Czechoslovak Mathematical Journal

Giuseppe Da Prato; Luciano Tubaro

Some results about dissipativity of Kolmogorov operators

Czechoslovak Mathematical Journal, Vol. 51 (2001), No. 4, 685–699

Persistent URL: http://dml.cz/dmlcz/127680

Czechoslovak Mathematical Journal, 51 (126) (2001), 685–699

SOME RESULTS ABOUT DISSIPATIVITY OF KOLMOGOROV OPERATORS

Giuseppe Da Prato, Pisa, and Luciano Tubaro, Trento

(Received May 10, 2001)

Dedicated to Ivo Vrkoč on the occasion of his 70th birthday

Abstract. Given a Hilbert space H with a Borel probability measure ν, we prove the m-dissipativity in $L^1(H,\nu)$ of a Kolmogorov operator K that is a perturbation, not necessarily of gradient type, of an Ornstein-Uhlenbeck operator.

Keywords: Kolmogorov equations, invatiant measures, m-dissipativity

MSC 2000: 47B25, 81S20, 37L40, 35K57, 70H15

1. Introduction

Let H be a real separable Hilbert space and ν a Borel probability measure on H. We are given a linear operator $A\colon \mathfrak{D}(A) \subset H \to H$ that we suppose to be the infinitesimal generator of a strongly continuous semigroup e^{tA} on H, a linear operator $B \in L(H)$ and a nonlinear Borel mapping $F\colon H \to H$. We set $C = BB^*$.

Let us introduce the function space $\mathcal{E}_A(H)$ as the linear span of all real and imaginary parts of functions on H of the form $x \to \mathrm{e}^{i\langle h,x\rangle}$, where $h \in \mathfrak{D}(A^*)$ and A^* is the adjoint of A. It is well known that this space is dense in $L^p(H,\nu)$ for any $p \geqslant 1$.

We are concerned with the linear operator

$$\mathring{K}\varphi = L\varphi + \langle F(x), C^{1/2}D\varphi\rangle, \quad \varphi \in \mathcal{E}_A(H),$$

where L is the Ornstein-Uhlenbeck operator

$$L\varphi = \frac{1}{2}\operatorname{Tr}[CD^2\varphi] + \langle x, A^*D\varphi\rangle, \quad \varphi \in \mathcal{E}_A(H).$$

In a sense this paper is a continuation of the paper [4]. The main difference is that here we do not assume that ν is absolutely continuous with respect to a Gaussian measure.

Let us state our assumptions. Concerning A and C we will assume

Hypothesis 1.

(i) *There exists* $\omega \geqslant 0$ *such that*

$$\langle Ax, x\rangle \leqslant -\omega|x|^2, \quad x \in \mathfrak{D}(A), \tag{1.1}$$

(ii) $\operatorname{Tr} Q < +\infty$, *where*

$$Qx = \int_0^{+\infty} \mathrm{e}^{tA} C \mathrm{e}^{tA^*} x \, \mathrm{d}t, \quad x \in H,$$

and concerning F we will assume

Hypothesis 2.

(i) *There exists a constant* $c > 0$ *such that*

$$\int_H (|x|^2 + |F(x)|^2)\, \nu(\mathrm{d}x) \leqslant c, \tag{1.2}$$

(ii) *for any* $\varphi \in \mathcal{E}_A(H)$ *we suppose*

$$\int_H \mathring{K}\varphi \, \mathrm{d}\nu = 0, \tag{1.3}$$

(iii) $\mathring{K}$ *is dissipative in* $L^p(H,\nu)$, $\forall p \geqslant 1$,

(iv) *there exist a sequence* $(F_n) \subset \mathcal{C}_b^2(H;H)$ *such that* $F_n(x) \to F(x)$ ν-a.e. *and a constant* $c_1 > 0$ *such that*

$$\int_H |F_n(x)|^2\, \nu(\mathrm{d}x) \leqslant c_1.$$

It is well known that the operator $\mathring{K}$ is closable in $L^p(H,\nu)$ since it is dissipative in it, as stated in (iii). Let us denote its closure in $L^p(H,\nu)$ by K_p. Our goal is to show that K_p is dissipative on $L^p(H,\nu)$, $p \geqslant 1$ and that ν is an infinitesimally invariant measure for K_p. The main result of the paper is Theorem 3.6, where we show that K_1 is m-dissipative on $L^1(H,\nu)$.

In this section we assume that Hypothesis 1 holds. Let $\mathcal{C}_b(H)$ be the space of uniformly continuous and bounded functions $\varphi\colon H \to \mathbb{R}$. Moreover, for any integer $k \geqslant 0$ let us define $\mathcal{C}_{b,k}(H)$ as the space of all $\varphi\colon H \to \mathbb{R}$ such that the mapping

$$H \to \mathbb{R}, \quad x \to \frac{\varphi(x)}{1+|x|^k}$$

belongs to $\mathcal{C}_b(H)$. We set

$$\|\varphi\|_{b,k} = \sup_{x\in H} \frac{|\varphi(x)|}{1+|x|^k}.$$

Obviously one has $\mathcal{C}_{b,k}(H) \subset \mathcal{C}_{b,k+1}(H)$.

