Czechoslovak Mathematical Journal

U-Hang Ki; Makoto Kimura; Sadahiro Maeda

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Czechoslovak Mathematical Journal, Vol. 51 (2001), No. 1, 197–204

Persistent URL: http://dml.cz/dmlcz/127639

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GEOMETRY OF HOLOMORPHIC DISTRIBUTIONS OF REAL HYPERSURFACES IN A COMPLEX PROJECTIVE SPACE

U-Hang Ki*, Taegu, Makoto Kimura**, Ibaraki,
and Sadahiro Maeda,* Matsue

(Received May 15, 1998)

Abstract. We characterize homogeneous real hypersurfaces M's of type (A_1), (A_2) and (B) of a complex projective space in the class of real hypersurfaces by studying the holomorphic distribution T^0M of M.

Keywords: complex projective space, real hypersurfaces, holomorphic distribution

MSC 2000: 53B25, 53C40

0. Introduction

Let $P_n(\mathbb{C})$ be an n-dimensional complex projective space with Fubini-Study metric G of constant holomorphic sectional curvature 4, and let M^{2n-1} be a real hypersurface of $P_n(\mathbb{C})$. Then M has an almost contact metric structure (φ, ξ, η, g) induced by the complex structure J of $P_n(\mathbb{C})$. This structure is a useful tool in the study of real hypersurfaces M's in $P_n(\mathbb{C})$ (for examples, see [1], [4], [7]). In this paper we study the holomorphic distribution T^0M which is defined by $(T^0M)_p = \{X \in T_p(M) \mid X \perp \xi\}$ for $p \in M$.

It is known that if the structure vector ξ of a real hypersurface M is a principal curvature vector, the holomorphic distribution T^0M is not integrable (see [3]). This implies that the holomorphic distribution of any homogeneous real hypersurfaces in

* Supported by TGRC-KOSEF
** Supported by Grant-in-Aid for Scientific Research (No.09740050), Ministry of Education, Science and Culture

$P_n(\mathbb{C})$, that is any of the real hypersurfaces given as orbits under subgroups of the projective unitary group $PU(n+1)$, is not integrable.

Takagi ([7]) classified homogeneous real hypersurfaces in $P_n(\mathbb{C})$. By virtue of his work, we find that a homogeneous real hypersurface in $P_n(\mathbb{C})$ is locally congruent to one of the six model spaces of type A_1, A_2, B, C, D and E. They are realized as tubes of constant radii over compact Hermitian symmetric spaces of rank 1 or rank 2 (see Theorem A). A homogeneous real hypersurface of type A_1 is usually called a *geodesic hypersphere*. In the study of real hypersurfaces in $P_n(\mathbb{C})$, many differential geometers have considered the following two problems:

(I) Give a characterization of homogeneous real hypersurfaces in $P_n(\mathbb{C})$.

(II) Construct non-homogeneous *nice* real hypersurfaces in $P_n(\mathbb{C})$ and characterize such examples.

We first investigate in detail the distribution T^0M of any homogeneous real hypersurface M in $P_n(\mathbb{C})$. From the viewpoint of Problem (I) we establish the following two theorems.

Theorem 1. *Let M be a real hypersurface of $P_n(\mathbb{C})$. Then M is locally congruent to a homogeneous real hypersurface of type A_1 or type A_2 if and only if the holomorphic distribution T^0M satisfies the following two conditions:*

(1) *T^0M is decomposed as the direct sum of principal foliations V_{λ_i}'s of M in $P_n(\mathbb{C})$.*

(2) *For each principal foliation V_{λ_i} in condition* (1), *the distribution $V_{\lambda_i} \oplus \{\xi\}_{\mathbb{R}}$ is integrable.*

Theorem 2. *Let M be a real hypersurface of $P_n(\mathbb{C})$, $n \geqslant 3$. Then M is locally congruent to a homogeneous real hypersurface of type B if and only if the holomorphic distribution T^0M satisfies the following three conditions:*

(1) *T^0M is decomposed as the direct sum of principal foliations V_{λ_i}'s of M in $P_n(\mathbb{C})$ with* $\dim V_{\lambda_i} \geqslant 2$.

(2) *Every principal foliation V_{λ_i} in condition* (1) *is integrable.*

(3) *Every leaf of any principal foliation V_{λ_i} in condition* (1) *is a totally geodesic submanifold of the real hypersurface M.*

We remark that if we omit the condition (3), Theorem 2 is not true. We will construct a certain class of non-homogeneous real hypersurfaces M's (in $P_n(\mathbb{C})$) satisfying the conditions (1), (2) in Theorem 2. In this paper, a real hypersurface satisfying the conditions (1), (2) in Theorem 2 is called a *real hypersurface of Dupin type*. Needless to say, the characteristic vector ξ of any real hypersurface M of Dupin type is a principal curvature vector of M in $P_n(\mathbb{C})$.

