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SEQUENTIAL CONVERGENCES ON BOOLEAN ALGEBRAS DEFINED BY SYSTEMS OF MAXIMAL FILTERS

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Abstract. We study sequential convergences defined on a Boolean algebra by systems of maximal filters. We describe the order properties of the system of all such convergences. We introduce the category of **2**-generated convergence Boolean algebras and generalize the construction of Novák sequential envelope to such algebras.

Keywords: sequential convergence on Boolean algebras, **2**-generated convergence, **2**-embedded Boolean algebra, absolutely sequentially closed Boolean algebra

MSC 2000: 54A20, 54H12, 06E15, 54B30

1. Introduction

The system $\operatorname{Conv} B$ of all sequential convergences on a Boolean algebra $B \neq \{0\}$ which are compatible with the structure of B was investigated in [10], [11] and [15]. The analogously defined system $\operatorname{Conv} L$ where L is a lattice was dealt with in [12] and [13].

It is well-known that the notions of Boolean algebra and of sequential convergence are basic tools for constructing the fundaments of probability theory ([8], [16], [17]). The connection between Boolean algebras and probability was, indeed, appreciated by Boole himself. Let us quote from [4]: "The design of the following treatise is to investigate the fundamental laws of those operations of the mind by which reasoning is performed; to give expression to them in the symbolic language of a calculus, and upon this foundation to establish the science of Logic and construct its method, to make that method itself the basis of a general method for the application of the mathematical doctrine of Probabilities ..." (cf. also the quotation of this text in [2]).

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When studying the foundations of probability theory a specific subset of $\operatorname{Conv} B$ occurs to be relevant ([5], [7]; see also [18], [20]). Namely, let S be a system of homomorphisms of B onto the two-element Boolean algebra $\mathbf{2}$ (the set $\{0,1\}$ carrying the usual Boolean operations), such that for any two distinct elements $x, y \in B$ there exists $\varphi \in S$ with $\varphi(x) \neq \varphi(y)$. If $\langle x_n \rangle$ is a sequence in B and $x \in B$, then we put

$$x_n \to_{\alpha(S)} x$$

if for each $\varphi \in S$ there exists a positive integer m such that $\varphi(x_n) = \varphi(x)$ whenever $n \geqslant m$. We denote

$$\alpha(S) = \{(\langle x_n \rangle, x)\colon\ x_n \to_{\alpha(S)} x\}.$$

Then $\alpha(S)$ is a compatible convergence on B (see Lemma 3.1) and each $\varphi \in S$ is a sequentially continuous homomorphism of B onto $\mathbf{2}$ (carrying a natural convergence in which only almost constant sequences converge). Furthermore, if a sequence $\langle x_n \rangle$ does not converge to x with respect to $\alpha(S)$, then there exists $\varphi \in S$ such that $\varphi(x_n) \neq \varphi(x)$ for infinitely many indexes n. Hence $\alpha(S)$ is an initial convergence with respect to the system of all $\alpha(S)$-sequentially continuous homomorphisms of B onto $\mathbf{2}$. The system of all sequential convergences on B which can be constructed in this way will be denoted by $\operatorname{Conv} B(\mathbf{2})$.

Since there is a one-to-one relationship between the homomorphisms of B onto $\mathbf{2}$ and maximal filters of B (cf., e.g., [22]), each $\alpha(S)$ can be defined by a system of maximal filters of B. In fact, $\alpha(S)$ can be viewed as a pointwise convergence on B (via the Stone representation of B).

A variant of pointwise convergence on an archimedean lattice ordered group G with a weak unit (applying Yosida representation of G) was dealt with by Ball and Hager [1].

In [3], Boolean algebras carrying sequential convergence were applied to the abstract integration.

Sample results:

The partially ordered set $\operatorname{Conv} B(\mathbf{2})$ is a $\wedge$-semilattice having the least element. Each interval of $\operatorname{Conv} B(\mathbf{2})$ is a complete lattice. If B is infinite, then

(i) $\operatorname{card}(\operatorname{Conv} B \setminus \operatorname{Conv} B(\mathbf{2})) \geqslant \mathfrak{c}$,

(ii) the least element of $\operatorname{Conv} B$ is not equal to the least element of $\operatorname{Conv} B(\mathbf{2})$.

If B is complete and completely distributive, then $\operatorname{Conv} B(\mathbf{2})$ is a complete lattice. In general, $\operatorname{Conv} B(\mathbf{2})$ need not contain any maximal element. We introduce the category of $\mathbf{2}$-generated convergence Boolean algebras and prove that the absolutely sequentially closed objects form its epireflective subcategory.

In their monograph, Riečan and Neubrunn [21] developed a probability theory on MV-algebras (Chapter 9). Sequential convergences on MV-algebras were studied in [14]. The methods of the present paper could be relevant also for the theory of MV-algebras.

2. Preliminaries

We recall the definitions of $\operatorname{Conv} B$ and $\operatorname{Conv}_0 B$ as given in [10].

Let $B^{\mathbb{N}}$ be the system of all sequences in B and let α be a subset of $B^{\mathbb{N}} \times B$. If $(\langle x_n\rangle, x) \in \alpha$, then we write $x_n \to_\alpha x$. If $\langle x_n\rangle \in B^{\mathbb{N}}$, $a \in B$ and $x_n = a$ for each $n \in \mathbb{N}$, then we put $\langle x_n\rangle = \operatorname{const} a$.

2.1. Definition. A subset α of $B^{\mathbb{N}} \times B$ is said to be a convergence in B if the following conditions are satisfied:

(i) If $x_n \to_\alpha x$ and $\langle y_n\rangle$ is a subsequence of $\langle x_n\rangle$, then $y_n \to_\alpha x$.