Denoting by N_{Q_t} the Gaussian measure with mean 0 and covariance operator

$$Q_t x = \int_0^t \mathrm{e}^{sA} C \mathrm{e}^{sA^*} x \,\mathrm{d}s, \quad x \in H,$$

let $\mathcal{R}_t$ be the Ornstein-Uhlenbeck "semigroup"

$$\mathcal{R}_t\varphi(x) = \int_H \varphi(\mathrm{e}^{tA}x + y)\, N_{Q_t}(\mathrm{d}y), \quad \varphi \in \mathcal{C}_{b,k}(H),\ \ k \geqslant 0. \tag{2.1}$$

It is not difficult to show that for all $t \geqslant 0$ and for all $k \geqslant 0$, $\mathcal{R}_t$ maps $\mathcal{C}_{b,k}(H)$ into itself, see [1]. Following [1][1], we define the infinitesimal generator L of $\mathcal{R}_t$ through its resolvent

$$(\lambda - L)^{-1}\varphi(x) = \int_0^{+\infty} \mathrm{e}^{-\lambda t}\mathcal{R}_t\varphi(x)\,\mathrm{d}t, \quad x \in H,\ \ \lambda > 0. \tag{2.2}$$

Thus for any $\lambda > 0$, $(\lambda - L)^{-1}$ maps $\mathcal{C}_{b,k}(H)$ into itself. Since the image of the resolvent is independent of λ we can set, see [1],

$$\mathfrak{D}\big(L, \mathcal{C}_{b,k}(H)\big) = (\lambda - L)^{-1}\big(\mathcal{C}_{b,k}(H)\big), \quad k \geqslant 0.$$

As noticed in [1], $\mathcal{R}_t$ is not a strongly continuous semigroup on $\mathcal{C}_{b,k}(H)$ for any $k \geqslant 0$. Let us denote by $\mathfrak{X}_k$ the maximal closed subspace of $\mathcal{C}_{b,k}(H)$ where $\mathcal{R}_t$ is strongly continuous, that is

$$\mathfrak{X}_k = \big\{\varphi \in \mathcal{C}_{b,k}(H)\colon \lim_{t\to 0} \mathcal{R}_t\varphi = \varphi \ \text{ in } \ \mathcal{C}_{b,k}(H)\big\}.$$

[1] See also [2] and [7].

To characterize $\mathfrak{X}_k$ it is useful to introduce an auxiliary family $(\mathcal{G}_t)$ of linear operators on $\mathcal{C}_{b,k}(H)$:

$$\mathcal{G}_t\varphi(x) = \int_H \varphi(x+y)\, N_{Q_t}(\mathrm{d}y), \quad \varphi \in \mathcal{C}_{b,k}(H).$$

They are related to $(\mathcal{R}_t)$ by

$$\mathcal{R}_t\varphi(x) = (\mathcal{G}_t\varphi)(\mathrm{e}^{tA}x), \quad \varphi \in \mathcal{C}_{b,k}(H).$$

Proposition 2.1. *Let $\varphi \in \mathcal{C}_{b,k}(H)$. Then the following statements are equivalent:*

(i) $\lim\limits_{t\to 0} \mathcal{R}_t\varphi = \varphi$ *in* $\mathcal{C}_{b,k}(H)$.

(ii) $\lim\limits_{t\to 0} \varphi(\mathrm{e}^{tA}\cdot) = \varphi$ *in* $\mathcal{C}_{b,k}(H)$.

P r o o f. We first show that for any $\varphi \in \mathcal{C}_{b,k}(H)$ we have

$$\lim_{t\to 0} \mathcal{G}_t\varphi = \varphi \ \text{ in } \ \mathcal{C}_{b,k}(H). \tag{2.3}$$

Let $\varphi \in \mathcal{C}_{b,k}(H)$ and set $\psi(x) = \varphi(x)/(1+|x|^k)$. We may assume that $\psi \in \mathcal{C}_b^1(H)$.Then we have

$$\mathcal{G}_t\varphi(x) - \varphi(x) = \int_H \big[(1+|x+y|^k)\psi(x+y) - (1+|x|^k)\psi(x)\big]\, N_{Q_t}(\mathrm{d}y).$$

Consequently,

$$\begin{aligned}\frac{|\mathcal{G}_t\varphi(x)-\varphi(x)|}{1+|x|^k} \leqslant &\int_H \left|\frac{1+|x+y|^k}{1+|x|^k} - 1\right| \|\psi\|_0\, N_{Q_t}(\mathrm{d}y)\\ &+ \|\psi\|_1 \int_H |y|\, N_{Q_t}(\mathrm{d}y).\end{aligned}$$

Therefore (2.3) follows.

We now prove that (i) $\Rightarrow$ (ii). In fact we have

$$|\varphi(\mathrm{e}^{tA}x) - \varphi(x)| \leqslant |\varphi(\mathrm{e}^{tA}x) - \mathcal{G}_t\varphi(\mathrm{e}^{tA}x)| + |\mathcal{R}_t\varphi(x) - \varphi(x)|.$$

So (i) $\Rightarrow$ (ii). The converse can be proved similarly. □

Remark 2.2. Since for any $\varphi_h = \mathrm{e}^{i\langle h,x\rangle}$ we have

$$\mathcal{R}_t\varphi_h = \mathrm{e}^{-1/2\langle Q_t h,h\rangle}\varphi_{\mathrm{e}^{tA^*}h},$$

it follows that $\mathcal{R}_t$ maps $\mathcal{E}_A(H)$ into itself. Properties of the space $\mathcal{E}_A(H)$ follow also from the results in [3] and [10].

