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ON RUSAKOV'S n-ARY rs-GROUPS

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Abstract. Properties of n-ary groups connected with the affine geometry are considered. Some conditions for an n-ary rs-group to be derived from a binary group are given. Necessary and sufficient conditions for an n-ary group $\langle\theta, b\rangle$-derived from an additive group of a field to be an rs-group are obtained. The existence of non-commutative n-ary rs-groups which are not derived from any group of arity $m < n$ for every $n \geqslant 3$, $r > 2$ is proved.

Keywords: n-ary group, symmetry

MSC 2000: 20N15, 51A25, 51D15

1. Introduction

If the standard (affine) geometry has a fixed point O, then any point P of this geometry is uniquely determined by the vector $\vec{p} = \overrightarrow{OP}$, and conversely, the vector $\vec{p} = \overrightarrow{OP}$ uniquely determines the point P. Moreover, any interval $\overline{PQ}$ may be interpreted as the vector $\vec{q} - \vec{p}$ or as the vector $\vec{p} - \vec{q}$. In the latter case,

$$\overline{AB} = \overline{CD} \quad \text{if and only if} \quad \vec{a} - \vec{b} + \vec{d} = \vec{c},$$

or in other words

$$\overline{AB} = \overline{CD} \quad \text{if and only if} \quad f(a, b, d) = c,$$

where any vector $\vec{v}$ is treated as an element v of a commutative group $(G, +)$. The operation f has the form $f(x, y, z) = x - y + z$. Groups (also non-commutative) with a ternary operation defined in this way were considered by J. Certaine (cf. [3]) as a special case of *ternary heaps* first described by H. Prüfer (cf. [11]). Ternary heaps have interesting applications to projective geometry (cf. [1]), affine geometry (cf. [2]), theory of nets (webs), theory of knots and even to the differential geometry.

Moreover, all affine geometries may be treated as geometries defined by some ternary relations (cf. for example [15]). Such geometries may be defined also by some n-ary ($n > 3$) relations (cf. [16]). The class of affine geometries defined by n-ary groups, which are a natural generalization of the notion of groups, was introduced by S. A. Rusakov (cf. [13], [14]) and in detail described by J. I. Kulachgenko (cf. [10]).

Note that n-ary structures are interesting also for their applications to problems of modern mathematical physics (cf. for example [17], [18]).

2. Preliminaries

Traditionally in the theory of n-ary groups we use the following abbreviated notation: the sequence $x_i, \ldots, x_j$ is denoted by x_i^j (for $j < i$ this symbol is empty). If $x_{i+1} = \ldots = x_{i+k} = x$, then instead of x_{i+1}^{i+k} we write $\overset{(k)}{x}$. Obviously $\overset{(0)}{x}$ is the empty symbol. In this notation the formula

$$f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_{i+k}, x_{i+k+1}, \ldots, x_n),$$

where $x_{i+1} = \ldots = x_{i+k} = x$, will be written as $f(x_1^i, \overset{(k)}{x}, x_{i+k+1}^n)$.

By an n-ary group (G, f) we mean (cf. [4]) a non-empty set G together with one n-ary operation $f\colon\ G^n \to G$, which for all $i = 1, 2, \ldots, n$ satisfies the following two conditions:

1° *the associative law*:

$$f(f(x_1^n), x_{n+1}^{2n-1}) = f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}),$$

2° for all $a_1, a_2, \ldots, a_n, b \in G$ there exits a unique $x_i \in G$ such that

$$f(a_1^{i-1}, x_i, a_{i+1}^n) = b.$$

Such an n-ary group may be considered also (see for example [5], [6] or [8]) as an algebra (G, f, g) with one associative n-ary operation f and one unary operation g satisfying some identities. In particular, an n-ary group may be treated as an algebra $(G, f, {}^{[-2]})$ with one associative n-ary operation f and one unary operation ${}^{[-2]}\colon\ x \mapsto x^{[-2]}$ in which the identities

$$(1) \qquad f(x^{[-2]}, \overset{(n-2)}{x}, f(\overset{(n-1)}{x}, y)) = f(f(y, \overset{(n-1)}{x}), \overset{(n-2)}{x}, x^{[-2]}) = y,$$

hold (cf. [12]).

