f$. On the other hand, if (S, R) is an ordered set such that L is a linear extension of R, (F, N) the natural representative of (S, R), f its natural representation, then $R = f^{-1} \circ N \circ f$ and $L = f^{-1} \circ A(L) \circ f$ by 1.3, so that f is an isomorphism of (S, L) onto $(F, A(L))$. □

1.10. Example. Let $\mathbf{Q}_1$ be the set of all rational numbers greater than 0 and less than 1, let L be the usual linear ordering on $\mathbf{Q}_1$. For any $q \in \mathbf{Q}_1$ there exist uniquely determined positive integers m, n such that $m < n$, $q = \frac{m}{n}$ and the greatest common divisor of m, n equals 1. We put $n = I(q)$. Let us define a binary relation M on $\mathbf{Q}_1$ as follows: For $q, r \in \mathbf{Q}_1$ we put $(q, r) \in M$ if either $I(q) < I(r)$ or $I(q) = I(r)$ and $(q, r) \in L$. It is easy to see that M is a linear ordering on $\mathbf{Q}_1$. We now assign a 01-function on $\mathbf{Q}_1$ to any $q \in \mathbf{Q}_1$. For any $q \in \mathbf{Q}_1$ put $K(q) = \{r \in \mathbf{Q}_1;\ I(r) \leqslant I(q)\}$, $O(q) = \{r \in K(q);\ (q, r) \in L\}$ and

$$f[q](r) = \begin{cases} 0 \text{ if } r \in O(q), \\ 1 \text{ if either } r \in K(q) - O(q) \text{ or } r \in \mathbf{Q}_1 - K(q). \end{cases}$$

Let $F = \{f[q];\ q \in \mathbf{Q}_1\}$ and let N be the natural ordering on F. We show that f is an isomorphism of $(\mathbf{Q}_1, L)$ onto $(F, A(M))$. By definition, $f\colon \mathbf{Q}_1 \to F$ is a surjection. Let us have $q, r \in \mathbf{Q}_1$, $q \neq r$, $(q, r) \in L$. Put $p = \min\{I(q), I(r)\}$, $J(p) = \{t \in \mathbf{Q}_1;\ I(t) \leqslant p\}$. It is easy to see that $J(p)$ is an initial interval in $(\mathbf{Q}_1, M)$. By definition, $O(q) \cap J(p)$ and $O(r) \cap J(p)$ are final intervals in $J(p)$ with respect to L. Since $(q, r) \in L$, we obtain $O(r) \cap J(p) \subseteq O(q) \cap J(p)$. It follows that $f[r](t) = 0$ implies $f[q](t) = 0$ for any $t \in J(p)$, thus

$$(*) \qquad f[q](t) \leqslant f[r](t) \quad \text{for any } t \in J(p).$$

Two cases are now possible.

(1) $p = I(q) \leqslant I(r)$. As $q \in O(q)$, we have $f[q](q) = 0$. Since $(q, r) \in L$, $q \neq r$, we obtain $q \notin O(r)$ and hence $f[r](q) = 1$. Thus, $q \in J(p)$ and $f[q](q) < f[r](q)$. Regarding $(*)$, we obtain $(f[q], f[r]) \in A(M)$, $f[q] \neq f[r]$.

(2) $p = I(r) < I(q)$. If $t \in K(q) - K(r)$, then $f[r](t) = 1$, which implies $f[q](t) \leqslant f[r](t)$ for any $t \in K(q)$ by virtue of $(*)$. By definition, we have $f[q](q) = 0$, $f[r](q) = 1$; thus $(f[q], f[r]) \in A(M)$, $f[q] \neq f[r]$.

We have proved that f is a homomorphism of $(\mathbf{Q}_1, L)$ onto $(F, A(M))$ which is a bijection. Thus, $(F, A(M))$ is linearly ordered and f is an isomorphism.