From the viewpoint of Problem (II) it is interesting to study non-homogeneous real hypersurfaces of Dupin type in $P_n(\mathbb{C})$. We here review the definition of a Dupin hypersurface M^n of a real space form $\widetilde{M}^{n+1}(c)$ of constant curvature c (that is, $\widetilde{M}^{n+1}(c) = \mathbb{R}^{n+1}$, $S^{n+1}(c)$ or $H^{n+1}(c)$ as the curvature c is zero, positive or negative). A hypersurface M^n in $\widetilde{M}^{n+1}(c)$ is called a *Dupin hypersurface* if each of its principal curvatures has constant multiplicity and is constant along the leaves of its principal foliation. So every leaf of its principal foliation is totally umbilic in $\widetilde{M}^{n+1}(c)$, but generally it is not totally geodesic in the hypersurface M^n.

Finally, we will construct non-homogeneous real hypersurfaces of Dupin type in $P_n(\mathbb{C})$.

1. Preliminaries

Let M be a real hypersurface of $P_n(\mathbb{C})$ and let N be a unit normal local vector field on M. The Riemannian connections $\widetilde{\nabla}$ of $P_n(\mathbb{C})$ and ∇ of M are related by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N \tag{1.1}$$

and

$$\widetilde{\nabla}_X N = -AX, \tag{1.2}$$

where g denotes the Riemannian metric of M induced by the Fubini-Study metric G of $P_n(\mathbb{C})$ and A is the shape operator of M in $P_n(\mathbb{C})$. Eigenvalues and eigenvectors of the shape operator A are called *principal curvatures* and *principal curvature vectors*, respectively. In what follows, we denote by V_λ the eigenspace of A associated with the eigenvalue λ. It is known that M admits an almost contact metric structure (φ, ξ, η, g) induced by the complex structure of $P_n(\mathbb{C})$, which satisfies

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

It follows from (1.1) and (1.2) that

$$(\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{1.3}$$

and

$$\nabla_X \xi = \varphi AX. \tag{1.4}$$

Let $\tilde{R}$ and R be the curvature tensors of $P_n(\mathbb{C})$ and M, respectively. Since the curvature tensor $\tilde{R}$ has a nice form, we have the following Gauss and Codazzi equations:

$$\begin{aligned} g(R(X,Y)Z,W) &= g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ &\quad + g(\varphi Y,Z)g(\varphi X,W) - g(\varphi X,Z)g(\varphi Y,W) \\ &\quad - 2g(\varphi X,Y)g(\varphi Z,W) + g(AY,Z)g(AX,W) \\ &\quad - g(AX,Z)g(AY,W), \end{aligned} \tag{1.5}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X,Y)\xi. \tag{1.6}$$

In the following, we use the same terminology and notation as above unless otherwise stated. Now we present without proof the following results in order to prove our theorems:

Theorem A ([7]). *Let M be a homogeneous real hypersurface of $P_n(\mathbb{C})$. Then M is a tube of radius r over one of the following Kaehler submanifolds:*

(A_1) *hyperplane $P_{n-1}(\mathbb{C})$, where $0 < r < \frac{\pi}{2}$,*
(A_2) *totally geodesic $P_k(\mathbb{C})$ $(1 \leqslant k \leqslant n-2)$, where $0 < r < \frac{\pi}{2}$,*
(B) *complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$,*
(C) *$P_1(\mathbb{C}) \times P_{\frac{n-1}{2}}(\mathbb{C})$, where $0 < r < \frac{\pi}{4}$ and n $(\geqslant 5)$ is odd,*
(D) *complex Grassmann $G_{2,5}(\mathbb{C})$, where $0 < r < \frac{\pi}{4}$ and $n = 9$,*
(E) *Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $n = 15$.*

The number of distinct principal curvatures of these homogeneous real hypersurfaces is 2, 3, 3, 5, 5, 5, respectively.

