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# BOUNDED OSCILLATION OF NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS OF ARBITRARY ORDER

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*Abstract.* The paper is concerned with oscillation properties of $n$-th order neutral differential equations of the form

$$[x(t)+cx(\tau(t))]^{(n)}+q(t)f\big(x(\sigma(t))\big)=0, \quad t\geqslant t_0>0,$$

where $c$ is a real number with $|c|\neq 1$, $q\in C([t_0,\infty),\mathbb{R})$, $f\in C(\mathbb{R},\mathbb{R})$, $\tau,\sigma\in C([t_0,\infty),\mathbb{R}_+)$ with $\tau(t)<t$ and $\lim\limits_{t\to\infty}\tau(t)=\lim\limits_{t\to\infty}\sigma(t)=\infty$.

Sufficient conditions are established for the existence of positive solutions and for oscillation of bounded solutions of the above equation. Combination of these conditions provides necessary and sufficient conditions for oscillation of bounded solutions of the equation. Furthermore, the results are generalized to equations in which $c$ is a function of $t$ and a certain type of a forcing term is present.

*Keywords*: oscillation, positive solutions, neutral equation

*MSC 2000*: 34K11, 34K40

## 1. Introduction

The oscillation behavior of differential equations with deviating arguments has received a great deal of attention in the last few decades. Recently, there has been an increasing interest in studying the oscillation character of neutral type differential equations. Among numerous papers dealing with the subject we refer in particular to [1]–[3], [6], [8]–[11] and the references cited therein. We note that these equations occur in applications such as problems in economics where demand depends on current price but supply depends on the price at an earlier time and in the study of systems which communicate through lossless channels.

In this paper we consider $n$-th order neutral type differential equations of the form

$$[x(t)+cx(\tau(t))]^{(n)}+q(t)f(x(\sigma(t)))=0, \quad t\geqslant t_0\geqslant 0, \tag{1}$$

where $c$ is a real number with $|c|\neq 1$, $q\in C([t_0,\infty),\mathbb{R})$, $f\in C(\mathbb{R},\mathbb{R})$, $\tau,\sigma\in C([t_0,\infty),\mathbb{R}_+)$ with $\tau(t)<t$ and $\lim\limits_{t\to\infty}\tau(t)=\lim\limits_{t\to\infty}\sigma(t)=\infty$. It is also assumed that $\tau(t)$ is monotone.

The following theorem, Theorem 8.4.2 in [8], which generalizes and extends some of the results obtained by Zhang and Yu [11] for second order linear neutral differential equations, motivates our study in this paper.

**Theorem A.** *Suppose that $|c|\neq 1$, $|f(x)|\leqslant|f(y)|$ for $|x|\leqslant|y|$, $xy>0$, and $xf(x)>0$ for $x\neq 0$. Let $\tau(t)=t-r$ and $\sigma(t)=t-s(t)$, where $r$ is a positive real number and $s\in C([t_0,\infty),\mathbb{R}_+)$. If*

$$\int^{\infty} t^{n-1}|q(t)|\,\mathrm{d}t<\infty,$$

*then equation (1) has a bounded nonoscillatory solution.*

Note that the monotonicity condition imposed on the function $f$ is quite restrictive and therefore the above theorem applies only to a special class of neutral type differential equations. Also, $\tau(t)=t-r$, $c$ being just a constant, and nonexistence of any forcing term in (1) cause further restrictions.

Our purpose here, first of all, is to show that the conclusion of Theorem A is valid without the monotonicity condition on $f$ and the requirement $\tau(t)=t-r$. We also remove the sign condition imposed on $f$. Next, we obtain necessary and sufficient conditions for oscillation of bounded solutions of equation (1) under some very mild conditions. Several examples are inserted into the text to illustrate the results of the paper.

As is customary, a solution $x(t)$ of (1) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

The paper is organized as follows. Section 2 deals with the existence of positive solutions of (1). We use the contraction mapping principle for this purpose instead of Krasnosel'skii's fixed point theorem which was employed in the proof of Theorem A and also in [11]. In Section 3 we tackle the oscillation problem of bounded solutions of (1) and establish, with help of the results in Section 2, necessary and sufficient conditions for oscillation. Finally, in the last section we discuss equation (1) when a certain type of a forcing term is present and $c$ is a function of $t$. Some suggestions on further research are also made in this section.