By 1.9, $(\mathbf{Q}_1, f^{-1} \circ N \circ f)$ is an ordered set such that L is a linear extension of $f^{-1} \circ N \circ f$. Note that $f^{-1} \circ N \circ f$ is not a linear ordering. E.g., we have $f\left[\frac{1}{3}\right]\left(\frac{1}{3}\right) = 0$, $f\left[\frac{1}{3}\right]\left(\frac{3}{4}\right) = 1$, $f\left[\frac{3}{4}\right]\left(\frac{1}{3}\right) = 1$, $f\left[\frac{3}{4}\right]\left(\frac{3}{4}\right) = 0$, which implies that $f\left[\frac{1}{3}\right]$, $f\left[\frac{3}{4}\right]$ are incomparable with respect to N and thus $\frac{1}{3}, \frac{3}{4}$ are incomparable with respect to $f^{-1} \circ N \circ f$.

2. Dense subsets

Let (S, R) be an ordered set, $H \subseteq S$. The set H will be called *dense* in (S, R) if for any $s_1, s_2 \in S$ such that $s_1 \neq s_2$ and $(s_1, s_2) \in R$ there exist elements $h_1, h_2 \in H$ with the properties $h_1 \neq h_2$, $(s_1, h_1) \in R$, $(h_1, h_2) \in R$, $(h_2, s_2) \in R$.

2.1. Examples. (i) If (S, R) is an antichain, i.e. $R = \mathrm{id}_S$, then $\emptyset$ is dense in (S, R).

(ii) If (S, R) is a linearly ordered set and $H \subseteq S$, then H is dense in (S, R) in the sense of the above formulated definition if and only if it is dense in (S, R) in the sense of Hausdorff ([4], p. 89).

(iii) If (S, R) is a finite connected ordered set, $|S| \geqslant 2$, then the only set that is dense in (S, R) is equal to S.

The last example may create the impression that the use of dense subsets in constructions has only a very limited impact. But there are some cases where the use of dense subsets is advantageous. To this aim, we mention the definition of a lexicographic sum of ordered sets [2].

Let (P,T) be an ordered set and let (S_p,R_p) be an ordered set for any $p\in P$. The *lexicographic sum* $\sum\limits_{p\in(P,T)}(S_p,R_p)$ is the set of all pairs (p,s) where $p\in P$, $s\in S_p$, together with the binary relation R such that $((p_1,s_1),(p_2,s_2))\in R$ if either $p_1\neq p_2$, $(p_1,p_2)\in T$ or $p_1=p_2=p$ and $(s_1,s_2)\in R_p$. It is well known that $\sum\limits_{p\in(P,T)}(S_p,R_p)$ is an ordered set [2]. If (P,T) is an antichain, i.e. $T=\mathrm{id}_P$, we obtain the *cardinal (direct) sum* which will be denoted by $\sum\limits_{p\in P}(S_p,R_p)$.

2.2. Lemma. *Let P be a set, (S_p,R_p) an ordered set for any $p\in P$. Let H_p be a dense subset of (S_p,R_p) for any $p\in P$. Then $\bigcup\limits_{p\in P}(\{p\}\times H_p)$ is dense in the cardinal sum $\sum\limits_{p\in P}(S_p,R_p)$.*

Proof. Put $\sum\limits_{p\in P}(S_p,R_p)=(S,R)$. If $(p_1,s_1,),(p_2,s_2)\in S$, $(p_1,s_1)\neq(p_2,s_2)$ and $((p_1,s_1),(p_2,s_2))\in R$, then $p_1=p_2=p$ and $(s_1,s_2)\in R_p$. Thus $s_1\neq s_2$ and hence there exist elements $h_1,h_2\in H_p$ such that $h_1\neq h_2$, $(s_1,h_1)\in R_p$, $(h_1,h_2)\in R_p$, $(h_2,s_2)\in R_p$. Then $((p,s_1),(p,h_1))\in R$, $((p,h_1),(p,h_2))\in R$, $((p,h_2),(p,s_2))\in R$ and $(p,h_1),(p,h_2)\in\bigcup\limits_{p\in P}(\{p\}\times H_p)$. $\square$

Let (S,R) be an ordered set, $H\subseteq S$. The set H will be called *strongly dense* in (S,R) if it is dense in (S,R) and if it contains all maximal and all minimal elements in (S,R).