Theorem B ([2]). *Let M be a real hypersurface of $P_n(\mathbb{C})$. Then M has constant principal curvatures and ξ is a principal curvature vector if and only if M is locally congruent to a homogeneous real hypersurface.*

Proposition A ([5]). *Assume that ξ is a principal curvature vector and the corresponding principal curvature is α. Then α is locally constant. In addition, $A\varphi X = \frac{\alpha\lambda+2}{2\lambda-\alpha}\varphi X$ holds for any $X(\perp\xi) \in V_\lambda$.*

2. Proof of Theorems

Proof of Theorem 1. Let M be a real hypersurface satisfying the conditions (1), (2) in Theorem 1. We shall show that our manifold is of type A_1 or type A_2. T^0M is decomposed as $T^0M = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \ldots \oplus V_{\lambda_d}$, where d is the number of distinct principal curvatures λ_i corresponding to the principal curvature vectors $v_{\lambda_i}(\perp\xi)$. Then for any $X(=\Sigma_{i=1}^d X^i v_{\lambda_i}) \in T^0M$, $g(A\xi, X) = g(\xi, AX) = \Sigma_{i=1}^d g(\xi, X^i\lambda_i v_{\lambda_i}) = 0$, so that ξ is principal. By hypothesis, for any $X(\in V_{\lambda_i})$ we have $\nabla_X\xi - \nabla_\xi X \in V_{\lambda_i} \oplus \{\xi\}_{\mathbb{R}}$ $(i = 1, \ldots, d)$. We note that $\nabla_X\xi - \nabla_\xi X$ is perpendicular to ξ for any $X(\in V_{\lambda_i})$, because ξ is a principal curvature (unit) vector, so that

$$A(\nabla_X\xi - \nabla_\xi X) = \lambda_i(\nabla_X\xi - \nabla_\xi X) \quad \text{for any } \ X \in V_{\lambda_i}.$$

This, together with (1.4) and Proposition A, shows

$$(A - \lambda_i I)\nabla_\xi X = \lambda_i\left(\frac{\alpha\lambda_i + 2}{2\lambda_i - \alpha} - \lambda_i\right)\varphi X. \tag{2.1}$$

It follows from (1.4), (2.1) and Proposition A that

$$\begin{aligned}(\nabla_X A)\xi - (\nabla_\xi A)X &= \nabla_X(\alpha\xi) - A\nabla_X\xi - \nabla_\xi(AX) + A\nabla_\xi X \\ &= \alpha\varphi AX - A\varphi AX - (\xi\lambda_i)X + (A - \lambda_i I)\nabla_\xi X \\ &= \lambda_i\left(\alpha - \frac{\alpha\lambda_i + 2}{2\lambda_i - \alpha}\right)\varphi X - (\xi\lambda_i)X + \lambda_i\left(\frac{\alpha\lambda_i + 2}{2\lambda_i - \alpha} - \lambda_i\right)\varphi X \\ &= \lambda_i(\alpha - \lambda_i)\varphi X - (\xi\lambda_i)X.\end{aligned}$$

On the other hand, the Codazzi equation (1.6) implies

$$(\nabla_X A)\xi - (\nabla_\xi A)X = -\varphi X.$$

Hence, the principal curvature λ_i is a solution of the quadratic equation

$$\lambda_i^2 - \alpha\lambda_i - 1 = 0. \tag{2.2}$$

Then our manifold M is locally congruent to one of the homogeneous real hypersurfaces in $P_n(\mathbb{C})$ (see Theorem B). Moreover, again by using (2.2) we find that $\varphi V_{\lambda_i} = V_{\lambda_i}$ $(i = 1, \ldots, d)$, which yields that M is of type A_1 or type A_2 (cf. [5]).

Our theorem is obvious for type A_1. So, let M be of type A_2 (which is a tube of radius r). Let $x = \cot r$ $(0 < r < \frac{\pi}{2})$. Then at any point p of M, $T_p(M)$ is decomposed as $T_p(M) = \{\xi\}_{\mathbb{R}} \oplus V_{\lambda_1} \oplus V_{\lambda_2}$, where $\lambda_1 = x$, $\lambda_2 = -\frac{1}{x}$, $\alpha = x - \frac{1}{x}$. Note that $\varphi V_{\lambda_i} = V_{\lambda_i}$ $(i = 1, 2)$ (for details, see [8]). We remark that neither V_{λ_1} nor V_{λ_2} is integrable. Our aim here is to prove that $V_{\lambda_1} \oplus \{\xi\}_{\mathbb{R}}$ $(V_{\lambda_2} \oplus \{\xi\}_{\mathbb{R}})$ is integrable and moreover, that any leaf of the distribution $V_{\lambda_1} \oplus \{\xi\}_{\mathbb{R}}$ (respectively, $V_{\lambda_2} \oplus \{\xi\}_{\mathbb{R}}$) is a totally geodesic submanifold of M. Let $\mathcal{T} = V_{\lambda_1} \oplus \{\xi\}_{\mathbb{R}}$. Then we can show the following:

$$\nabla_\xi \xi \in \mathcal{T}, \ \nabla_X \xi \in \mathcal{T}, \ \nabla_\xi X \in \mathcal{T} \quad \text{and} \quad \nabla_X Y \in \mathcal{T} \quad \text{for any } X, Y \in V_{\lambda_1}.$$