Corollary 2.3.
(i) $\mathcal{E}_A(H) \subset \mathfrak{D}\big(L, \mathcal{C}_{b,k}(H)\big)$ *for all* $k \geqslant 1$,
(ii) $\mathcal{E}_A(H) \subset \mathfrak{X}_1$, *and consequently,*

$$L\varphi = \frac{1}{2}\operatorname{Tr}[CD^2\varphi] + \langle x, A^* D\varphi\rangle, \quad \varphi \in \mathcal{E}_A(H). \tag{2.4}$$

(iii) *If* $\varphi \in \mathcal{E}_A(H)$, *then we have* $L\varphi \in \mathfrak{X}_2$.

P r o o f. Taking in account the definition of $\mathcal{E}_A(H)$, we need only to prove the corollary in the case of the functions $\sin[\langle x, h\rangle]$ and $\cos[\langle x, h\rangle]$. Moreover, since the proof for the cosine function is just the same as for the sine, we are reduced to make the proof only for $\varphi_h(x) = \sin[\langle x, h\rangle]$. Hence we have

$$L\varphi_h = -\frac{1}{2}\sin[\langle x, h\rangle]\,|h|^2 + \cos[\langle x, h\rangle]\langle x, A^* h\rangle, \tag{2.5}$$

which yields (i). Let us prove (ii). We have

$$\frac{\varphi_h(\mathrm{e}^{tA}x) - \varphi_h(x)}{1+|x|} = \frac{\sin[\langle \mathrm{e}^{tA}x, h\rangle] - \sin[\langle x, h\rangle]}{1+|x|}.$$

Consequently,

$$\frac{|\varphi_h(\mathrm{e}^{tA}x) - \varphi_h(x)|}{1+|x|} \leqslant \frac{|\langle x, \mathrm{e}^{tA^*}h\rangle - \langle x, h\rangle|}{1+|x|} \leqslant \frac{|x|}{1+|x|}|\mathrm{e}^{tA^*}h - h|.$$

This implies

$$\lim_{t\to 0}\sup_{x\in H}\frac{|\varphi_h(\mathrm{e}^{tA}x) - \varphi_h(x)|}{1+|x|} = 0,$$

and so $\varphi_h \in \mathfrak{X}_1$ by Proposition 2.1.

Finally, (iii) follows by a similar argument, when taking into account (2.5). □

2.1. Approximations by exponential functions.

This subsection is devoted to the study of a kind of approximations of functions of $\mathcal{C}_b(H)$, and moreover of $\mathfrak{D}\big(L, \mathcal{C}_b(H)\big)$, by functions of $\mathcal{E}_A(H)$, which we need in the sequel.

These approximations are not possible by using simple sequences, but k-sequences, $k \in \mathbb{N}$, that is sequences $\{\varphi_n\} = \{\varphi_{n_1,\ldots,n_k}\}$ depending on k indices. We say that $\{\varphi_n\}$ is convergent to φ if

$$\lim_{n\to\infty}\varphi_n := \lim_{n_1\to\infty}\ldots\lim_{n_k\to\infty}\varphi_{n_1,\ldots,n_k}(x) = \varphi(x), \quad x \in H.$$

Lemma 2.4. *For any $\varphi \in \mathcal{C}_b(H)$ there exists a 3-sequence $(\varphi_n) = (\varphi_{n_1,n_2,n_3}) \subset \mathcal{E}_A(H)$ such that*

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x), \quad \forall x \in H, \tag{2.6}$$

and

$$\|\varphi_n\|_{b,0} \leqslant \|\varphi\|_{b,0}. \tag{2.7}$$

Proof. Let $\varphi \in \mathcal{C}_b(H)$ and let $(P_{n_1})_{n_1\in\mathbb{N}}$ be a sequence of finite dimensional projection operators on H strongly convergent to the identity. Then for each $n_1 \in \mathbb{N}$ there exists[2] a sequence $(\varphi_{n_1,n_2})_{n_2\in\mathbb{N}} \subset \mathcal{E}(H)$ such that

$$\lim_{n_2\to\infty} \varphi_{n_1,n_2}(x) = \varphi(P_{n_1}x), \quad x \in H,$$

and

$$|\varphi_{n_1,n_2}(x)| \leqslant |\varphi(P_{n_1}x)| \leqslant \|\varphi\|_{b,0}.$$

Now set

$$\varphi_{n_1,n_2,n_3}(x) = \varphi_{n_1,n_2}\big(n_3(n_3 - A^*)^{-1}x\big), \quad x \in H.$$

Then $\varphi_n = \varphi_{n_1,n_2,n_3} \subset \mathcal{E}_A(H)$, $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$, $\forall x \in H$, and

$$|\varphi_{n_1,n_2,n_3}(x)| = \big|\varphi_{n_1,n_2}\big(n_3(n_3 - A^*)^{-1}x\big)\big| \leqslant \|\varphi_{n_1,n_2}\|_{b,0} \leqslant \|\varphi\|_{b,0}.$$

Therefore the 3-sequence (φ_{n_1,n_2,n_3}) fulfils (2.6) and (2.7) as required. □

Now we want to show that any function $\varphi \in \mathfrak{D}\big(L, \mathcal{C}_b(H)\big)$ can be approximated pointwise in the graph norm by functions in $\mathcal{E}_A(H)$ with uniformly bounded norm.