2.3. Theorem. *Let (P,T) be an ordered set, let (S_p,R_p) be an ordered set for any $p\in P$. Let H_p be a strongly dense subset of (S_p,R_p) for any $p\in P$. Then $\bigcup\limits_{p\in P}(\{p\}\times H_p)$ is dense in the lexicographic sum $\sum\limits_{p\in(P,T)}(S_p,R_p)$.*

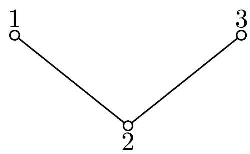
Proof. Put $\sum\limits_{p\in(P,T)}(S_p,R_p)=(S,R)$. Let $(p_1,s_1),(p_2,s_2)\in S$, $(p_1,s_1)\neq(p_2,s_2)$ and $((p_1,s_1),(p_2,s_2))\in R$. Then either $p_1\neq p_2$, $(p_1,p_2)\in T$ or $p_1=p_2=p$ and $(s_1,s_2)\in R_p$.

Consider the first case. If s_1 is maximal in (S_{p_1},R_{p_1}) then $s_1\in H_{p_1}$. In the opposite case there exists $q_1\in S_{p_1}$ such that $s_1\neq q_1$, $(s_1,q_1)\in R_{p_1}$. Then there exist $h_1,h_2\in H_{p_1}$ such that $h_1\neq h_2$, $(s_1,h_1)\in R_{p_1}$, $(h_1,h_2)\in R_{p_1}$, $(h_2,q_1)\in R_{p_1}$. In both cases there exists $h_1\in H_{p_1}$ such that $(s_1,h_1)\in R_{p_1}$. Similarly, either s_2 is minimal in (S_{p_2},R_{p_2}) or not; in both cases there exists $h_2\in H_{p_2}$ such that $(h_2,s_2)\in R_{p_2}$. Then $(p_1,h_1),(p_2,h_2)\in\bigcup\limits_{p\in P}(\{p\}\times H_p)$, $(p_1,h_1)\neq(p_2,h_2)$ and $((p_1,s_1),(p_1,h_1))\in R$, $((p_1,h_1),(p_2,h_2))\in R$, $((p_2,h_2),(p_2,s_2))\in R$.

If $p_1 = p_2 = p$ and $(s_1, s_2) \in R_p$, then $s_1 \neq s_2$ and thus there exist $h_1, h_2 \in H_p$ such that $h_1 \neq h_2$ and $(s_1, h_1) \in R_p$, $(h_1, h_2) \in R_p$, $(h_2, s_2) \in R_p$. Then $(p, h_1), (p, h_2) \in \bigcup_{p \in P} (\{p\} \times H_p)$, $(p, h_1) \neq (p, h_2)$ and $((p, s_1), (p, h_1)) \in R$, $((p, h_1), (p, h_2)) \in R$, $((p, h_2), (p, s_2)) \in R$. $\square$

2.4. Example. Let (P, T) be an ordered set. For any $p \in P$ put $(S_p, R_p) = (\mathbf{R}, \leqslant)$ where $\mathbf{R}$ is the set of all real numbers and $\leqslant$ is the usual ordering on $\mathbf{R}$. Furthermore, set $H_p = \mathbf{Q}$ for any $p \in P$ where $\mathbf{Q}$ is the set of all rational numbers. As $\mathbf{Q}$ is strongly dense in $(\mathbf{R}, \leqslant)$, the set $\bigcup_{p \in P} (\{p\} \times \mathbf{Q})$ is dense in the lexicographic sum $\sum_{p \in (P,T)} (S_p, R_p)$; the last set is uncountable while the set $\bigcup_{p \in P} (\{p\} \times \mathbf{Q})$ is countable for a denumerable set P.