(ii) If $\langle x_n\rangle \in B^{\mathbb{N}}$, $x \in B$ and if for each subsequence $\langle y_n\rangle$ of $\langle x_n\rangle$ there is a subsequence $\langle z_n\rangle$ of $\langle y_n\rangle$ such that $z_n \to_\alpha x$, then $x_n \to_\alpha x$.

(iii) If $a \in B$ and $\langle x_n\rangle = \operatorname{const} a$, then $x_n \to_\alpha a$.

(iv) If $x_n \to_\alpha x$ and $x_n \to_\alpha y$, then $x = y$.

(v) If $x_n \to_\alpha x$ and $y_n \to_\alpha y$, then $x_n \vee y_n \to_\alpha x \vee y$, $x_n \wedge y_n \to_\alpha x \wedge y$ and $x'_n \to_\alpha x'$.

(vi) If $x_n \leqslant y_n \leqslant z_n$ is valid for each $n \in \mathbb{N}$ and $x_n \to_\alpha x$, $z_n \to_\alpha x$, then $y_n \to_\alpha x$.

The system of all convergences on B is denoted by $\operatorname{Conv} B$.

2.2. Definition. Let $\alpha \in \operatorname{Conv} B$. We put $\alpha_0 = \{\langle x_n\rangle \in B^{\mathbb{N}} \colon x_n \to_\alpha 0\}$, $\operatorname{Conv}_0 B = \{\alpha_0 \colon \alpha \in \operatorname{Conv} B\}$.

Both the systems $\operatorname{Conv} B$ and $\operatorname{Conv}_0 B$ are partially ordered by the set-theoretical inclusion.

For each $\alpha \in \operatorname{Conv} B$ we put $f(\alpha) = \alpha_0$.

2.3. Lemma ([15]). *The mapping f is an isomorphism of the partially ordered set $\operatorname{Conv} B$ onto the partially ordered set $\operatorname{Conv}_0 B$.*

We denote by d the set of all $(\langle x_n\rangle, x) \in B^{\mathbb{N}} \times B$ having the property that there exists $m \in \mathbb{N}$ such that $x_n = x$ for each $m \in \mathbb{N}$ with $n \geqslant m$. Then d is the least element of $\operatorname{Conv} B$ and d_0 is the least element of $\operatorname{Conv}_0 B$. We say that d is a discrete convergence.

2.4. Proposition (cf. [10]). *$\operatorname{Conv}_0 B$ is a $\wedge$-semilattice and each interval of $\operatorname{Conv}_0 B$ is a complete lattice.*

2.5. Proposition (cf. [15]). *Let $[\mu_1, \mu_2]$ be an interval of $\operatorname{Conv}_0 B$ and $\beta \in [\mu_1, \mu_2]$, $\emptyset \neq \{\alpha_i\}_{i \in I} \subseteq [\mu_1, \mu_2]$. Then*

$$\Bigl(\bigvee_{i \in I} \alpha_i\Bigr) \wedge \beta = \bigvee_{i \in I} (\alpha_i \wedge \beta).$$

3. The system $\operatorname{Conv} B(\mathbf{2})$

Assume that S, $\alpha(S)$ and $\operatorname{Conv} B(\mathbf{2})$ are as in Section 1.

3.1. Lemma. $\alpha(S) \in \operatorname{Conv} B$.

Proof. Put $\alpha = \alpha(S)$. Consider the conditions (i)–(vi) from 2.1. It is obvious that the conditions (i) and (iii)–(vi) are satisfied. Suppose that the assumption of (ii) are valid. By way of contradiction, assume that $(\langle x_n\rangle, x)$ does not belong to α. Thus there exists $\varphi \in S$ such that for each $n \in \mathbb{N}$ there is $m(n) \in \mathbb{N}$ with $m(n) \geqslant n$ and $\varphi(x_{m(n)}) \neq \varphi(x)$. Put $y_n = x_{m(n)}$ for each $n \in \mathbb{N}$. If $\langle z_n\rangle$ is a subsequence of $\langle y_n\rangle$, then for each $n \in \mathbb{N}$ we have $\varphi(z_n) \neq \varphi(x)$, whence the relation $z_n \rightarrow_\alpha x$ fails to hold. We arrived at a contradiction. $\square$

3.2. Corollary. $\operatorname{Conv} B(\mathbf{2}) \subseteq \operatorname{Conv} B$.

If $\{\alpha_i\}_{i\in I} \subseteq \operatorname{Conv} B(\mathbf{2})$ and if $\sup\{\alpha_i\}_{i\in I}$ exists in $\operatorname{Conv} B(\mathbf{2})$, then we denote it by $\overset{s}{\bigvee\limits_{i\in I}} \alpha_i$; the meaning of $\overset{s}{\bigwedge\limits_{i\in I}} \alpha_i$ is analogous. (The symbols $\wedge$ and $\vee$ will be applied for the partially ordered sets $\operatorname{Conv} B$ and $\operatorname{Conv} B(\mathbf{2})$.)

Let us denote by S^0 the system of all maximal filters of the Boolean algebra B. Let $p \in S^0$. For each $x \in B$ we put $\varphi_p(x) = 1$ if $x \in p$, and $\varphi_p(x) = 0$ otherwise. Let S_1^0 be the system of all mappings φ_p defined in this way. Then S_1^0 satisfies the assumptions for S in Section 1 and thus $\alpha(S_1^0) \in \operatorname{Conv} B(\mathbf{2})$. Hence $\operatorname{Conv} B(\mathbf{2}) \neq \emptyset$.