In the affine geometry defined on an n-ary group (G,f) (for details see [13]) four points $a,b,c,d\in G$ define a parallelogram if and only if

$$f(f(a,b^{[-2]},\overset{(n-2)}{b}),\overset{(n-2)}{b},c)=d.$$

Two points a and c are called *symmetric* if and only if there exists a uniquely determined point $x\in G$ such that

$$f(f(a,x^{[-2]},\overset{(n-2)}{x}),\overset{(n-2)}{x},c)=x.$$

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For $n\geqslant 3$ the above definitions can be modified. Indeed, since (1) and the associativity of f give

$$f(f(x^{[-2]},\overset{(n-1)}{x}),\overset{(n-2)}{x},y)=f(y,\overset{(n-2)}{x},f(\overset{(n-1)}{x},x^{[-2]}))=y,$$

the results obtained in [8] and [5] imply

$$f(x^{[-2]},\overset{(n-1)}{x})=f(\overset{(n-1)}{x},x^{[-2]})=\overline{x},$$

where $\overline{x}$ denotes *the skew element to* x (cf. [4], [5] or [8]). In general $\overline{x}\neq x$, but in the so-called *idempotent* n-ary groups, i.e. in n-ary groups (G,f) in which $f(\overset{(n)}{x})=x$ for all $x\in G$, we have $\overline{x}=x$.

Thus for $n\geqslant 3$ the above two definitions can be formulated in the following equivalent form:

Definition 1. Elements a, b, c, d of an n-ary group (G,f), where $n\geqslant 3$, define a parallelogram if and only if

$$f(a,\overline{b},\overset{(n-3)}{b},c)=d.$$

Definition 2. Two elements a and c of an n-ary group (G,f) are symmetric if and only if there exists one and only one $x\in G$ such that

$$f(a,\overline{x},\overset{(n-3)}{x},c)=x. \tag{2}$$

Thus for symmetric elements a and c there exist a uniquely determined $x\in G$ and a symmetry S_x such that $S_x(a)=c$. Since the definition of an n-ary group implies that in the equation (2) the element c is uniquely determined by a and x, then using the same method as in [5] and [8] one can prove that for $n \geqslant 3$ the symmetry S_x has the form

$$S_x(a) = f(x, \overline{a}, \overset{(n-3)}{a}, x).$$

An n-ary group (G, f) in which for any pair (a, c) of elements of G there exists a uniquely determined chain $x_1, x_2, \ldots, x_{r+1}$ $(r \geqslant 2)$ of elements of G such that $x_1 = a$, $x_{r+1} = c$ and $x_{i+2} = S_{x_{i+1}}(x_i)$ for $i = 1, 2, \ldots, r-1$ is called *an n-ary rs-group* (cf. [13]). The chain $a, x_2, x_3, \ldots, x_r, c$ is called *the rs-chain.*

The set $\mathbb{R}$ of reals with the operation $f(x_1^n) = x_1 + \ldots + x_n + 1$ is an example of an n-ary group in which any two elements are symmetric. The set $\mathbb{Z}$ of all integers with the same operation is an example of an n-ary group in which only some pairs of elements are symmetric. It is easy to see that $a, c \in Z$ are symmetric iff $a \equiv c \pmod 2$. Also it is not difficult to verify that in a ternary group (Z_3, f) with the operation $f(x, y, z) = x + y + z + 1 \pmod 3$ any two elements are symmetric. Any $2s$-chain has the form a, $2a + 2c$, c. Any $4s$-chain is defined by a, c, $2a + 2c$, a, c. Moreover, one can prove that in this group any pair (a, c) determines the rs-chain $x_{k+1} = a + k(2ja + jc) \pmod 3$, where $r \equiv j \pmod 3$ and $j = 1, 2$. For $r \equiv 0 \pmod 3$ such rs-chains do not exist.

Let (G, f) be an n-ary rs-group. Then for every $a, c \in G$ there exists a uniquely determined chain $x_1, x_2, \ldots, x_{r+1} \in G$ such that $x_1 = a$, $x_{r+1} = c$ and

$$(3) \qquad \begin{cases} x_2 = f(a, \overline{x}_2, \overset{(n-3)}{x_2}, x_3), \\ x_3 = f(x_2, \overline{x}_3, \overset{(n-3)}{x_3}, x_4), \\ x_4 = f(x_3, \overline{x}_4, \overset{(n-3)}{x_4}, x_5), \\ \quad \cdots\cdots\cdots\cdots\cdots\cdots \\ x_{r-1} = f(x_{r-2}, \overline{x}_{r-1}, \overset{(n-3)}{x_{r-1}}, x_r), \\ x_r = f(x_{r-1}, \overline{x}_r, \overset{(n-3)}{x_r}, c). \end{cases}$$

Replacing x_2 in the second identity by the first, we obtain

$$x_3 = f(f(a, \overline{x}_2, \overset{(n-3)}{x_2}, x_3), \overline{x}_3, \overset{(n-3)}{x_3}, x_4),$$

which by the associativity of f implies

$$x_3 = f(a, \overline{x}_2, \overset{(n-3)}{x_2}, f(x_3, \overline{x}_3, \overset{(n-3)}{x_3}, x_4)).$$