Proposition 2.5. *For any $\varphi \in \mathfrak{D}\big(L, \mathcal{C}_b(H)\big)$ there exist a 4-sequence $(\varphi_n) \subset \mathcal{E}_A(H)$ and $C(\varphi) > 0$ such that for all $x \in H$ we have*

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x), \quad \lim_{n\to\infty} L\varphi_n(x) = L\varphi(x), \tag{2.8}$$

and

$$\sup_{x\in H}\left\{\frac{|\varphi_n(x)| + |L\varphi_n(x)|}{1 + |x|^2}\right\} \leqslant C(\varphi). \tag{2.9}$$

[2] For example, first we can approximate φ_{n_1} by functions with support contained in squares larger and larger, for each of which we can use multiple Fourier series; then we can apply a diagonal procedure.

Proof. Set $f = \varphi - L\varphi$ and let $(f_n) = (f_{n_1,n_2,n_3}) \subset \mathcal{E}_A(H)$ be a 3-sequence fulfilling (2.6) and (2.7) (with φ replaced by f). Setting $\varphi_n = (1-L)^{-1} f_n$, we have

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x), \qquad \forall x \in H,$$
$$\lim_{n\to\infty} L\varphi_n(x) = L\varphi(x), \quad \forall x \in H,$$

and

$$\|\varphi_n\|_{b,0} \leqslant \|f\|_{b,0} \leqslant (2\|\varphi\|_{b,0} + \|L\varphi\|_{b,0}),$$
$$\|L\varphi_n\|_{b,0} \leqslant (\|\varphi\|_{b,0} + \|L\varphi\|_{b,0}).$$

Next, set for any $M, N \in \mathbb{N}$

$$\varphi_{n,M,N}(x) = \frac{1}{M} \sum_{h=1}^{N} \sum_{k=1}^{M} \mathrm{e}^{-(h+k/M)} \mathcal{R}_{h+k/M} f_n(x),$$

so that

$$|\varphi_{n,M,N}(x)| \leqslant \|f\|_0$$

and

$$L\varphi_{n,M,N}(x) = \frac{1}{M} \sum_{h=1}^{N} \sum_{k=1}^{M} \mathrm{e}^{-(h+k/M)} \mathcal{R}_{h+k/M} L f_n(x).$$

Now, by Corollary 2.3 it follows that $Lf_n \in \mathfrak{X}_2$ so that $\mathcal{R}_t f_n$ is continuous on t in the topology of $\mathcal{C}_{b,2}(H)$. Therefore for any $n = (n_1, n_2, n_3)$ we have

$$\lim_{M,N\to\infty} \sup_{x\in H} \frac{1}{1+|x|^2} \left| \int_0^{+\infty} \mathrm{e}^{-t} \mathcal{R}_t L f_n(x) \,\mathrm{d}t - \frac{1}{M} \sum_{h=1}^{N} \sum_{k=1}^{M} \mathrm{e}^{-(h+\frac{k}{M})} \mathcal{R}_{h+\frac{k}{M}} L f_n(x) \right| = 0.$$

Therefore for any $\varepsilon \in (0,1]$ there exist $M_\varepsilon, N_\varepsilon$ such that

$$|L\varphi_n(x) - L\varphi_{n,M_\varepsilon,N_\varepsilon}(x)| \leqslant \varepsilon(1+|x|^2), \quad x \in H.$$

Consequently,

$$\lim_{\varepsilon\downarrow 0} L\varphi_{n,M_\varepsilon,N_\varepsilon}(x) = L\varphi_n(x),$$

and

$$|L\varphi_{n,M_\varepsilon,N_\varepsilon}(x)| \leqslant |L\varphi_n(x)| + \varepsilon(1+|x|^2) \leqslant 2\|f\|_0 + |x|^2.$$

Now the conclusion follows easily. □

In a similar way we prove

Proposition 2.6. *For any $\varphi \in \mathfrak{D}\big(L, \mathcal{C}_{b,1}(H)\big)$ there exist a 4-sequence $(\varphi_n) = (\varphi_{n_1,n_2,n_3,n_4}) \subset \mathcal{E}_A(H)$ and $C(1,\varphi) > 0$ such that for all $x \in H$ we have*

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x), \quad \lim_{n\to\infty} D\varphi_n(x) = D\varphi(x), \quad \lim_{n\to\infty} L\varphi_n(x) = L\varphi(x) \tag{2.10}$$

and

$$\sup_{x\in H} \left\{ \frac{|\varphi_n(x)| + |D\varphi_n(x)| + |L\varphi_n(x)|}{1+|x|^2} \right\} \leqslant C(1,\varphi). \tag{2.11}$$

Proposition 2.7. *Assume in addition that C^{-1} is bounded. Then for any $\varphi \in \mathfrak{D}\big(L, \mathcal{C}_b(H)\big)$ there exist a 4-sequence $(\varphi_n) \subset \mathcal{E}_A(H)$ and $C(\varphi) > 0$ such that for all $x \in H$ we have*