The collection of all systems S which satisfy the assumptions from Section 1 will be denoted by $C(B)$.

From the definition of $\alpha(S)$ we obtain immediately:

3.3. Lemma. *Let $S_1, S_2 \in C(B)$, $S_1 \subseteq S_2$. Then $\alpha(S_1) \geqslant \alpha(S_2)$.*

3.4. Lemma. *Let $\emptyset \neq \{S_i\}_{i\in I} \subseteq C(B)$. Put $S = \bigcup\limits_{i\in I} S_i$. Then*

(i) *$S \in C(B)$;*

(ii) *$\alpha(S) = \overset{s}{\bigwedge\limits_{i\in I}} \alpha(S_i) = \bigwedge\limits_{i\in I} \alpha(S_i)$.*

Proof. The assertion (i) is a consequence of the definition of $C(B)$. In view of 3.3 we have $\alpha(S) \leqslant \alpha(S_i)$ for each $i \in I$. Hence $\alpha(S) \subseteq \bigcap\limits_{i\in I} \alpha(S_i)$.

Let $(\langle x_n\rangle, x) \in \bigcap\limits_{i\in I} \alpha(S_i)$. Thus $x_n \rightarrow_{\alpha(S_i)} x$ for each $i \in I$. According to the definition of S we obtain $x_n \rightarrow_{\alpha(S)} x$, i.e., $(\langle x_n\rangle, x) \in \alpha(S)$. Therefore

$$\alpha(S) = \bigcap_{i\in I} \alpha(S_i).$$

Thus in view of the definition of the operation $\wedge$ in $\operatorname{Conv} B$ we obtain

$$\alpha(S) = \bigwedge_{i \in I} \alpha(S_i).$$

Then from 3.2 we conclude

$$\alpha(S) = \bigwedge_{i \in I}^{s} \alpha(S_i).$$

$\square$

The set S_1^0 is the largest element of the system $C(B)$ (partially ordered by the set-theoretical inclusion). Hence by applying 3.3 we get

3.5. Lemma. *$\alpha(S_1^0)$ is the least element of* $\operatorname{Conv} B(\mathbf{2})$.

3.6. Proposition. *Each interval of* $\operatorname{Conv} B(\mathbf{2})$ *is a complete lattice.*

P r o o f. Let $\alpha, \beta \in \operatorname{Conv} B(\mathbf{2})$, $\alpha \leqslant \beta$. Suppose that $\{\alpha_i\}_{i \in I}$ is a nonempty subset of $\operatorname{Conv} B(\mathbf{2})$ such that $\alpha \leqslant \alpha_i \leqslant \beta$ for each $i \in I$. Let $\{\beta_j\}_{j \in J}$ be the set of all upper bounds of the set $\{\alpha_i\}_{i \in I}$ with $\beta_j \leqslant \beta$. In view of 3.4 there exists $\beta^0 \in \operatorname{Conv} B(\mathbf{2})$ such that

$$\beta^0 = \bigwedge_{j \in J}^{s} \beta_j.$$

Then we have $\alpha \leqslant \beta^0 \leqslant \beta$ and

$$\beta^0 = \bigvee_{i \in I}^{s} \alpha_i.$$

The proof is finished by applying 3.4. $\square$

From 3.5 and 3.6 we obtain

3.7. Corollary. $\operatorname{Conv} B(\mathbf{2})$ *is a $\wedge$-semilattice.*

In general, $\operatorname{Conv}(\mathbf{2})$ does not have even maximal elements. The well-known Boolean algebra B_0 generated by semi-open intervals (see § 8E in [22]) provides a counterexample.

3.8. Example. Let B_0 be the least field of subsets of the unit interval $0 \leqslant x < 1$ containing all intervals $0 \leqslant x < a$, $0 < a \leqslant 1$, i.e. the class of all finite unions of left-closed right-open subintervals of this interval. The Stone space of B_0 is the set X obtained from the closed unit interval $0 \leqslant x \leqslant 1$ by splitting every interior point x into two parts, x^- and x^+. We consider X as an ordered set with the natural order:

$$0 < x^- < x^+ < y^- < y^+ < 1 \quad \text{whenever} \quad 0 < x < y < 1.$$

The set X with the topology determined by this order is compact and totally disconnected. The Boolean algebra B_0 is isomorphic to the field $\mathbf{F}$ of all open-closed subsets of X (associate, with every interval $a \leqslant x < b$, the set composed of a^+, b^- and all x^-, x^+ where $a < x < b$), therefore X is the Stone space of B_0; points of X can be considered homomorphisms of B_0 onto the two-element Boolean algebra $\mathbf{2}$. For $D \subset [0,1]$ denote $D^- = \{d^-;\ d \in D\}$, where $0^+ = 0$ and $1^- = 1$.

Straightforward proofs of the next two lemmas are omitted.