Put, e.g., $P = \{1, 2, 3\}$ and $T = \{(1,1), (2,1), (2,2), (2,3), (3,3)\}$, i.e. (P, T) has the following Hasse diagram:



Then $\sum_{p \in (P,T)} (S_p, R_p) = (S, R)$ is an ordered set where $S = \bigcup_{i=1}^{3} (\{i\} \times \mathbf{R})$ and R is such that $((2, r), (k, t)) \in R$ for any $r, t \in \mathbf{R}$ and $k \in \{1, 3\}$, $((i, r), (i, t)) \in R$ for any $i \in \{1, 2, 3\}$ and $r, t \in \mathbf{R}$, $r \leqslant t$ while $(1, r), (3, t)$ are incomparable with respect to R for any $r, t \in \mathbf{R}$. The set $H = \bigcup_{i=1}^{3} (\{i\} \times \mathbf{Q})$ is dense in (S, R); this set is countable.

3. Linear extensions and dense subsets

Let (S, R) be an ordered set, H its dense subset. Let L be a linear extension of the ordering $R \cap (H \times H)$ on H. Denote by $I(H, L)$ the set of all initial intervals (ideals) in (H, L); this set is ordered by the set inclusion.

Let $s \in S$ be an element, $i \in I(H, L)$ an initial interval in (H, L). The interval i will be called *corresponding to* s if it has the following properties:

(α) If $h \in H$ and there exists $g \in H$ such that $(h, g) \in L$, $(g, s) \in R$, then $h \in i$.

(β) If $h \in H$, $h \neq s$ and there exists $g \in H$ such that $(s, g) \in R$, $(g, h) \in L$, then $h \notin i$.

3.1. Lemma. *If $s \in H$, then the principal ideal $\{h \in H;\ (h, s) \in L\}$ is the only interval in $I(H, L)$ corresponding to s.*

P r o o f. Put $i(s) = \{h \in H;\ (h,s) \in L\}$. If $h \in H$ and there exists $g \in H$ such that $(h,g) \in L$, $(g,s) \in R$, then $(g,s) \in L$, thus $(h,s) \in L$ and $h \in i(s)$. Therefore, $i(s)$ has property (α). If $h \in H$, $h \neq s$ and there exists $g \in H$ such that $(s,g) \in R$, $(g,h) \in L$, then $(s,h) \in L$, $s \neq h$, which implies $(h,s) \notin L$ and $h \notin i(s)$. It follows that $i(s)$ has property (β), too, and hence $i(s)$ is corresponding to s. Let $i \in I(H,L)$ be an arbitrary interval corresponding to s. If $h \in i(s)$, then $(h,s) \in L$, $(s,s) \in R$, which implies $h \in i$ by (α). Thus, $i(s) \subseteq i$. If $h \in H$, $h \notin i(s)$, then $(h,s) \notin L$ and consequently $(s,h) \in L$, $s \neq h$. Since $(s,s) \in R$, we obtain $h \notin i$ by (β). Hence $i = i(s)$ and thus $i(s)$ is the only interval in $I(H,L)$ corresponding to s. □

In the sequel we assume that for any $s \in S$ an interval $i(s) \in I(H,L)$ is given which is corresponding to s. In other words, a mapping $i\colon\ S \to I(H,L)$ is given such that $i(s)$ is corresponding to s for any $s \in S$.

3.2. Lemma. *If $s_1, s_2 \in S$, $s_1 \neq s_2$ and $i(s_1) = i(s_2)$, then $(s_1, s_2) \notin R$.*

P r o o f. Suppose $s_1 \neq s_2$, $i(s_1) = i(s_2)$ and $(s_1,s_2) \in R$. As H is dense in (S,R), there exist $h_1, h_2 \in H$ such that $h_1 \neq h_2$, $(s_1,h_1) \in R$, $(h_1,h_2) \in R$, $(h_2,s_2) \in R$. Note that $h_2 \neq s_1$; otherwise $(h_2,h_1) \in R$ would hold, thus $h_1 = h_2$, a contradiction. Since $(h_2,h_2) \in L$, we obtain $h_2 \in i(s_2)$ by (α), and $h_2 \notin i(s_1)$ by (β). This contradicts the hypothesis $i(s_1) = i(s_2)$. □

It follows from 3.2 that the conditions $s_1, s_2 \in S$, $s_1 \neq s_2$, $i(s_1) = i(s_2)$ imply that s_1, s_2 are incomparable elements in (S,R). In other words, any block of the decomposition $S/_{i^{-1} \circ i}$ is an antichain with respect to R.