## 2. Existence of Positive Solutions

In this section we are concerned with the existence of positive solutions of neutral type differential equations of the form (1). It will be proved that equation (1) has a bounded positive solution when $|c| \neq 1$. The monotonicity condition imposed on $f$ in Theorem A is replaced by a condition that $f$ satisfies a Lipschitz condition on an interval $[a, b]$, where $a$ and $b$ are arbitrary positive real numbers. We should note that this condition holds for almost all functions; it will always be satisfied for any function which is differentiable in some interval $[a, b]$.

**Theorem 1.** *Let $|c| \neq 1$, and suppose that for some positive numbers $a$ and $b$, the function $f$ satisfies the Lipschitz condition with a constant $L$ on the interval $[a, b]$. If*

$$\int^{\infty} s^{n-1}|q(s)|\,\mathrm{d}s < \infty \tag{2}$$

*then equation (1) has a bounded positive solution.*

Proof. Let $K = \max\{|f(x)|/|x|\colon\ a \leqslant x \leqslant b\}$ and $M = \max\{K, L\}$.

We first consider the case $|c| < 1$.

Clearly, there exists a sufficiently large real number $t_1 \geqslant t_0$ such that due to (2),

$$\int_{t_1}^{\infty} s^{n-1}|q(s)|\,\mathrm{d}s < \frac{(n-1)!}{Mb}\beta, \tag{3}$$

for all $t \geqslant t_1$, $\tau(t) \geqslant t_0$, $\sigma(t) \geqslant t_0$, where $\beta = (b-a)(1-|c|)/2$.

We introduce the Banach space $Y = \{x\colon\ x \in C([T,\infty),\mathbb{R}), \sup\limits_{t\geqslant T}|x(t)| < \infty\}$ with the supremum norm $\|x\| = \sup\limits_{t\geqslant T}|x(t)|$, where $T = \inf\limits_{t\geqslant t_1}\{\tau(t), \sigma(t)\}$.

Set $X = \{x \in Y\colon\ a \leqslant x \leqslant b\}$. It follows that $X$ is a bounded, convex and closed subset of $Y$.

Define an operator $S\colon\ X \to Y$ by

$$\begin{aligned}(Sx)(t) &= \alpha - cx(\tau(t)) + \frac{(-1)^{n-1}}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}q(s)f\bigl(x(\sigma(s))\bigr)\,\mathrm{d}s, \quad t \geqslant t_1\\ &= (Sx)(t_1), \quad T \leqslant t \leqslant t_1,\end{aligned}$$

where $\alpha = (b+a)(1+c)/2$.

We shall show that $S$ is a contraction mapping on $X$. We prove this when $0 \leqslant c < 1$, the case $-1 < c < 0$ being similar. It is easy to see that $S$ maps $X$

into itself. In fact, for $x \in X$, $t \geqslant t_1$, using (3) it follows that

$$(Sx)(t) \geqslant \alpha - cb - \beta = a$$

and

$$(Sx)(t) \leqslant \alpha - ca + \beta = b.$$

Therefore $S$ maps $X$ into itself. To show that $S$ is a contraction, let $x, y \in X$. Then

$$\begin{aligned}|(Sx)(t) - (Sy)(t)| &\leqslant c|x(\tau(t)) - y(\tau(t))| \\ &\quad + \frac{M}{(n-1)!} \int_t^\infty s^{n-1} |q(s)|\, |x(\sigma(s)) - y(\sigma(s))| \,\mathrm{d}s \\ &\leqslant c\|x - y\| + \frac{\beta}{b}\|x - y\| = \Big(c + \frac{\beta}{b}\Big)\|x - y\|.\end{aligned}$$

It is easy to verify that

$$c + \frac{\beta}{b} < 1.$$

Thus $S$ is a contraction on $X$, and therefore there exists a point $x \in X$ such that $Sx = x$. It is easy to check that $x$ is a bounded positive solution of equation (1).