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x), \quad \lim_{n\to\infty} D\varphi_n(x) = D\varphi(x), \quad \lim_{n\to\infty} L\varphi_n(x) = L\varphi(x) \tag{2.12}$$

and

$$\sup_{x\in H} \left\{ \frac{|\varphi_n(x)| + |D\varphi_n(x)| + |L\varphi_n(x)|}{1+|x|^2} \right\} \leqslant C(\varphi). \tag{2.13}$$

P r o o f. Let $\varphi \in \mathfrak{D}\big(L, \mathcal{C}_b(H)\big)$. By Proposition 2.5 we know that there exist a 4-sequence $(\varphi_n) \subset \mathcal{E}_A(H)$ and $C(\varphi) > 0$ such that (2.8) and (2.9) hold. Moreover, if C^{-1} is bounded then $\mathcal{R}_t$ is strong Feller and, for any $k = 0, 1, \ldots$, there exists $c_k > 0$ such that

$$\frac{|D\mathcal{R}_t f(x)|}{1+|x|^k} \leqslant c_k t^{-1/2} \|f\|_{b,k}, \quad k = 0, 1, \ldots$$

By the Laplace transform we obtain

$$\frac{|D(\lambda - L)^{-1} f(x)|}{1+|x|^k} \leqslant \sqrt{\pi/\lambda}\, c_k \|f\|_{b,k}, \quad k = 0, 1, \ldots$$

Now set $\varphi_n - L\varphi_n = f_n$. Then we have

$$\frac{|D\varphi_n(x)|}{1+|x|^2} \leqslant \sqrt{\pi}\, c_2 \|f\|_{b,2}.$$

Since

$$\|f\|_{b,2} \leqslant \|\varphi_n\|_{b,2} + \|L\varphi_n\|_{b,2},$$

the conclusion follows from (2.8) and (2.9). □

3. m-dissipativity of K_1 on $L^1(H,\nu)$

Proposition 3.1. *For all $\varphi \in \mathcal{E}_A(H)$ we have*

$$\int_H \mathring{K}\varphi\,\varphi\,\mathrm{d}\nu = -\frac{1}{2}\int_H |C^{1/2}D\varphi|^2\,\mathrm{d}\nu. \tag{3.1}$$

Proof. In fact, if $\varphi \in \mathcal{E}_A(H)$ then we have $\varphi^2 \in \mathcal{E}_A(H)$ and

$$\mathring{K}(\varphi^2) = 2\varphi\mathring{K}\varphi + |C^{1/2}D\varphi|^2.$$

Then integrating both sides with respect to ν and using (1.3), the conclusion follows. □

Since, by definition, $\mathcal{E}_A(H)$ is a core for K_2, (3.1) implies that the linear operator

$$D_C\colon \mathcal{E}_A(H) \subset \mathfrak{D}(K_2) \to L^2(H,\nu;H), \quad \varphi \to C^{1/2}D\varphi,$$

is continuous and consequently can be extended to all $\mathfrak{D}(K_2)$. We denote again by D_C the extension. By Proposition 3.1 we get

Corollary 3.2. *For all $\varphi \in \mathfrak{D}(K_2)$ we have*

$$\int_H K_2\varphi\,\varphi\,\mathrm{d}\nu = -\frac{1}{2}\int_H |D_C\varphi|^2\,\mathrm{d}\nu. \tag{3.2}$$

Let us now consider the problem

$$\mathrm{d}X_n = \big(AX_n + C^{1/2}F_n(X_n)\big)\,\mathrm{d}t + B\,\mathrm{d}W_t, \quad X_n(0) = x. \tag{3.3}$$

Since $F_n \in \mathcal{C}_b^2(H)$, problem (3.3) has a unique mild solution that we will denote by $X_n(t,x)$, see e.g. [5]. Moreover, $X_n(t,x)$ is differentiable with respect to x and, setting $\eta_n^h(t,x) = DX_n(t,x)h$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta_n^h(t,x) = A\eta_n^h(t,x) + C^{1/2}DF_n\big(X_n(t,x)\big)\eta_n^h(t,x), \quad \eta_n^h(t,x) = h. \tag{3.4}$$

Now we consider the equation

$$\lambda\varphi_n - L\varphi_n - \langle F_n(x), C^{1/2}D\varphi_n\rangle = f. \tag{3.5}$$

Lemma 3.3. *Let $f \in \mathcal{C}_b^2(H)$ and $\lambda > 0$. Then equation (3.5) has a unique solution $\varphi_n \in \mathfrak{D}\big(L, \mathcal{C}_b^1(H)\big) \cap \mathcal{C}_b^1(H)$ given by*

$$\varphi_n(x) = \int_0^{+\infty} \mathrm{e}^{-\lambda t} \mathbb{E}\big[f\big(X_n(t,x)\big)\big]\,\mathrm{d}t, \quad x \in H. \tag{3.6}$$

P r o o f. Let $f \in \mathcal{C}_b^1(H)$ and

$$\varphi_n(x) = \int_0^{+\infty} \mathrm{e}^{-\lambda t} \mathbb{E}\big[f\big(X_n(t,x)\big)\big]\,\mathrm{d}t.$$

Clearly $\varphi_n \in \mathcal{C}_b^1(H)$ since $|\eta_n^h(t,x)| \leqslant \mathrm{e}^{t\|C^{1/2}F_n\|_0}$, and we have

$$\langle D\varphi_n(x), h\rangle = \int_0^{+\infty} \mathrm{e}^{-\lambda t} \mathbb{E}\big[\big\langle Df\big(X_n(t,x)\big), \eta_n^h(t,x)\big\rangle\big]\,\mathrm{d}t.$$