3.3. Construction. *Let (S,R) be an ordered set, H a subset of S which is dense in (S,R), L a linear extension of the ordering $R \cap (H \times H)$ on H and $i\colon\ S \to I(H,L)$ a mapping such that $i(s)$ is a corresponding interval to s for any $s \in S$.*

To any block $B \in S/_{i^{-1} \circ i}$, assign an arbitrary linear ordering E_B on B.

For any $s_1, s_2 \in S$ put $(s_1,s_2) \in E$ if and only if one of the following cases occurs:

(a) *$i(s_1) \subseteq i(s_2)$ and $i(s_1) \neq i(s_2)$;*

(b) *$i(s_1) = i(s_2)$ and $(s_1,s_2) \in E_B$ where $B \in S/_{i^{-1} \circ i}$ and $s_1, s_2 \in B$.*

Then E is a linear ordering on S that is a linear extension of R.

Any linear extension of R may be obtained in this way by choosing L, i and E_B appropriately for any $B \in S/_{i^{-1} \circ i}$.

The last two assertions must be proved.

3.4. Theorem. *E is a linear ordering on S that is a linear extension of R.*

P r o o f. Let $s_1, s_2 \in S$. As the set of all initial intervals in the linearly ordered set (H, L) is linearly ordered by the set inclusion, either $i(s_1) \subseteq i(s_2)$ or $i(s_2) \subseteq i(s_1)$ holds; suppose $i(s_1) \subseteq i(s_2)$. If $i(s_1) \neq i(s_2)$, we have $(s_1, s_2) \in E$; if $i(s_1) = i(s_2)$ and $B \in S/_{i^{-1} \circ i}$ is such that $s_1, s_2 \in B$, then $(s_1, s_2) \in E_B$ or $(s_2, s_1) \in E_B$, thus $(s_1, s_2) \in E$ or $(s_2, s_1) \in E$. It follows that E is a linear ordering on S.

Suppose $s_1, s_2 \in S$, $(s_1, s_2) \in R$ and $s_1 \neq s_2$. Then $i(s_1) \neq i(s_2)$ by 3.2 and there exist elements $h_1, h_2 \in H$ such that $h_1 \neq h_2$ and $(s_1, h_1) \in R$, $(h_1, h_2) \in R$, $(h_2, s_2) \in R$. Then necessarily $h_2 \neq s_1$ so that $h_2 \notin i(s_1)$ by (β), and $(h_2, h_2) \in L$ implies $h_2 \in i(s_2)$ by (α). Thus $i(s_2) \subseteq i(s_1)$ is impossible, so that $i(s_1) \subseteq i(s_2)$ and $(s_1, s_2) \in E$. We have proved that E is an extension of R. $\square$

3.5. Theorem. *Let (S, R) be an ordered set, H a subset of S which is dense in (S, R). Then any linear extension of R may be obtained by the construction described in* 3.3 *by a suitable choice of L, i and E_B for any $B \in S/_{i^{-1} \circ i}$.*

P r o o f. Let E' be a linear extension of R onto S. Put $L = E' \cap (H \times H)$ and $i(s) = \{h \in H;\ (h, s) \in E'\}$ for any $s \in S$. Then L is a linear extension of $R \cap (H \times H)$ and $i(s)$ is an initial interval in (H, L) for any $s \in S$. We prove that $i(s)$ is corresponding to s. If $h \in H$ and there exists $g \in H$ such that $(h, g) \in L$, $(g, s) \in R$, then $(h, g) \in E'$, $(g, s) \in E'$ and, therefore, $(h, s) \in E'$, which implies $h \in i(s)$. Thus, $i(s)$ has property (α). If $h \in H$, $h \neq s$ and there exists $g \in H$ such that $(s, g) \in R$, $(g, h) \in L$, then $(s, g) \in E'$, $(g, h) \in E'$, and therefore $(s, h) \in E'$. Thus $(h, s) \notin E'$ and $h \notin i(s)$; hence $i(s)$ has property (β). Now put $E_B = E' \cap (B \times B)$ for any $B \in S/_{i^{-1} \circ i}$ and let E be the linear extension of R obtained by Construction 3.3, where L, i and E_B for any $B \in S/_{i^{-1} \circ i}$ have been defined above. Let $s_1, s_2 \in S$, $(s_1, s_2) \in E$. If $i(s_1) = i(s_2)$ and $B \in S/_{i^{-1} \circ i}$ is such that $s_1, s_2 \in B$, then $(s_1, s_2) \in E_B = E' \cap (B \times B)$, thus $(s_1, s_2) \in E'$. If $i(s_1) \neq i(s_2)$, then $i(s_1) \subseteq i(s_2)$ and there exists $h \in H$ such that $h \in i(s_2) - i(s_1)$. By definition of i, this means $(h, s_2) \in E'$, $(h, s_1) \notin E'$, which entails $(s_1, h) \in E'$. Thus $(s_1, s_2) \in E'$ holds. We have proved $E \subseteq E'$; since both orderings E, E' are linear, $E = E'$ holds, which completes the proof of the theorem. $\square$