3.9. Lemma. *Let $S \subset X$. The following are equivalent:*

(i) $S \in C(B_0)$;

(ii) *There exists a dense subset $D \subset [0,1]$ such that $S \subset D^- \cup D^+$ and for each $d \in D$ we have $S \cap \{d^-, d^+\} \neq \emptyset$.*

3.10. Lemma. *Let $S \in C(B_0)$ and $d \in [0,1]$. Then $(S \setminus \{d^+\}) \in C(B_0)$ and $(S \setminus \{d^-\}) \in C(B_0)$.*

3.11. Lemma. *Let $S_1, S_2 \in C(B_0)$, $S_1 \neq S_2$. Then $\alpha(X_1) \neq \alpha(S_2)$.*

P r o o f. Assume that $d^+ \in S_1 \setminus S_2$. Then intervals $[d, d + 1/n]$ converge to $\emptyset$ under $\alpha(S_2)$ but do not converge to under $\alpha(S_1)$. Similarly, if $d^- \in S_1 \setminus S_2$, then intervals $[d - 1/n, d)$ converge to $\emptyset$ under $\alpha(S_2)$, but do not converge to $\emptyset$ under $\alpha(S_1)$. $\square$

3.12. Proposition. $\operatorname{Conv} B_0(\mathbf{2})$ *does not contain any maximal element.*

P r o o f. The assertion follows from Lemma 3.9 and Lemma 3.10. $\square$

3.13. Proposition. $\operatorname{card}(\operatorname{Conv} B_0(\mathbf{2})) = 2^{\mathfrak{c}}$.

P r o o f. On the one hand, since $\alpha_0(S) \subset B_0^{\mathbb{N}}$, $S \in C(B_0)$, $\operatorname{card}(\operatorname{Conv} B_0(\mathbf{2}))$ cannot exceed $2^{\mathfrak{c}}$. On the other hand, $Q \cap [0,1]$ is a dense subset of $[0,1]$ and for each $M \subset ([0,1] \setminus Q)$ the set $(Q \cap [0,1]) \cup M$ is dense, too. According to Lemma 3.9 and Lemma 3.11, for each set $M_1 = (Q \cap [0,1]) \cup M$ we have $\alpha(M_1) \in \operatorname{Conv} B_0(\mathbf{2})$ and $\alpha(M_1) \neq \alpha(M_1')$ whenever $M \neq M'$. Thus $\operatorname{card}(\operatorname{Conv} B_0(\mathbf{2})) = 2^{\mathfrak{c}}$. $\square$

For the notion of complete distributivity cf., e.g., [22].

In the remaining part of this section we assume that B is complete and completely distributive. Then each nonzero element of B can be uniquely represented as a join of a subset of A, where A is the set of all atoms of B.

For each $a \in A$ and each $x \in B$ we put $\varphi_a(x) = 1$ if $x \geqslant a$, and $\varphi_a(x) = 0$ otherwise. Let $S_A = \{\varphi_a \colon a \in A\}$. It is easy to verify that S_A satisfies the conditions from Section 1; hence we have

3.14. Lemma. *The set $\alpha(S_A)$ belongs to* $\operatorname{Conv} B(\mathbf{2})$.

3.15. Lemma. *Let $a \in A$ and $S \in C(B)$. Then $\varphi_a \in S$.*

P r o o f. By way of contradiction, suppose that φ_a does not belong to S. Let $\varphi \in S$. Put $F_\varphi = \{x \in B\colon \varphi(x) = 1\}$. Thus F_φ is a maximal filter in B. If $a \in F_\varphi$, then $F_\varphi = \{y \in B\colon y \geqslant a\}$, whence $\varphi = \varphi_a$, which is a contradiction. Thus $a \notin F_\varphi$ and hence $\varphi(a) = 0$ for each $\varphi \in S$. Put $\langle x_n \rangle = \operatorname{const} a$. We obtain $x_n \to_{\alpha(S)} 0$, which is impossible. □

3.16. Lemma. *$\alpha(S_A)$ is the greatest element of* $\operatorname{Conv} B(\mathbf{2})$.

P r o o f. In view of 3.8 we have $\alpha(S_A) \in \operatorname{Conv} B(\mathbf{2})$. Let $\alpha \in \operatorname{Conv} B(\mathbf{2})$. There exists $S \in C(B)$ with $\alpha = \alpha(S)$. According to 3.9, $S_A \subseteq S$. Hence 3.3 yields that $\alpha(S_A) \geqslant \alpha(S)$. □

3.17. Proposition. *Assume that B is complete and completely distributive. Then* $\operatorname{Conv} B(\mathbf{2})$ *is a complete lattice.*

P r o o f. It suffices to apply 3.10, 3.5 and 3.6. □

4. Disjoint sequences

A sequence $\langle x_n \rangle$ in B is called disjoint if $x_{n(1)} \wedge x_{n(2)} = 0$ whenever $n(1)$ and $n(2)$ are distinct positive integers. We denote by $D(B)$ the system of all disjoint sequences $\langle x_n \rangle$ in B such that $x_n > 0$ for each $n \in \mathbb{N}$. It is easy to verify that $D(B) \neq \emptyset$ if and only if B is infinite.

The following two lemmas are consequences of [10], Section 5.

4.1. Lemma. *Let $\langle x_n \rangle \in D(B)$. Then there exists $\alpha \in \operatorname{Conv} B$ such that*

(i) $x_n \to_\alpha 0$;

(ii) *if $\beta \in \operatorname{Conv} B$ and $x_n \to_\beta 0$, then $\beta \geqslant \alpha$.*

Under the notation as in 4.1, α is said to be generated by the sequence $\langle x_n \rangle$.