Let us prove that $\varphi_n \in \mathfrak{D}\big(L, \mathcal{C}_b(H)\big)$. Set

$$Z(t,x) = \mathrm{e}^{tA}x + \int_0^t \mathrm{e}^{(t-s)A} B\,\mathrm{d}W(s),$$

so that

$$X_n(t,x) = Z(t,x) + \int_0^t \mathrm{e}^{(t-s)A} C^{1/2} F_n\big(X_n(s,x)\big)\,\mathrm{d}s, \quad t \geqslant 0.$$

For any $h > 0$ we have

$$\begin{aligned}
&\frac{1}{h}\big(\mathcal{R}_h\varphi_n(x) - \varphi_n(x)\big)\\
&\quad = \frac{1}{h}\mathbb{E}\big[\varphi_n\big(Z(h,x)\big) - \varphi_n(x)\big]\\
&\quad = \frac{1}{h}\mathbb{E}\bigg[\varphi_n\bigg(X_n(h,x) - \int_0^h \mathrm{e}^{(h-s)A} C^{1/2} F_n\big(X_n(s,x)\big)\,\mathrm{d}s\bigg) - \varphi_n(x)\bigg]\\
&\quad = \frac{1}{h}\mathbb{E}\big[\varphi_n\big(X_n(h,x)\big) - \varphi_n(x)\big]\\
&\qquad - \frac{1}{h}\mathbb{E}\bigg[\bigg\langle D\varphi_n\big(X_n(h,x)\big), \int_0^h \mathrm{e}^{(h-s)A} C^{1/2} F_n\big(X_n(s,x)\big)\,\mathrm{d}s\bigg\rangle\bigg] + o(h).
\end{aligned}$$

As $h \to 0$ we find that $\varphi_n \in \mathfrak{D}\big(L, \mathcal{C}_b(H)\big)$ and

$$L\varphi_n = \lambda\varphi_n - \langle C^{1/2}F_n, D\varphi_n\rangle.$$

If $f \in \mathcal{C}_b^2(H)$ we prove, by proceeding in the same way as above, that $\varphi_n \in \mathfrak{D}\big(L, \mathcal{C}_b^1(H)\big)$. □

Lemma 3.4. *Let $\varphi \in \mathfrak{D}(L, \mathcal{C}_b^1(H))$. Then $\varphi \in \mathfrak{D}(K_1)$ and*

$$K_1\varphi = L\varphi + \langle F, C^{1/2}D\varphi\rangle. \tag{3.7}$$

P r o o f. By Proposition 2.6 there exist a 4-sequence $(\varphi_k) \subset \mathcal{E}_A(H)$ and $M > 0$ such that

$$\varphi_k(x) \to \varphi(x), \;\; D\varphi_k(x) \to D\varphi(x), \;\; L\varphi_k(x) \to L\varphi(x), \quad x \in H,$$

and

$$|\varphi_k(x)| + |D\varphi_k(x)| \leqslant M, \quad |L\varphi_k(x)| \leqslant M(1 + |x|^2), \quad x \in H.$$

It follows that

$$K_1\varphi_k(x) \to L\varphi(x) + \langle F(x), C^{1/2}D\varphi(x)\rangle, \quad x \in H,$$

and

$$|K_1\varphi_k(x)| \leqslant M(1 + |x|^2) + M|F(x)|\|C^{1/2}\|.$$

Now the conclusion follows from (1.2) and the dominated convergence theorem. □

Lemma 3.5. *Let $f \in \mathcal{C}_b^1(H)$ and $\lambda > 0$. Then the solution φ_n to* (3.5) *belongs to $\mathfrak{D}(K_1)$ and we have*

$$K_1\varphi_n = L\varphi_n + \langle F_n(x), C^{1/2}D\varphi_n\rangle. \tag{3.8}$$

P r o o f. By Lemma 3.3 we have $\varphi_n \in \mathfrak{D}(L, \mathcal{C}_b^1(H))$ and by Lemma 3.4 we know that $\varphi_n \in \mathfrak{D}(K_1)$. Thus the conclusion follows. □

Theorem 3.6. *K_1 is m-dissipative on $L^1(H, \nu)$.*

P r o o f. Let $f \in \mathcal{C}_b^2(H)$ and let φ_n be the solution to (3.5):

$$\lambda\varphi_n - L\varphi_n - \langle F_n(x), C^{1/2}D\varphi_n\rangle = f.$$

Then Lemma 3.5 yields $\varphi_n \in \mathfrak{D}(K_1)$ and

$$K_1\varphi_n = L\varphi_n + \langle F(x), C^{1/2}D\varphi_n\rangle.$$

Therefore

$$\lambda\varphi_n - K_1\varphi_n = f + \langle F_n(x) - F(x), C^{1/2}D\varphi_n\rangle. \tag{3.9}$$

Taking into account 3.2 we obtain

$$\lambda\int_H \varphi_n^2\,\mathrm{d}\nu + \frac{1}{2}\int_H |C^{1/2}D\varphi_n|^2\,\mathrm{d}\nu = \int_H f\varphi_n\,\mathrm{d}\nu + \int_H \varphi_n\langle F_n - F, C^{1/2}D\varphi_n\rangle\,\mathrm{d}\nu.$$