3.6. Example. Let (S, R) be the ordered set from Example 2.4, i.e. $S = \bigcup\limits_{i=1}^{3} (\{i\} \times \mathbf{R})$ and R is such that $((2, r), (k, t)) \in R$ for any $r, t \in \mathbf{R}$ and $k \in \{1, 3\}$, $((i, r), (i, t)) \in R$ for any $i \in \{1, 2, 3\}$ and $r, t \in \mathbf{R}$, $r \leqslant t$ and $(1, r), (3, t)$ are incomparable with respect to R for any $r, t \in \mathbf{R}$. The set $H = \bigcup\limits_{i=1}^{3} (\{i\} \times \mathbf{Q})$ is dense in (S, R). Let L be the following linear ordering on H: $((2, p), (k, q)) \in L$ for any $p, q \in \mathbf{Q}$ and $k \in \{1, 3\}$, $((k, p), (j, q)) \in L$ if $k \in \{1, 3\}$, $j \in \{1, 3\}$, $p \in \mathbf{Q}$, $q \in \mathbf{Q}$, $p < q$, $((1, q), (3, q)) \in L$ for any $q \in \mathbf{Q}$ and $((i, q), (i, q)) \in L$ for any $i \in \{1, 2, 3\}$,

$q \in \mathbf{Q}$. It is easy to see that L is in fact a linear ordering on H which is an extension of $R \cap (H \times H)$.

Let us choose the following mapping $i\colon S \to I(H, L)$.

$$
\begin{aligned}
&i((2,r)) = \{(2,q);\ q \in \mathbf{Q}, q \leqslant r\},\\
&i((1,r)) = \{(2,q);\ q \in \mathbf{Q}\} \cup \{(1,q);\ q \in \mathbf{Q}, q \leqslant r\} \cup \{(3,q);\ q \in \mathbf{Q}, q < r\},\\
&i((3,r)) = \{(2,q);\ q \in \mathbf{Q}\} \cup \{(1,q);\ q \in \mathbf{Q}, q \leqslant r\} \cup \{(3,q);\ q \in \mathbf{Q}, q \leqslant r\}
\end{aligned}
$$

for any $r \in \mathbf{R}$.

It is not difficult to prove that $i((k,r))$ is a corresponding interval to (k,r) for any $k \in \{1,2,3\}$, $r \in \mathbf{R}$.

If $i((k,r)) = i((j,s))$, then $r = s \in \mathbf{R} - \mathbf{Q}$ and $k, j \in \{1,3\}$, i.e. any block $B \in S/_{i^{-1} \circ i}$ with $|B| \geqslant 2$ is of the form $B = \{(1,r),(3,r)\}$ where $r \in \mathbf{R} - \mathbf{Q}$. Let us choose $E_B = \{((1,r),(3,r)),(1,r),(1,r)),((3,r),(3,r))\}$ for any such B. Then the linear extension E of R obtained by Construction 3.3 by choosing L, i and E_B for $B \in S/_{i^{-1} \circ i}$ as above is described as follows:

$$
\begin{aligned}
E = {} &\{((2,r),(j,t));\ j \in \{1,3\}, r \in \mathbf{R}, t \in \mathbf{R}\}\\
&\cup \{((k,r),(j,t));\ k,j \in \{1,3\}, r \in \mathbf{R}, t \in \mathbf{R}, r < t\}\\
&\cup \{((i,r),(i,t));\ i \in \{1,2,3\}, r \in \mathbf{R}, t \in \mathbf{R}, r \leqslant t\}\\
&\cup \{((1,r),(3,r));\ r \in \mathbf{R}\}.
\end{aligned}
$$