4.2. Lemma. *Let $\langle x_n \rangle, \langle y_n \rangle \in D(B)$. Suppose that α is generated by $\langle x_n \rangle$ and β is generated by $\langle y_n \rangle$. Further assume that $x_{n(1)} \wedge y_{n(2)} = 0$ whenever $n(1), n(2) \in \mathbb{N}$. Then $\alpha \wedge \beta = d$. In particular, the relation $x_n \to_\beta 0$ fails to be valid.*

4.3. Lemma. *Let $\langle x_n \rangle \in D(B)$ and $\alpha \in \operatorname{Conv} B(\mathbf{2})$. Then $x_n \to_\alpha 0$.*

P r o o f. By way of contradiction, assume that the relation $x_n \to_\alpha 0$ does not hold. Since $\alpha \in \operatorname{Conv} B(\mathbf{2})$ there exists $S \in C(B)$ such that $\alpha = \alpha(S)$. Hence there is $\varphi \in S$ such that for each $n \in \mathbb{N}$ there is $m(n) \in \mathbb{N}$ with $m(n) \geqslant n$ and

$$\varphi(x_n) \neq \varphi(0) = 0.$$

Choose $n_1 \in \mathbb{N}$ and put $m_1 = m(n_1)$. Further choose $n_2 \in \mathbb{N}$ with $n_2 > m_1$ and denote $m_2 = m(n_2)$. Then $m_2 > m_1$ and

$$\varphi(x_{m_1}) = \varphi(x_{m_2}) = 1.$$

Thus $\varphi(x_{m_1} \wedge x_{m_2}) = 1$. Since $\langle x_n \rangle \in D(B)$ we have $x_{m_1} \wedge x_{m_2} = 0$, thus $\varphi(0) = 1$, which is a contradiction. $\square$

Now, 4.1, 4.3 and 3.5 yield

4.4. Lemma. *Let $\langle x_n \rangle \in D(B)$ and let $\alpha \in S(B)$ be generated by $\langle x_n \rangle$. Then $\alpha(S_1^0) \geqslant \alpha$.*

It is obvious that if α is as in 4.4, then $\alpha = d$. Thus from 4.4 we obtain as a corollary

4.5. Proposition. *Let the Boolean algebra B be an infinite Boolean algebra. Then the last element of* $\operatorname{Conv} B$ *does not coincide with the last element of* $\operatorname{Conv} B(\mathbf{2})$.

Again, suppose that the Boolean algebra B is infinite. Hence there exists $\langle x_n \rangle \in D(B)$. There are subsets $\mathbb{N}_j$ $(j \in \mathbb{N})$ of $\mathbb{N}$ such that

(i) each $\mathbb{N}_j$ is infinite;

(ii) if $j(1) \neq j(2)$, then $\mathbb{N}_{j(1)} \cap \mathbb{N}_{j(2)} = \emptyset$.

Hence for each $j \in \mathbb{N}$, $\langle x_n \rangle_{n \in \mathbb{N}_j}$ is a subsequence of $\langle x_n \rangle$. If $n(1) \in \mathbb{N}_{j(1)}$, $n(2) \in \mathbb{N}_{j(2)}$ and $j(1) \neq j(2)$, then $x_{n(1)} \wedge x_{n(2)} = 0$.

For each $j \in \mathbb{N}$ let α_j be the element of $\operatorname{Conv} B$ which is generated by the sequence $\langle x_n \rangle_{n \in \mathbb{N}_j}$. The condition (ii) and 4.2 yield that whenever $j(1)$ and $j(2)$ are distinct elements of $\mathbb{N}$, then

$$\alpha_{j(1)} \wedge \alpha_{j(2)} = d. \tag{1}$$

For each nonempty subset M of $\mathbb{N}$ we put

$$\alpha_M = \bigvee_{j \in M} \alpha_j.$$

The existence of α_M is implied by 4.4, by the existence of the least element in $\operatorname{Conv} B$ and by the fact that each interval of $\operatorname{Conv} B$ is a complete lattice.

Let $j(0) \in \mathbb{N}$, $j(0) \notin M$. Consider the interval $[\alpha, \alpha(S_1^0)]$ of $\operatorname{Conv} B$. In view of 2.3, 2.5 and (1) we get

$$\alpha_{j(0)} \wedge \alpha_M = d. \tag{2}$$

Hence if $M(1)$ and $M(2)$ are distinct subsets of $\mathbb{N}$, then (2) yields

$$\alpha_{M(1)} \neq \alpha_{M(2)}. \tag{3}$$

Moreover, in view of 4.4, for each M with $\emptyset \neq M \subseteq \mathbb{N}$ we have $\alpha(M) \leqslant \alpha(S_1^0)$. Since the number of distinct sets M with the given properties is equal to c and at most one α_M belongs to $\operatorname{Conv} B(\mathbf{2})$, we obtain

4.6. Proposition. *Let B be an infinite Boolean algebra. Then*

$$\operatorname{card}(\operatorname{Conv} B \setminus \operatorname{Conv} B(\mathbf{2})) \geqslant \mathfrak{c}.$$

5. Convergence Boolean algebras

5.1. Definition. Let B be a Boolean algebra and let $\alpha \in \operatorname{Conv} B$. Then (B, α) is said to be a *convergence Boolean algebra*. If $\alpha \in \operatorname{Conv} B(\mathbf{2})$, then (B, α) is said to be **2**-*generated*.

Denote by $\mathscr{B}$ the category whose objects are convergence Boolean algebras and whose morphisms are sequentially continuous Boolean homomorphisms. Denote by $\hom((B_1, \alpha_1), (B_2, \alpha_2))$ the set of all morphisms from (B_1, α_1) into (B_2, α_2). Denote by $\mathscr{B}(\mathbf{2})$ the full subcategory of $\mathscr{B}$ consisting of **2**-generated convergence Boolean algebras. Convergence Boolean subalgebras are defined in the obvious way.