Moreover, in view of 3.6, $\|\varphi_n\|_0 \leqslant \lambda^{-1}\|f\|_0$,

$$\begin{aligned}\lambda\int_H \varphi_n^2\,\mathrm{d}\nu &+ \frac{1}{2}\int_H |C^{1/2}D\varphi_n|^2\,\mathrm{d}\nu \\ &\leqslant \frac{1}{\lambda}\|f\|_0^2 + \frac{1}{\lambda}\|f\|_0\int_H |F_n - F|\,|C^{1/2}D\varphi_n|\,\mathrm{d}\nu \\ &\leqslant \frac{1}{\lambda}\|f\|_0^2 + \frac{1}{4}\int_H |C^{1/2}D\varphi_n|^2\,\mathrm{d}\nu + \frac{4}{\lambda^2}\|f\|_0^2\int_H |F_n - F|^2\,\mathrm{d}\nu.\end{aligned}$$

Consequently, there exists a constant M_1 independent of n and such that

$$\int_H |C^{1/2}D\varphi_n|^2\,\mathrm{d}\nu \leqslant M_1.$$

It follows that

$$\lim_{n\to\infty}\langle F_n(x) - F(x), C^{1/2}D\varphi_n\rangle = 0$$

in $L^1(H,\nu)$ and so

$$\lim_{n\to\infty}\lambda\varphi_n - K_1\varphi_n = f.$$

Therefore the closure of the image of $\lambda - \overline{K}$ contains $\mathcal{C}_b^2(H)$ and so it is dense in $L^1(H,\nu)$. Now the conclusion follows from a classical result due to Lumer and Phillips. □

4. Gradient systems

We assume here, in addition to Hypotheses 1 and 2, that A is self-adjoint and commuting with C. In this case the Ornstein-Uhlenbeck semigroup $\mathcal{R}_t$ is symmetric. We will denote by μ the Gaussian measure N_Q of mean 0 and covariance operator Q. Moreover, we recall that for any $\varphi \in \mathfrak{D}(L)$ and any $\psi \in W_C^{1,2}(H,\mu)$ the following identity holds:

$$\int_H L\varphi\psi\,\mathrm{d}\mu = -\frac{1}{2}\int_H \langle C^{1/2}D\varphi, C^{1/2}D\psi\rangle\,\mathrm{d}\mu. \tag{4.1}$$

We are given a probability measure ν of the form

$$\nu(\mathrm{d}x) = \varrho(x)\,\mu(\mathrm{d}x),$$

where ϱ fulfils

Hypothesis 3.
(i) $\varrho \geqslant 0, \quad \varrho \in L^1(H,\mu) \quad |x|^2\varrho \in L^1(H,\mu)$
(ii) $\sqrt{\varrho} \in W_C^{1,2}(H,\mu)$ *and* $\varrho \in W_C^{1,2}(H,\mu)$.

We notice that under Hypothesis 3 we have

$$C^{1/2}D\log\varrho \in L^2(H,\nu;H). \tag{4.2}$$

In fact,

$$\int_H |C^{1/2}D\log\varrho|^2\,\mathrm{d}\nu = \int_H \frac{|C^{1/2}D\varrho|^2}{\varrho}\,\mathrm{d}\mu = 4\int_H |C^{1/2}D\sqrt{\varrho}|^2\,\mathrm{d}\mu.$$

We set

$$U = -\frac{1}{2}\log\varrho, \quad F = -C^{1/2}DU = \frac{1}{2}C^{1/2}D\log\varrho.$$

We are going to show that Hypothesis 2 is fulfilled.
(i) follows from (4.2) and the assumption $|x|^2\varrho \in L^1(H,\mu)$.
(ii) is established by the following Proposition:

Proposition 4.1. *Under Hypothesis* 3 *we have*

$$\int_H \mathring{K}\varphi\,\mathrm{d}\nu = 0, \quad \varphi \in \mathcal{E}_A(H). \tag{4.3}$$

P r o o f. We have

$$\int_H \mathring{K}\varphi\,\mathrm{d}\nu = \int_H L\varphi\varrho\,\mathrm{d}\mu - \int_H \langle C^{1/2}DU, C^{1/2}D\varphi\rangle\,\mathrm{d}\nu.$$

However, in view of (4.1) we have

$$\int_H L\varphi\,\varrho\,\mathrm{d}\mu = -\frac{1}{2}\int_H \langle C^{1/2}D\varphi, C^{1/2}D\varrho\rangle\,\mathrm{d}\mu = \int_H \langle C^{1/2}DU, C^{1/2}D\varphi\rangle\,\mathrm{d}\nu,$$

and the conclusion follows. □

(iii) follows from the following Proposition:

Proposition 4.2. *Under Hypothesis* 3, $\mathring{K}$ *is symmetric. Moreover,*

$$\int_H (\mathring{K}\varphi)\psi \, \mathrm{d}\nu = -\frac{1}{2}\int_H \langle C^{1/2}D\varphi, C^{1/2}D\psi\rangle \, \mathrm{d}\nu, \quad \varphi,\psi \in \mathcal{E}_A(H). \tag{4.4}$$