This by the so-called *Dörnte's identity*

$$f(\overset{(i-2)}{x},\overline{x},\overset{(n-i)}{x},y)=y,$$

which for $i=2,3,\ldots,n$ holds in any n-ary group (for details see [4], [5], [6] or [8]), gives

$$x_3=f(a,\overline{x}_2,\overset{(n-3)}{x_2},x_4).$$

In a similar way, replacing x_3 in the third identity by the above formula, we get

$$x_4=f(a,\overline{x}_2,\overset{(n-3)}{x_2},x_5).$$

This together with the fourth identity gives

$$x_5=f(a,\overline{x}_2,\overset{(n-3)}{x_2},x_6).$$

Continuing the above procedure we obtain

$$x_k=f(a,\overline{x}_2,\overset{(n-3)}{x_2},x_{k+1})$$

for $k=2,3,\ldots,r$. Thus

$$x_r=f(a,\overline{x}_2,\overset{(n-3)}{x_2},c)$$

and

$$x_{r-1}=f(a,\overline{x}_2,\overset{(n-3)}{x_2},x_r)=f(a,\overline{x}_2,\overset{(n-3)}{x_2},f(a,\overline{x}_2,\overset{(n-3)}{x_2},c)).$$

Analogously we obtain

$$x_k=f_{(r-k+1)}(\underbrace{\underbrace{a,\overline{x}_2,\overset{(n-3)}{x_2}},\ldots,\underbrace{a,\overline{x}_2,\overset{(n-3)}{x_2}}}_{r-k+1\text{-times}},c) \tag{4}$$

for all $k=2,3,\ldots,r$, where $f_{(r-k+1)}$ means that the n-ary operation f is used $(r-k+1)$-times.

However, by Hosszú's theorem every n-ary group (G,f) is $\langle\theta,b\rangle$-derived from some binary group, i.e. every n-ary group has the form

$$f(x_1^n)=x_1\cdot\theta x_2\cdot\theta^2x_3\cdot\ldots\cdot\theta^{n-1}x_n\cdot b,$$

where $(G,\cdot)$ is a binary group (called *the creating group*), $b\in G$, θ is an automorphism of $(G,\cdot)$ such that $\theta b=b$ and $\theta^{n-1}x\cdot b=b\cdot x$ for all $x\in G$.

Since this form may be written also as

$$f(x_1^n) = x_1 \cdot \theta x_2 \cdot \theta^2 x_3 \cdot \ldots \cdot \theta^{n-2} x_{n-1} \cdot b \cdot x_n, \tag{5}$$

and $f(z, \overline{z}, \overset{(n-2)}{z}) = z$ for every $z \in G$, hence $\theta\overline{z} \cdot \theta^2 z \cdot \ldots \cdot \theta^{n-2} z \cdot b \cdot z$ is the identity of $(G, \cdot)$ and $\theta\overline{z} \cdot \theta^2 z \cdot \ldots \cdot \theta^{n-2} z \cdot b$ is the inverse of z in $(G, \cdot)$. Thus the identity (2) may be written in $(G, \cdot)$ as $a \cdot x^{-1} \cdot c = x$. Similarly (4), for $k = 2$ and $a = e$, where e is the identity of $(G, \cdot)$, may be written as $x_2 = (x_2^{-1})^{r-1} \cdot c$, which proves that in a creating group of an n-ary rs-group $(G, \cdot)$ for any element c there exists only one x such that $c = x^r$. In particular, there exists only one x such that $e = x^r$.

Thus the following theorem is true.

Theorem 1. *An n-ary rs-group is $\langle \theta, b \rangle$-derived from a binary group without elements of order r.*

Corollary 1. *An n-ary $2s$-group is $\langle \theta, b \rangle$-derived from a binary group without elements of order two.*

Corollary 2. *A finite n-ary $2s$-group is $\langle \theta, b \rangle$-derived from a binary group with an odd exponent.*

Corollary 3. *There are no n-ary $2s$-groups of an even order.*

Since all binary retracts of a given n-ary group (G, f) $\langle \theta, b \rangle$-derived from a group $(G, \cdot)$, i.e. groups $(G, *_a)$ where $x *_a y = f(x, \overset{(n-2)}{a}, y)$ and $a \in G$ is fixed are isomorphic to $(G, \cdot)$ (see [9]), we have

Corollary 4. *If a binary retract of an n-ary group (G, f) has an element of order $d|r$, then (G, f) is not a rs-group.*

Corollary 5. *Commutative idempotent n-ary rs-groups do not exist for $r \equiv 0 \pmod{(n-1)}$.*