Suppose that $|c| > 1$. Fix $\beta = (b - a)(|c| - 1)/2|c|$ and let $t_1$ be large enough so that

$$\int_{\tau^{-1}(t_1)}^\infty s^{n-1} |q(s)| \,\mathrm{d}s < \frac{(n-1)!}{Mb}|c|\beta.$$

We define an operator $S$ on $X$ as follows:

$$\begin{aligned}(Sx)(t) &= \frac{1}{c}\bigg[\alpha - x(\tau^{-1}(t)) + \frac{(-1)^n}{(n-1)!} \\ &\qquad \times \int_{\tau^{-1}(t)}^\infty \big(s - \tau^{-1}(t)\big)^{n-1} q(s) f(x(\sigma(s))) \,\mathrm{d}s\bigg], \quad t \geqslant t_1 \\ &= (Sx)(t_1), \quad T \leqslant t \leqslant t_1,\end{aligned}$$

where $\alpha = (b + a)(1 + c)/2$.

It can be shown similarly that $SX \subseteq X$ and

$$\|Sx - Sy\| \leqslant \Big(\frac{1}{c} + \frac{\beta}{b}\Big)\|x - y\| < \|x - y\|,$$

and therefore $S$ is a contraction on $X$. This completes the proof. □

We now give examples to which Theorem 1 is applicable, while Theorem A is not.

**Example 1.** Consider the neutral delay differential equation

$$[x(t)+2x(t/2)]' + \frac{4\mathrm{e}^{-2t}}{1+\mathrm{e}^{-2t}} x^2(t/2) = 0.$$

Here $f(x)=x^2$ is of fixed sign and $\tau(t)=t-t/2$, and so Theorem A does not apply. But, according to Theorem 1, there is a positive solution. In fact, $x(t)=1+\mathrm{e}^{-4t}$ is a positive solution of the equation.

**Example 2.** By Theorem 1, the equation

$$[x(t)+4x(t/2)]'' - 16\mathrm{e}^{-2t}(2+2\mathrm{e}^{-2t}+\mathrm{e}^{-4t})\frac{x(t/2)}{1+x^2(t/2)} = 0$$

has a positive solution. In fact, $x(t)=1+\mathrm{e}^{-4t}$ is such a solution. We note that since $f(x)=x/(1+x^2)$ is not monotone, Theorem A is not applicable here.

## 3. Oscillation of Bounded Solutions

In this section we investigate the oscillation behavior of bounded solutions of (1) and establish necessary and sufficient conditions under which every solution $x(t)$ of (1) is either oscillatory or else satisfies $\lim\limits_{t\to\infty} x(t)=0$.

The following lemma, which may have further applications in analysis (cf. [3]), will be needed.

**Lemma 1.** *Let $g$ be a continuous monotone function with $\lim\limits_{t\to\infty} g(t)=\infty$. Set*

$$z(t) = x(t) + c(t)x(g(t)). \tag{4}$$

*If $x(t)$ is eventually positive, $\liminf\limits_{t\to\infty} x(t)=0$ and $\lim\limits_{t\to\infty} z(t)=L\in\mathbb{R}$ exists, then $L=0$ provided that for some real numbers $c_1$, $c_2$, $c_3$, and $c_4$ the function $c(t)$ is in one of the following ranges:*

(a) $c_1 \leqslant c(t) \leqslant 0$,

(b) $0 \leqslant c(t) \leqslant c_2 < 1$,

(c) $1 < c_3 \leqslant c(t) \leqslant c_4$.

Proof. Clearly, by (4) we have

$$z(g^{-1}(t)) - z(t) = x(g^{-1}(t)) + c(g^{-1}(t))x(t) - x(t) - c(t)x(g(t))$$

and so

$$\lim_{t\to\infty}\{x(g^{-1}(t))+c(g^{-1}(t))x(t)-x(t)-c(t)x(g(t))\}=0. \tag{5}$$

Let $\{t_n\}$ be a sequence of real numbers such that

$$\lim_{n\to\infty} t_n=\infty \quad \text{and} \quad \lim_{n\to\infty} x(t_n)=0. \tag{6}$$

Assume that (a) holds. From (5) and (6) we obtain

$$\lim_{n\to\infty}\{x(g^{-1}(t_n))-c(t_n)x(g(t_n))\}=0.$$

As $x(g^{-1}(t_n))>0$ and $-c(t_n)x(g(t_n))\geqslant 0$, it follows that

$$\lim_{n\to\infty} x(g^{-1}(t_n))=0$$

and so from (4),

$$L=\lim_{n\to\infty} z(g^{-1}(t_n))=\lim_{n\to\infty}\{x(g^{-1}(t_n))+c(g^{-1}(t_n))x(t_n)\}=0.$$

In the case of (b), by replacing $t$ by $g(t)$ in (5) and using (6), we obtain

$$\lim_{n\to\infty}\{[c(t_n)-1]x(g(t_n))-c(g(t_n))x(g(g(t_n)))\}=0.$$

Since this implies that $\lim_{n\to\infty} x(g(t_n))=0$, it follows from (4) that $L=0$.