Let $\{(B_t, \alpha_t);\ t \in T\}$ be a set of convergence Boolean algebras. Recall that the direct product $B = \prod_{t \in T} B_t$ and the usual (coordinatewise) product convergence $\alpha = \prod_{t \in T} \alpha_t$, together with the projections π_t of B into B_t, $t \in T$, are the categorical products of $\{B_t;\ t \in T\}$ and $\{\alpha_t;\ t \in T\}$, respectively. It is easy to verify that (B, α) is the categorical product of $\{(B_t, \alpha_t);\ t \in T\}$: for each (B', α') in $\mathscr{B}$ and for each set $\{\varphi_t;\ t \in T\}$ of morphisms from (B', α') into (B_t, α_t), $t \in T$, there exists a unique morphism φ from (B', α') into (B, α) such that $\varphi_t = \pi_t \circ \varphi$ for all $t \in T$.

Finally, observe that if φ is a morphism from (B_1, α_1) into (B_2, α_2) and B_2 is the smallest sequentially closed subset in (B_2, α_2) containing the image $\varphi(B_1)$, then φ

is an epimorphism in $\mathscr{B}$. This follows from the fact that (in sequential convergence spaces with unique sequential limits) if two sequentially continuous mappings agree on a topologically dense subset, then they agree on the whole domain (cf. Lemma 5 in [19]).

5.2. Example. Let $X \neq \emptyset$ be a set and let B be a reduced field of subsets of X. Then $X \in C(B)$ and $\alpha(X)$ is the usual convergence of subsets of X (restricted to B). In general, not every morphism from $(B, \alpha(X))$ onto $\mathbf{2}$ is generated by a point $x \in X$. Indeed, if $\operatorname{card} X = \omega_1$ and B consists of all finite and all co-finite subsets, then the set of all infinite elements of B forms a free ultrafilter and the induced homomorphism of B onto $\mathbf{2}$ is sequentially continuous.

5.3. Lemma. *Let (B, α) be a nondiscrete convergence Boolean algebra and let φ be a Boolean homomorphism of B onto $\mathbf{2}$. The following are equivalent.*

(i) *φ fails to be a sequentially continuous homomorphism from (B, α) onto $\mathbf{2}$;*

(ii) *There exists a sequence $\langle y_n \rangle$ of elements of B such that $y_n \to_\alpha 0$ and $\varphi(y) = 1$ for infinitely many $n \in \mathbb{N}$.*

P r o o f. (i) implies (ii). Since α is nondiscrete, there exists a one-to-one sequence $\langle z_n \rangle$ of elements of B and $z \in B$ such that $z_n \to_\alpha z$ and $\varphi(z_n) \not\to \varphi(z)$. Then also $z_n^c \to_\alpha z^c$, $z_n \wedge z^c \to_\alpha 0$, $z_n^c \wedge z \to_\alpha 0$. Put $y_n = z_n \wedge z^c$ if $\varphi(z) = 0$ and $y_n = z_n^c \wedge z$ otherwise, $n \in \mathbb{N}$. Clearly, then (ii) holds true. Since (ii) always implies (i), the proof is complete. $\square$

5.4. Proposition. *Let B be a Boolean algebra and let $S \in C(B)$. Let φ be a homomorphism of B onto $\mathbf{2}$ and let $\mathscr{F} = \{b \in B;\ \ \varphi(b) = 1\}$ be the ultrafilter induced by φ. The following are equivalent.*

(i) *$\varphi \in \operatorname{hom}((B, \alpha(S)), \mathbf{2})$;*

(ii) *For each sequence $\langle x_n \rangle$ of elements of $\mathscr{F}$, there exists $\chi \in S$ such that $\chi(x_n) = 1$ for infinitely many $n \in \mathbb{N}$.*

P r o o f. (i) implies (ii). Let $\langle x_n \rangle$ be a sequence of elements of $\mathscr{F}$. Assume (i) and, contrariwise, suppose that for each $\chi \in S$ we have $\chi(x_n) = 0$ for infinitely many $n \in \mathbb{N}$. Put $y_n = \bigwedge\limits_{k=1}^{n} x_k$. Then, for each $\chi \in S$, necessarily $\chi(y_n) = 0$ for all but finitely many $n \in \mathbb{N}$. Hence $y_n \to_{\alpha(S)} 0$. Since all y_n belong to $\mathscr{F}$, we have a contradiction with the sequential continuity of φ.

(ii) implies (i). Assume (ii) and, contrariwise, suppose that φ fails to be sequentially continuous. According to 5.3 Lemma, there exists a sequence $\langle y_n \rangle$ of elements of B such that $y_n \to_{\alpha(S)} 0$ and $\varphi(y_n) = 1$ for infinitely many $n \in \mathbb{N}$. Let $\langle x_n \rangle$ be a subsequence of $\langle y_n \rangle$ such that $\varphi(x_n) = 1$ for all $n \in \mathbb{N}$. Since $x_n \to_{\alpha(S)} 0$, i.e. $\chi(x_m) \to \chi(0)$ for all $\chi \in S$, we have a contradiction with (ii). $\square$

5.5. Remark. It is easy to see that condition (ii) in 5.4. Proposition can be replaced by the following one:

(iii) For each sequence $\langle x_n \rangle$ of element of $\mathscr{F}$, there exists $\chi \in S$ such that $\chi(x_n) = 1$ for all $n \in \mathbb{N}$.

Indeed, (ii) implies (iii). Assume (ii). Let $\langle x_n \rangle$ be a sequence of elements of $\mathscr{F}$. Put $y_n = \bigwedge\limits_{k=1}^{n} x_k$. Then $\langle y_n \rangle$ is a nonincreasing sequence of elements of $\mathscr{F}$ and $y_n \leqslant x_k$ whenever $n \geqslant k$. According to (ii), there exists $\chi \in S$ such that $\chi(y_n) = 1$ for all but finitely many $n \in \mathbb{N}$. Clearly, $\chi(y_n) = \chi(x_n)$ for all $n \in \mathbb{N}$ and hence (iii) holds true. Since (iii) implies (ii), the assertion follows.