P r o o f. For all $\varphi,\psi \in \mathcal{E}_A(H)$ we have

$$\int_H (\mathring{K}\varphi)\psi \, \mathrm{d}\nu = \int_H L\varphi(\psi\varrho)\, \mathrm{d}\mu - \int_H \langle DU, C^{1/2}D\varphi\rangle\psi \, \mathrm{d}\nu.$$

However, $\psi\varrho \in W^{1,2}_C(H,\nu)$, and so by (4.1) we have

$$\int_H (L\varphi)\psi\varrho \, \mathrm{d}\mu = -\frac{1}{2}\int_H \langle C^{1/2}D\varphi, C^{1/2}D\psi\rangle \, \mathrm{d}\mu - \frac{1}{2}\int_H \langle C^{1/2}D\varphi, D\log\varrho\rangle \, \mathrm{d}\mu,$$

and the conclusion follows easily □

Remark 4.3. By Proposition 4.2 it follows that K_2 is dissipative in $L^2(H,\mu)$. By [6] it follows that K_p is dissipative in $L^p(H,\nu)$ for all $p \geqslant 1$.

Finally, to prove (iv) we need suitable approximations for F. To this end it is convenient to introduce the Sobolev space $W^{1,2}(H,\nu)$, in which D_h, the partial derivative in the direction e_h, is closable.

We need the following integration-by-parts formula.

Proposition 4.4. *Assume that Hypotheses* 1, 2 *and* 3 *hold. Let* $\varphi,\psi \in \mathcal{E}_A(H)$, $h \in \mathbb{N}$. *Then we have*

$$\int_H (D_h\varphi)\psi \, \mathrm{d}\nu = -\int_H \varphi(D_h\psi)\, \mathrm{d}\nu + \frac{1}{\lambda_h}\int_H x_h\varphi\psi \, \mathrm{d}\nu + 2\int_H \varphi\psi(D_hU)\, \mathrm{d}\nu. \tag{4.5}$$

P r o o f. In fact we have

$$\int_H (D_h\varphi)\psi \, \mathrm{d}\nu = \int_H (D_h\varphi)\psi\varrho \, \mathrm{d}\mu.$$

Since $\psi\varrho \in W^{1,2}(H,\mu)$ we have

$$\int_H (D_h\varphi)\psi \, \mathrm{d}\nu - \int_H (D_h\varphi)\psi\varrho \, \mathrm{d}\mu - \int_H \varphi D_h(\psi\varrho)\, \mathrm{d}\mu + \frac{1}{\lambda_h}\int_H x_h\varphi\psi \, \mathrm{d}\nu,$$

and the conclusion follows. □

Proposition 4.4 implies, by standard arguments, that the mapping

$$D\colon \mathcal{E}_A(H) \subset L^2(H,\nu) \to L^2(H,\nu;H)$$

is closable; we denote its closure again by D.

Let us define the space $W^{1,2}(H,\nu)$ as the subspace of $L^2(H,\nu)$ consisting of all functions $\varphi \in \mathfrak{D}(D)$ such that

$$\int_H |D\varphi|^2 \,\mathrm{d}\nu < +\infty.$$

Now, since $U \in W^{1,2}(H,\nu)$, there is a sequence $(U_N) \subset \mathcal{E}_A(H)$ such that

$$U_N \to U \text{ in } L^2(H,\nu), \quad DU_N \to DU \text{ in } L^2(H,\nu;H).$$

Hence we can apply the previous results.

References

[1] *S. Cerrai*: A Hille-Yosida theorem for weakly continuous semigroups. Semigroup Forum *49* (1994), 349–367.

[2] *S. Cerrai*: Weakly continuous semigroups in the space of functions with polynomial growth. Dynam. Systems Appl. *4* (1995), 351–371.

[3] *S. Cerrai and F. Gozzi*: Strong solutions of Cauchy problems associated to weakly continuous semigroups. Differential Integral Equations (1995), 465–486.

[4] *G. Da Prato and L. Tubaro*: Self-adjointness of some infinite dimensional elliptic operators and application to stochastic quantization. Probab. Theory Relat. Fields *118* (2000), 131–145.

[5] *G. Da Prato and J. Zabczyk*: Ergodicity for Infinite Dimensional Systems. Cambridge University Press, Cambridge, 1996.

[6] *A. Eberle*: Uniqueness and Non-uniqueness of Semigroups Generated by Singular Diffusion Operators. Lecture Notes in Mathematics Vol. 1718. Springer-Verlag, Berlin, 1999.

[7] *B. Goldys and M. Kocan*: Diffusion semigroups in spaces of continuous functions with mixed topology. J. Differential Equations *173* (2001), 17–39.

[8] *V. Liskevich and M. Röckner*: Strong uniqueness for certain infinite-dimensional Dirichlet operators and applications to stochastic quantization. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) *27* (1999), 69–91.

[9] *Z. M. Ma and M. Röckner*: Introduction to the Theory of (Non-Symmetric) Dirichlet Forms. Springer-Verlag, Berlin, 1992.

[10] *E. Priola*: π-semigroups and applications. Preprint No. 9 of the Scuola Normale Superiore di Pisa. 1998.

Authors' addresses: G. Da Prato, Scuola Normale Superiore di Pisa, Piazza dei Cavalieri 7, I-56126 Pisa, Italy, e-mail: daprato@sns.it; L. Tubaro, Department of Mathematics, University of Trento, via Sommarive 14, I-38050 Povo (Trento), Italy, e-mail: tubaro@science.unitn.it.