In fact, (1.4) yields $\nabla_\xi \xi = 0 \in \mathcal{T}$ and $\nabla_X \xi = \varphi AX = \lambda_1 \varphi X \in V_{\lambda_1} \subset \mathcal{T}$. Next,

$$\begin{aligned}
(\nabla_\xi A)X - (\nabla_X A)\xi &= \nabla_\xi (AX) - A\nabla_\xi X - \nabla_X (A\xi) + A\nabla_X \xi \\
&= (\lambda_1 I - A)\nabla_\xi X - \alpha\varphi AX + A\varphi AX \\
&= (\lambda_1 I - A)\nabla_\xi X + \lambda_1(\lambda_1 - \alpha)\varphi X.
\end{aligned}$$

On the other hand, it follows from (1.6) that

$$(\nabla_\xi A)X - (\nabla_X A)\xi = \varphi X \in V_{\lambda_1}.$$

Thus for any $Z \in V_\lambda$ $(\lambda = \lambda_2, \alpha)$ we find $g\big((\lambda_1 I - A)\nabla_\xi X, Z\big) = 0$, so that $\nabla_\xi X \in V_{\lambda_1} \subset \mathcal{T}$. Finally, for any $X, Y \in V_{\lambda_1}$ and for any $Z \in V_{\lambda_2}$ we get

$$\begin{aligned}
g((\nabla_X A)Y, Z) &= g(\nabla_X (AY) - A\nabla_X Y, Z) \\
&= g((\lambda_1 I - A)\nabla_X Y, Z) \\
&= (\lambda_1 - \lambda_2) g(\nabla_X Y, Z).
\end{aligned}$$

On the other hand, it follows from (1.6) that

$$\begin{aligned}
g((\nabla_X A)Y, Z) &= g((\nabla_X A)Z, Y) \\
&= g((\nabla_Z A)X, Y) \\
&= g(\nabla_Z (AX) - A\nabla_Z X, Y) \\
&= g((\lambda_1 I - A)\nabla_Z X, Y) = 0.
\end{aligned}$$

Hence, $\nabla_X Y \in \mathcal{T}$. Thus we can see that every leaf L of the distribution $\mathcal{T}$ is a totally geodesic submanifold of M. The manifold L is locally congruent to a homogeneous real hypersurface (with the unit vector $-N$) of type A_1 of radius $(\frac{\pi}{2} - r)$ in $P_{m+1}(\mathbb{C})$ which is a holomorphic totally geodesic submanifold of $P_n(\mathbb{C})$, where $2m = \dim V_{\lambda_1}$ (see Theorem 1 in [1]). The same discussion as above yields that the distribution $\mathcal{S} = V_{\lambda_2} \oplus \{\xi\}_{\mathbb{R}}$ is integrable and moreover, that every leaf K of the distribution $\mathcal{S}$ is a totally geodesic submanifold of M. The manifold K is locally congruent to a homogeneous real hypersurface (with the unit normal vector N) of type A_1 of radius r in $P_{k+1}(\mathbb{C})$ which is a holomorphic totally geodesic submanifold of $P_n(\mathbb{C})$, where $2k = \dim V_{\lambda_2}$. $\square$

P r o o f of Theorem 2. Let M be a real hypersurface satisfying the conditions (1), (2), (3) in Theorem 2. We shall show that the manifold M is of type B. From the condition (1) we can set $T^0M = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \ldots \oplus V_{\lambda_d}$. It follows from the conditions (2), (3) that $A\nabla_X Y = \lambda_i \nabla_X Y$ for any $X, Y \in V_{\lambda_i}$ $(i = 1, 2, \ldots, d)$. Hence, for any $X, Y \in V_{\lambda_i}$ we get $(\nabla_X A)Y = (X\lambda_i)Y$. On the other hand, it follows from the condition (3) and (1.1) that every leaf of V_{λ_i} is a totally umbilic submanifold of $P_n(\mathbb{C})$. Needless to say, the mean curvature of any totally umbilic submanifold whose dimension is greater than 1 in $P_n(\mathbb{C})$ is constant, which implies that $X\lambda_i = 0$ for any $X \in V_{\lambda_i}$. Hence,