Finally, if (c) holds, then by replacing $t$ by $g^{-1}(t)$ in (5) we see that

$$\lim_{n\to\infty}\{x(g^{-1}(g^{-1}(t_n)))+[c(g^{-1}(g^{-1}(t_n)))-1]x(g^{-1}(t_n))\}=0,$$

and so

$$\lim_{n\to\infty} x(g^{-1}(t_n))=0. \tag{7}$$

Now using (4) and (7), we have

$$L=\lim_{n\to\infty} z(g^{-1}(t_n))=0.$$

Thus the proof is complete. □

**Theorem 2.** *Assume that $xf(x) > 0$ for $x \neq 0$ and that $q(t)$ is nonnegative for $t \geqslant t_0$ and not identically zero on any half-line of the form $[t_*, \infty)$ for some $t_* \geqslant t_0$. If*

$$\int^{\infty} t^{n-1} q(t)\,\mathrm{d}t = \infty, \tag{8}$$

*then*

(A) *for $c \geqslant 0$ with $c \neq 1$, every bounded solution $x(t)$ of (1) is oscillatory when $n$ is even, and is either oscillatory or $\lim_{t\to\infty} x(t) = 0$ when $n$ is odd,*

*and*

(B) *for $c < -1$ with $\inf_{t\geqslant t_0}[t-\tau(t)] > 0$, every bounded solution $x(t)$ of (1) is oscillatory when $n$ is odd, and is either oscillatory or $\lim_{t\to\infty} x(t) = 0$ when $n$ is even.*

Proof. Since $c < -1$ is a special case of a theorem in [9], we consider only the case $c \geqslant 0$ with $c \neq 1$. Suppose on the contrary that $x(t)$ is a nonoscillatory solution of (1). Without any loss of generality we may assume that $x(t)$ is eventually positive. Setting $z(t) = x(t) + cx(\tau(t))$, we see that $z(t)$ is also bounded and eventually positive. Furthermore, since

$$z^{(n)}(t) = -q(t)f(x(\sigma(t))) < 0$$

for $t$ sufficiently large implies that $z(t)z^{(n)}(t) < 0$, applying a lemma of Kiguradze [7] it follows that there exist $t_1$ and an integer $l \in \{0, 1\}$ with $n - l$ odd such that for $t \geqslant t_1$,

$$\begin{aligned} z^{(k)}(t) &> 0 \quad \text{for } k = 0, 1, \ldots, l, \\ (-1)^{k-l} z^{(k)}(t) &> 0 \quad \text{for } k = l, l+1, \ldots, n-1. \end{aligned} \tag{9}$$

Moreover, from (9),

$$\lim_{t\to\infty} z^{(k)}(t) = 0 \quad \text{for } k = 1, 2, \ldots, n-1. \tag{10}$$

In view of (10), integrating (1) repeatedly $(n-1)$ times from $t$ to $\infty$, we see that

$$(-1)^{n-1} z'(t) + \frac{1}{(n-2)!} \int_t^{\infty} (s-t)^{n-2} q(s) f(x(\sigma(s)))\,\mathrm{d}s = 0. \tag{11}$$

Let $L = \lim_{t\to\infty} z(t)$. Then, integrating (11) from $T$ to $\infty$, we obtain

$$\frac{1}{(n-1)!} \int_T^{\infty} (s-T)^{n-1} q(s) f(x(\sigma(s)))\,\mathrm{d}s = (-1)^n [L - z(T)]. \tag{12}$$