References

[1] *G. Birkhoff*: Lattice Theory. Providence, Rhode Island, 1967.
[2] *M. M. Day*: Arithmetic of ordered systems. Trans. Amer. Math. Soc. *58* (1945), 1–43.
[3] *T. Fofanova, I. Rival, A. Rutkowski*: Dimension two, fixed points nad dismantable ordered sets. Order *13* (1996), 245–253.
[4] *F. Hausdorff*: Grundzüge der Mengenlehre. Leipzig, 1914.
[5] *T. Hiraguchi*: On the dimension of partially ordered sets. Sci. Rep. Kanazawa Univ. *1* (1951), 77–94.
[6] *H. A. Kierstead, E. C. Milner*: The dimension of the finite subsets of K. Order *13* (1996), 227–231.
[7] *D. Kurepa*: Partitive sets and ordered chains. Rad Jugosl. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke *6 (302)* (1957), 197–235.
[8] *J. Loś, C. Ryll-Nardzewski*: On the application of Tychonoff's theorem in mathematical proofs. Fund. Math. *38* (1951), 233–237.
[9] *E. Mendelson*: Appendix. W. Sierpiński: Cardinal and Ordinal Numbers. Warszawa, 1958.
[10] *J. Novák*: On partition of an ordered continuum. Fund. Math. *39* (1952), 53–64.
[11] *V. Novák*: On the well dimension of ordered sets. Czechoslovak Math. J. *19 (94)* (1969), 1–16.

[12] *V. Novák*: Über Erweiterungen geordneter Mengen. Arch. Math. (Brno) *9* (1973), 141–146.
[13] *V. Novák*: Some cardinal characteristics of ordered sets. Czechoslovak Math. J. *48 (123)* (1998), 135–144.
[14] *M. Novotný*: O representaci částečně uspořádaných množin posloupnostmi nul a jedniček (On representation of partially ordered sets by means of sequences of 0's and 1's). Čas. pěst. mat. *78* (1953), 61–64.
[15] *M. Novotný*: Bemerkung über die Darstellung teilweise geordneter Mengen. Spisy přír. fak. MU Brno *369* (1955), 451–458.
[16] Ordered sets. Proc. NATO Adv. Study Inst. Banff (1981)(I. Rival, ed.).
[17] *M. Pouzet, I. Rival*: Which ordered sets have a complete linear extension?. Canad. J. Math. *33* (1981), 1245–1254.
[18] *A. Rutkowski*: Which countable ordered sets have a dense linear extension?. Math. Slovaca *46* (1996), 445–455.
[19] *J. Schmidt*: Lexikographische Operationen. Z. Math. Logik Grundlagen Math. *1* (1955), 127–170.
[20] *J. Schmidt*: Zur Kennzeichnung der Dedekind-Mac Neilleschen Hülle einer geordneten Menge. Arch. Math. *7* (1956), 241–249.
[21] *V. Sedmak*: Dimenzija djelomično uredenih skupova pridruženih poligonima i poliedrima (Dimension of partially ordered sets connected with polygons and polyhedra). Period. Math.-Phys. Astron. *7* (1952), 169–182.
[22] *W. Sierpiński*: Cardinal and Ordinal Numbers. Warszawa, 1958.
[23] *W. Sierpiński*: Sur une propriété des ensembles ordonnés. Fund. Math. *36* (1949), 56–67.
[24] *G. Szász*: Einführung in die Verbandstheorie. Leipzig, 1962.
[25] *E. Szpilrajn*: Sur l'extension de l'ordre partiel. Fund. Math. *16* (1930), 386–389.

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