Let B be a Boolean algebra. Consider the following relation on $C(B)$: $S \sim S'$ whenever $\alpha(S) = \alpha(S')$. A straightforward proof of the next proposition is omitted.

5.6. Proposition.

(i) $\sim$ *is an equivalence relation;*

(ii) *For each* $S \in C(B)$, $\hom((B, \alpha(S)), \mathbf{2})$ *is the largest element of the equivalence class* $[S]$ *containing* S.

5.7. Remark. Let B be a field of subsets of $X \neq \emptyset$. If each sequentially continuous homomorphism of B into $\mathbf{2}$ induces a fixed ultrafilter on B, then B is said to be s-perfect. The importance of s-perfect fields of subsets is given by the fact that if B is s-perfect, then each sequentially continuous homomorphism φ from B into a field B' of subsets of X' is induced by a mapping f of X' into X, i.e. $\varphi(F) = \{y \in X';\ f(y) \in F\}$ for all $F \in B$. Further, s-perfectness is preserved by the products and the formation of generated σ-field, and yields an interpretation of sequentially continuous homomorphisms as random variables (cf. [5], [7]).

6. Embedding of convergence Boolean algebras

In this section we generalize the notion of a sequential envelope of J. Novák (see [19], [20]) to $\mathbf{2}$-generated convergence Boolean algebras. Note that the construction can be further generalized to convergence rings carrying the initial convergence with respect to a given complete ring ([6]).

6.1. Definition. Let (B, α) be a $\mathbf{2}$-generated convergence Boolean algebra. If (B, α) is a convergence subalgebra of a $\mathbf{2}$-generated convergence Boolean algebra $(\overline{B}, \overline{\alpha})$ and each $\varphi \in \hom((B, \alpha), \mathbf{2})$ can be extended to $\overline{\varphi} \in \hom((\overline{B}, \overline{\alpha}), \mathbf{2})$, then (B, α) is said to be $\mathbf{2}$-*embedded* in $(\overline{B}, \overline{\alpha})$. If B is sequentially closed in each $\mathbf{2}$-generated convergence Boolean algebra in which it is $\mathbf{2}$-embedded, then (B, α) is said to be *absolutely sequentially closed* (with respect to $\hom((B, \alpha)\mathbf{2})$.

Denote by $\mathscr{A}\mathscr{B}(\mathbf{2})$ the subcategory of $\mathscr{B}(\mathbf{2})$ consisting of all absolutely sequentially closed convergence Boolean algebras. The next two propositions are a straightforward categorical bookkeeping and their proofs are omitted.

6.2. Proposition. *The category $\mathscr{B}(\mathbf{2})$ is closed with respect to products and subobjects.*

6.3. Proposition. *A convergence Boolean algebra is $\mathbf{2}$-generated iff it is isomorphic to a subobject of a power $\mathbf{2}^T$, $T \neq \emptyset$.*

6.4. Proposition. *$\mathscr{A}\mathscr{B}(\mathbf{2})$ is an epireflective subcategory of $\mathscr{B}(\mathbf{2})$.*

Proof. Let (B,α) be a $\mathbf{2}$-generated convergence Boolean algebra. First, we shall construct a $\mathbf{2}$-generated convergence Boolean albebra $(\overline{B},\overline{\alpha})$ such that:
(e$_1$) $(\overline{B},\overline{\alpha})$ is absolutely sequentially closed;
(e$_2$) (B,α) is a $\mathbf{2}$-embedded subalgebra of $(\overline{B},\overline{\alpha})$;
(e$_3$) $\overline{B}$ is the smallest sequentially closed subset in $(\overline{B},\overline{\alpha})$ containing B.
Second, we shall prove that $(\overline{B},\overline{\alpha})$ is the desired epireflection: for each sequentially continuous Boolean homomorphism from $(\overline{B},\overline{\alpha})$ into an absolutely sequentially closed convergence Boolean algebra (B',α') there exists a unique sequentially continuous Boolean homomorphism $\overline{\varphi}$ from $(\overline{B},\overline{\alpha})$ into (B',α') such that the restriction of $\overline{\varphi}$ to B is equal to φ (in symbols $\overline{\varphi} \restriction B = \varphi$); then, since B is topologically dense in $(\overline{B},\overline{\alpha})$, the embedding of (B,α) into $(\overline{B},\overline{\alpha})$ is an epimorphism.