In view of (8), one can conclude from (12) that $\liminf_{t\to\infty} f(x(t)) = 0$ or

$$\liminf_{t\to\infty} x(t) = 0,$$

and by Lemma 1, $L = 0$. But $L = 0$ is possible only when $n$ is odd. This means that bounded solutions of (1) must be oscillatory when $n$ is even. To complete the proof we note that if $n$ is odd, then the trivial inequality $0 < x(t) \leqslant z(t)$ implies $\lim_{t\to\infty} x(t) = 0$, a contradiction to our assumption. □

**Example 3.** By Theorem 2, every bounded solution of the following equation is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$:

$$\left[x(t) + \frac{1}{\mathrm{e}}x(t-3)\right]''' + \frac{2\mathrm{e}^{2t/3}}{27t}x^3(t - \ln t) = 0.$$

It is easy to verify that $x(t) = \mathrm{e}^{-t/3}$ is a solution of the equation.

**Example 4.** According to Theorem 2, every bounded solution of

$$[x(t) - \mathrm{e}x(t - \pi/2)]'' + 2(1+\mathrm{e})(3 - \cos 4t)\frac{x(t-\pi)}{1 + x^2(t-\pi)} = 0$$

is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. Indeed, $x(t) = \sin 2t$ is an oscillatory solution of the equation.

The following example shows that the condition $c \neq \pm 1$ cannot be dropped:

**Example 5.** All conditions, except $c \neq \pm 1$, of Theorem 2 are satisfied for

$$[x(t) \pm x(2t/3)]''' + x(t/3)[8/27 - x(t/3)]^2 = 0,$$

and $x(t) = 8/27 \pm \mathrm{e}^{-t}$ is a solution.

In view of Theorem 1 and Theorem 2, we obtain the following necessary and sufficient condition for oscillation of solutions of (1).

**Theorem 3.** *Let $xf(x) > 0$ for $x \neq 0$, and let $q(t)$ be nonnegative for $t \geqslant t_0$ and not identically zero on any half-line of the form $[t_*, \infty)$ for some $t_* \geqslant t_0$. Assume also that for some positive numbers $a$ and $b$, the function $f$ satisfies the Lipschitz condition on the interval $[a, b]$. Then the conclusion of Theorem 2 holds if and only if* (8) *is satisfied.*

## 4. Some generalizations

In this section we extend the results obtained in Sections 2 and 3 to neutral type differential equations of the form

$$[x(t)+c(t)x(\tau(t))]^{(n)}+q(t)f(x(\sigma(t)))=h(t). \tag{13}$$

The following conditions (F1) and/or (F2) are to be used:

(F1) $\int^{\infty} t^{n-1}|h(t)|<\infty$.

(F2) There exists an oscillatory function $\varrho(t)$ such that

$$\lim_{t\to\infty}\varrho(t)=0 \quad \text{and} \quad h(t)=\varrho^{(n)}(t).$$

Assuming (F1), one can prove that the following theorem is true. The proof is similar to that of Theorem 1 and therefore is omitted.

**Theorem 4.** *Let* (F1) *be satisfied, and assume that the function $f$ satisfies the Lipschitz condition with a Lipschitz constant $L$ on an interval $[a,b]$, where $a$ and $b$ are positive real numbers which depend on the range of $c(t)$ and may be chosen as follows:*

(i) $a/b<(c_2+1)/(c_1+1)$, *when* $c_1\leqslant c(t)\leqslant c_2<-1$,

(ii) $a/b<(c_1+1)$, *when* $-1<c_1\leqslant c(t)\leqslant 0$,

(iii) $a/b<(1-c_2)$, *when* $0\leqslant c(t)\leqslant c_2<1$,

(iv) $a/b<(c_1-1)/(c_2-1)$, *when* $1<c_1\leqslant c(t)\leqslant c_2$,

*where $c_1$ and $c_2$ are real numbers.*

*If* (2) *is true then equation* (13) *has a bounded positive solution.*

**Example 6.** Consider

$$[x(t)+(4+\mathrm{e}^{-2t})x(t/2)]''-16\mathrm{e}^{-2t}(2+2\mathrm{e}^{-2t}+\mathrm{e}^{-4t})\frac{2x(t/2)}{1+x^2(t/2)}=-12\mathrm{e}^{-2t}.$$

If we take $c_1=4$ and $c_2=5$ then (iv) will be satisfied when $4a<3b$. We may choose $a=1$ and $b=2$, and conclude from Theorem 4 that there is a positive solution $x(t)\in[1,2]$, $t\geqslant t_0\geqslant 0$. We note that $x(t)=1+\mathrm{e}^{-4t}$ is a solution of the equation.