$$(\nabla_X A)Y = 0 \quad \text{for any } X, Y \in V_{\lambda_i}. \tag{2.3}$$

Thus, for each unit $X \in V_{\lambda_i}$ and for any $Z \in TM$

$$\begin{aligned}
0 &= g((\nabla_X A)X, Z) \qquad \text{(from (2.3))} \\
&= g((\nabla_X A)Z, X) \\
&= g((\nabla_Z A)X + \eta(X)\varphi Z - \eta(Z)\varphi X - 2 \cdot g(\varphi X, Z)\xi, X) \quad \text{(from (1.6))} \\
&= g((\nabla_Z A)X, X) = g(\nabla_Z(AX) - A\nabla_Z X, X) \\
&= g((Z\lambda_i)X + (\lambda_i I - A)\nabla_Z X, X) \\
&= Z\lambda_i.
\end{aligned}$$

Then Theorem B tells us that the manifold M is homogeneous in $P_n(\mathbb{C})$. However, the principal foliation V_λ is not integrable in the case that $\varphi V_\lambda = V_\lambda$ (see (1.6)). Thus we can see that M is of type B (for details, see [8]).

Conversely, let M be of type B. Then at any point p of M, $T_p(M)$ is decomposed as $T_p(M) = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \{\xi\}_{\mathbb{R}}$, where $\lambda_1 = \frac{1+x}{1-x}$, $\lambda_2 = \frac{x-1}{x+1}$, $\alpha = x - \frac{1}{x}$ and $x = \cot r$ $(0 < r < \frac{\pi}{4})$ (cf. [8]). We note that $\varphi V_{\lambda_1} = V_{\lambda_2}$ (see Proposition A). We shall prove that the principal foliation V_{λ_1} (resp. V_{λ_2}) on M is integrable, and moreover that every leaf of the distribution V_{λ_1} (resp. V_{λ_2}) is a totally geodesic submanifold of M. It suffices to verify that $\nabla_X Y \in V_{\lambda_1}$ for any $X, Y \in V_{\lambda_1}$. We first have

$$\begin{aligned}
A\nabla_X Y &= \nabla_X(AY) - (\nabla_X A)Y \\
&= \lambda_1 \nabla_X Y - (\nabla_X A)Y.
\end{aligned}$$

For any $Z \in TM$, since A is symmetric, from (1.6) we find

$$\begin{aligned}
g((\nabla_X A)Y, Z) &= g((\nabla_X A)Z, Y) \\
&= g((\nabla_Z A)X + \eta(X)\varphi Z - \eta(Z)\varphi X - 2 \cdot g(\varphi X, Z)\xi, Y) \\
&= g((\nabla_Z A)X, Y) = g(\nabla_Z(AX) - A\nabla_Z X, Y) \\
&= g((Z\lambda_1)X + (\lambda_1 I - A)\nabla_Z X, Y) = 0,
\end{aligned}$$

so that $(\nabla_X A)Y = 0$ for any $X, Y \in V_{\lambda_1}$. This implies that every leaf L_{λ_1} of the principal foliation V_{λ_1} is a totally geodesic submanifold of the real hypersurface M. L_{λ_1} is locally congruent to a totally umbilic hypersurface of constant curvature c (with $\sqrt{c-1} = |\lambda_1|$) in $P^n(\mathbb{R})$ which is a totally real totally geodesic submanifold of $P_n(\mathbb{C})$. □

Example. Let V_{n-1} be a complex hypersurface of $P_n(\mathbb{C})$, $n \geqslant 3$ such that

(1) any principal curvature with respect to the shape operator A_ξ for any unit normal vector ξ of V_{n-1} is non zero, and

(2) multiplicity of each principal curvature with respect to A_ξ for any unit normal vector ξ of V_{n-1} is constant.

Then by [1, Proposition 3.1, p. 487] we can see that a real hypersurface M which lies on the tube of radius $r > 0$ over V_{n-1} satisfies the conditions (1), (2) in Theorem 2. We remark that there exists such complex hypersurfaces V_{n-1}.

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Authors' addresses: U-H. Ki, Department of Mathematics, Kyungpook University, Taegu 702-701, Korea, e-mail: uhang@bh.kyungpook.ac.kr; M. Kimura, Department of Mathematics, Ibaraki University, Mito, Ibaraki 310-0056, Japan, e-mail: mkimura@mito.ipc.ibaraki.ac.jp; (M. Kimura's current address: Dept. of Mathematics, Shimane University, Matsue 690-8504, Japan, e-mail: mkimura@math.shimane-u.ac.jp); S. Maeda, Department of Mathematics, Shimane University, Matsue 690-8504, Japan, e-mail: smaeda@math.shimane-u.ac.jp.