P r o o f. Let (G, f) be a commutative idempotent n-ary group. By Hosszú's theorem there exists a group $(G, \cdot)$ with the identity e such that (5) holds. Since the operation f is commutative and $x = \overline{x}$ for every $x \in G$, we have $f(y, e, \ldots, e) = f(e, y, e, \ldots, e) = y$ for all $y \in G$ (cf. [4] or [8]). This together with (5) gives $\theta y \cdot b = y$. Hence $b = e$ and $\theta y = y$. Therefore the operation f has the form $f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$. Thus for every x in $(G, \cdot)$ we have $x^{n-1} = e$, which proves (by Corollary 4) that for $r \equiv 0 \pmod{(n-1)}$ this n-ary group is not an rs-group. □

From the last part of the proof of our Theorem 1 it follows that in a given rs-group (G,f) all rs-chains depend only on the creating group $(G,\cdot)$. Moreover, by (2), the rs-chain determined by $a,c\in G$ has the form $a, x_2, yx_2, \dots, y^{r-1}x_2, c$, where $y = x_2a^{-1}$. Thus all n-ary groups (also non-isomorphic), which are $\langle\theta,b\rangle$-derived from $(G,\cdot)$, have the same rs-chains. If $t = exp(G,\cdot)$, then all such n-ary groups are also $(r+t)s$-groups.

Now let (K,f) be an n-ary group $\langle\theta,b\rangle$-derived from an additive group of a field K. For this group the system of equations (3) can be written as

$$(6)\qquad \begin{cases} x_2 = a - x_2 + x_3,\\ x_3 = x_2 - x_3 + x_4,\\ x_4 = x_3 - x_4 + x_5,\\ \dots\dots\dots\dots\dots\dots\\ x_{r-1} = x_{r-2} - x_{r-1} + x_r,\\ x_r = x_{r-1} - x_r + c, \end{cases}$$

i.e. as a system of $r-1$ linear equations with $r-1$ unknowns $x_2, x_3, \dots, x_r$.

The determinant of this system may be written in the form

$$D_{r-1} = \begin{vmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0\\ -1 & 2 & -1 & 0 & \dots & 0 & 0\\ 0 & -1 & 2 & -1 & \dots & 0 & 0\\ 0 & 0 & -1 & 2 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & \dots & 2 & -1\\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{vmatrix}.$$

Applying the Laplace formula to the last column (or to the last row) we obtain

$$D_{r-1} = 2D_{r-2} - D_{r-3}$$

for $r > 3$. Since $D_1 = 2$ and $D_2 = 3$, we obtain by induction $D_{r-1} = r$. This proves that the system of equations (6) has a unique solution if and only if we have $r\cdot x \neq 0$ for any non-zero element x of the group $(K,+)$. Thus we obtain the following theorem.

Theorem 2. *An n-ary group $\langle\theta,b\rangle$-derived from an additive group of a field K is an rs-group if and only if K has the characteristic* 0 *or p, where $p \nmid r$.*

The assumption that $(K,+)$ is an additive group of a field is essential. Indeed, as was mentioned above, an n-ary group (Z,f) derived from the additive group of an integral domain $(Z,+,\cdot)$ is not an rs-group.

As is well known (cf. for example [4] and [9]) some n-ary groups may be reduced to groups of arity $m<n$, i.e. for some n-ary groups (G,f) there exists an m-ary group (G,g), where $n=s(m-1)+1$, $s>1$, such that the following identity holds:

$$f(x_1^n)=g(g(\dots g(g(x_1^m),x_{m+1}^{2m-1}),\dots),x_{(s-1)(m-1)+2}^{s(m-1)+1}).$$

Such n-ary groups are not interesting since all results on such groups immediately follow from the results on the corresponding m-ary groups. Obviously the affine geometry defined by such n-ary groups is identical with the geometry defined by the corresponding m-ary groups.

Theorem 3. *For every $n\geqslant 3$ and $r\geqslant 2$ there exists a non-commutative n-ary rs-group which is not derived from any group of arity $m<n$.*

Proof. Let C be the set of complex numbers and let ω be the primitive $(n-1)$-st root of unity. Then $G=C^3$ with the operation

$$(x_1,x_2,x_3)\bullet(y_1,y_2,y_3)=(x_1+y_1,x_2+y_2+x_1y_3,x_3+y_3)$$

is a group and $\theta(x_1,x_2,x_3)=(\omega x_1,\omega^2x_2,\omega x_3)$ is its automorphism.

It is not difficult to verify that an n-ary group $\langle\omega,1\rangle$-derived from $(G,\bullet)$ is an rs-group for every $r\geqslant 2$. Since it is isomorphic to the n-ary group from the proof of Theorem 3 in [7], it is not reduced to any group of arity $m<n$. □

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