1. Consider the evaluation mapping ev of (B,α) into $\mathbf{2}^{\hom((B,\alpha),\mathbf{2})}$ defined by $ev(x) = (\varphi(x);\ \varphi \in \hom((B,\alpha),\mathbf{2}),\ x \in B)$. Clearly, it is an isomorphism into; to avoid complicated notation, we identify (B,α) with its image under ev and denote by $(\overline{B},\overline{\alpha})$ the smallest sequentially closed subspace of $\mathbf{2}^{\hom((B\alpha),\mathbf{2})}$ containing $ev(B)$. According to 6.2 Proposition and 6.3 Proposition, $(\overline{B},\overline{\alpha})$ is $\mathbf{2}$-generated. From the construction of $(\overline{B},\overline{\alpha})$ it follows directly that (e$_2$) and (e$_3$) are satisfied. Now, let $(\overline{B},\overline{\alpha})$ be $\mathbf{2}$-embedded in a $\mathbf{2}$-generated convergence Boolean algebra $(\overline{\overline{B}},\overline{\overline{\alpha}})$ and let $\langle x_n\rangle$ be a sequence in $(\overline{B},\overline{\alpha})$ converging in $(\overline{\overline{B}},\overline{\overline{\alpha}})$ to some element $x \in \overline{\overline{B}}$. Since for each morphism $\overline{\overline{\varphi}}$ from $(\overline{\overline{B}},\overline{\overline{\alpha}})$ into $\mathbf{2}$ its restriction $\overline{\overline{\varphi}} \restriction \overline{\overline{B}}$ is a morphism from $(\overline{B},\overline{\alpha})$ into $\mathbf{2}$ and $\overline{\overline{\varphi}} \restriction B$ belongs to $\hom((B,\alpha),\mathbf{2})$, the sequence $\langle x_n\rangle$ converges in $\mathbf{2}^{\hom((B,\alpha),\mathbf{2})}$. Thus $x \in \overline{B}$ and $(\overline{B},\overline{\alpha})$ is sequentially closed in $(\overline{\overline{B}},\overline{\overline{\alpha}})$. Consequently $(\overline{B},\overline{\alpha})$ is absolutely sequentially closed and (e$_1$) is satisfied, too.

2. Let φ be a mapping from (B,α) into an absolutely sequentially closed $\mathbf{2}$-generated convergence Boolean algebra (B',α'). Identify (B',α') with its isomorphic image under the evaluation map into $\mathbf{2}^{\hom((B',\alpha'),\mathbf{2})}$. Since (B',α') is $\mathbf{2}$-embedded in $\mathbf{2}^{\hom((B',\alpha'),\mathbf{2})}$, it is a sequentially closed subalgebra. Denote by $\mathbf{2}_t$ the t-th factor and π_t the t-th projection of $\mathbf{2}^{\hom((B',\alpha'),\mathbf{2})}$, $t \in T$. Since each $\pi_t \circ \varphi$ is

a morphism from (B,α) into $\mathbf{2}_t$ and (B,α) is $\mathbf{2}$-embedded in $(\overline{B},\overline{\alpha})$, for each $t\in T$, there exists a morphism $\overline{\pi_t\circ\varphi}$ from $(\overline{B},\overline{\alpha})$ into $\mathbf{2}^t$ such that $\overline{\pi_t\circ\varphi}\restriction B=\pi_t\circ\varphi$. But $\mathbf{2}^{\hom((B',\alpha'),\mathbf{2})}$ is the categorical product (see Section 5) and hence there exists a uniquely determined morphism $\overline{\varphi}$ from $(\overline{B},\overline{\alpha})$ into $\mathbf{2}^{\hom((B',\alpha'),\mathbf{2})}$ such that $\pi_t\circ\overline{\varphi}=\overline{\pi_t\circ\varphi}$. Clearly $\overline{\varphi}\restriction B=\varphi$ and $\overline{\varphi}(\overline{B})=B'$. This completes the proof. $\square$

6.5. Proposition. *Let (B,α) be a $\mathbf{2}$-generated convergence Boolean algebra. The following are equivalent:*

(i) *(B,α) is absolutely sequentially closed;*

(ii) *In (B,α) the following implication holds true: if a sequence $\langle x_n\rangle$ of elements of B does not converge under α, then there exists $\varphi\in\hom((B,\alpha),\mathbf{2})$ such that the sequence $\langle\varphi(x_n)\rangle$ does not converge in $\mathbf{2}$.*

Proof. (i) implies (ii). Assume (i) and let $\langle x_n\rangle$ be a sequence in (B,α) which does not converge. Contrariwise, suppose that for each $\varphi\in\hom((B,\alpha),\mathbf{2})$ the sequence $\langle\varphi(x_n)\rangle$ converges in $\mathbf{2}$. Since (B,α) is $\mathbf{2}$-generated, the evaluation map ev is an isomorphism of (B,α) onto a $\mathbf{2}$-embedded subalgebra of $\mathbf{2}^{\hom((B,\alpha),\mathbf{2})}$. The sequence $\langle ev(x_n)\rangle$ converges in $\mathbf{2}^{\hom((B,\alpha),\mathbf{2})}$ and it follows from (i) that the limit belongs to $ev(B)$. This is a contradiction.

(ii) implies (i). Assume (ii). Contrariwise, suppose that (i) does not hold. Then (B,α) can be embedded in a $\mathbf{2}$-generated convergence Boolean algebra $(\overline{B},\overline{\alpha})$ such that there exists a sequence $\langle x_n\rangle$ of elements of B converging in $(\overline{B},\overline{\alpha})$ to a point $x\in\overline{B}\setminus B$. This is a contradiction. Indeed, according to (ii) there exists $\varphi\in\hom((B,\alpha),\mathbf{2})$ such that the sequence $\langle\varphi(x_n)\rangle$ does not converge in $\mathbf{2}$ and hence φ cannot be continuously extended over $(\overline{B},\overline{\alpha})$. $\square$

6.6. Proposition. *The category $\mathscr{A}\mathscr{B}(\mathbf{2})$ is closed with respect to products and sequentially closed subobjects.*

Proof. Both assertions easily follow from 6.5 Proposition. We leave out the details. $\square$

6.7. Proposition. *A $\mathbf{2}$-generated convergence Boolean algebra is absolutely sequentially closed iff it is isomorphic to a sequentially closed subobject of a power $\mathbf{2}^T$, $T\neq\emptyset$.*

Proof. The assertion follows from 6.4 Proposition, 6.5 Proposition and 6.6 Proposition. $\square$

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