We should note that the preceding theorem cannot be applied if $c(t)$ is not bounded away from $\pm1$, which agrees with the results of Section 3. Notice also that the interval $[a,b]$ cannot be taken arbitrarily anymore as opposed to the situation in Theorem 1.

The next theorem shows that the oscillation behavior is not disturbed by a forcing term $h(t)$ satisfying (F2). In the absence of a forcing term, the proof proceeds exactly as the proof of Theorem 2 and therefore is omitted. If forcing terms satisfying (F2) are present, then the proof can easily be modified by employing the arguments developed in [4], [5].

**Theorem 5.** *In addition to* (8), *suppose that* (F2) *is true and that* $f$ *and* $q$ *satisfy the sign conditions specified in Theorem* 2.

(C) *If* $c_1 \leqslant c(t) \leqslant c_2 < -1$ *and* $\inf\limits_{t\geqslant t_0}[t-\tau(t)] > 0$, *every bounded solution* $x(t)$ *of* (13) *is oscillatory when* $n$ *is odd, and is either oscillatory or* $\lim\limits_{t\to\infty} x(t) = 0$ *when* $n$ *is even.*

(D) *If* $0 \leqslant c(t) \leqslant c_2 < 1$ *or* $1 < c_1 \leqslant c(t) \leqslant c_2$, *every bounded solution* $x(t)$ *of* (13) *is oscillatory when* $n$ *is even, and is either oscillatory or* $\lim\limits_{t\to\infty} x(t) = 0$ *when* $n$ *is odd.*

**Example 7.** All conditions of Theorem 5 are satisfied for

$$[x(t) - (6 + 2\sin t)x(t/2 + \pi/4)]''' + 6(1 + \sin^2 t)\frac{x(t/2)}{1 + x^2(t/2)} = 0,$$

and therefore we can conclude that every bounded solution is oscillatory. In fact, $x(t) = \sin 2t$ is an oscillatory solution of the equation.

**Example 8.** By Theorem 5, every bounded solution of

$$[x(t) + (\mathrm{e}^{-t} - 5)x(t - 2\pi)]'' + 4(1 + \cos^2 t)\frac{x(t - \pi)}{1 + x^2(t - \pi)} = 2\mathrm{e}^{-t}\sin t$$

is oscillatory. It is easy to check that $x(t) = \cos t$ is such a solution of the equation.

Finally, by combining Theorems 4 and 5 we obtain the following necessary and sufficient condition for oscillation of bounded solutions of (13).

**Theorem 6.** *Let* (F1) *and* (F2) *be true, and let* $f$ *and* $q$ *satisfy the sign conditions specified in Theorem* 2. *Also assume that* $f$ *satisfies the Lipschitz condition on an interval* $[a, b]$, *where* $a$ *and* $b$ *are as stated in* (i), (iii), *and* (iv).

*Then the conclusion of Theorem* 5 *holds if and only if* (8) *is satisfied.*

**Remark.** If $q(t)$ is replaced by $-q(t)$ in (1) or in (13), then it can be easily shown that the results of this paper still remain true provided the role of $n$ is changed; that is, the phrases "*$n$ is even*" and "*$n$ is odd*" should be interchanged.

As to some directions for future research, we first recall our critical assumption on $c$ and therefore ask the question of finding some additional conditions (if there are any) under which the restrictions $c \neq \pm 1$ and/or $c(t)$ being bounded away from $\pm 1$

could be eliminated. Moreover, to obtain sufficient conditions for oscillation when $-1 \leqslant c < 0$ ($-1 \leqslant c(t) < 0$), as this seems to require a different approach, would be very interesting. In this case, Lemma 1 still applies but unfortunately it is not possible to make a decision about the sign of $z(t)$, which was very crucial in our proofs. Next, we note that the extension of any of the results in this paper to the case where $c(t)$ and/or $q(t)$ are allowed to change sign would be of significant interest. And finally, the study of oscillatory and nonoscillatory behavior of unbounded solutions would also deserve attention for the sake of completeness.

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