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CHARACTERIZATIONS OF TOTALLY ORDERED SETS BY THEIR VARIOUS ENDOMORPHISMS

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Dedicated to the memory of Professor Josef Novák on the occasion of his 95th birthday.

Abstract. We characterize totally ordered sets within the class of all ordered sets containing at least three-element chains using a simple relationship between their isotone transformations and the so called 2-, 3-, 4-endomorphisms which are introduced in the paper. Another characterization of totally ordered sets within the class of ordered sets of a locally finite height with at least four-element chains in terms of the regular semigroup theory is also given.

Keywords: endomorphisms, totally ordered sets—chains, isotone mappings, regular semigroups

MSC 2000: 06A05, 20M20

Determination of structures according to their endomorphisms is one of the problems of standing interest. In this note we define new types of endomorphisms of ordered sets. The motivation for this paper came from the hypergroup theory (cf. [6]), where 2-, 3-, 4-morphisms play an important role [6, 7, 9]. Here we show that a quite simple relationship between such endomorphisms and isotone selfmappings provides the total ordering of ordered sets in question. Further, totally ordered sets are characterized within the class of ordered sets of a locally finite height in terms of the classical semigroup theory (regular semigroups of endomorphisms). The authors dedicate this contribution in deep respect to the memory of Professor Josef Novák whose scientific activities included also very inspirating investigations of totally ordered continua (e.g. [12]).

Let $(X,\leqslant)$ be an ordered set, $\emptyset \neq A \subseteq X$. By $[A)_{\leqslant}$ we denote an end of an ordered set $(X,\leqslant)$ generated by the subset A. If $A=\{a\}$, then we write $[a)_{\leqslant}$ which is a principal end generated by an element a, i.e. $[a)_{\leqslant} = \{y \in X \mid a \leqslant y\}$. Dually we define principal beginnings. By an isolated point of an ordered set $(X, \leqslant)$ we mean a point $x \in X$ such that $\{y;\, y \in X,\, y \nparallel x\} = \{x\}$. A monoid of all isotone selfmappings of an ordered set $(X, \leqslant)$, i.e. of mappings $f\colon (X, \leqslant) \longrightarrow (X, \leqslant)$ such that $x, y \in X$, $x \leqslant y$ implies $f(x) \leqslant f(y)$, which are called also endomorphisms of $(X, \leqslant)$, will be denoted by $\mathrm{End}(X, \leqslant)$. The symbol R^{-} stands for the opposite relation of R, especially $f^{-}(S)$ denotes the preimage of S under a mapping f, i.e. for $f\colon X \longrightarrow Y$, $S \subseteq Y$, $f^{-}(S) = \{x \in X \mid f(x) \in S\}$, [10]. For a totally ordered set we use a synonym chain.

Lemma 1. *Let $(X, \leqslant)$ be an ordered set containing at least a three-element chain. Then for any ordered pair (x, y) of $\leqslant$-incomparable elements $x, y \in X$ there exist isotone mappings $f\colon (X, \leqslant) \longrightarrow (X, \leqslant)$, $g\colon (X, \leqslant) \longrightarrow (X, \leqslant)$ such that*
(1) *$f(x) < f(y)$ and $\{x\} = f^{-}(f(x))$,*
(2) *$g(x) < g(y)$ and $\{y\} = g^{-}(g(y))$.*

Proof. (1): Suppose $(X, \leqslant)$ contains at least a three-element chain C. Consider $C_0 \subseteq C$ such that $C_0 = \{a, b, c\}$, $a < b < c$ and $x, y \in X$ are incomparable elements (one of them can be equal to a or b or c). Now let X^{xy}, X_{xy} be subsets of X such that $X^{xy} = \{z \mid z > x \text{ or } z > y\}$ and $X_{xy} = \{z \mid z < x \text{ or } z < y\}$. Let $f(x) = b$ and $f(y) = c$, which means $f(x) < f(y)$. Furthermore, let $f(t) = a$ for any $t \in X_{xy}$ and $f(s) = c$ for any $s \in X^{xy}$. If there exists $\emptyset \neq Y \subseteq X$ such that $Y = X \setminus \{X^{xy} \cup X_{xy} \cup \{x, y\}\}$ then let $f(r) = c$ for any $r \in Y$. Any $u \in Y$ is incomparable with each $v \in X^{xy} \cup X_{xy} \cup \{x, y\}$. Now $f(u) = f(v)$ for any pair $(u, v) \in X^{xy} \times X^{xy}$, $(u, v) \in X_{xy} \times X_{xy}$, $(u, v) \in Y \times Y$, and $f(u) < f(v)$ for any pair $(u, v) \in X_{xy} \times X^{xy}$, which implies f is isotone, because $p \leqslant q$ implies $f(p) \leqslant f(q)$ for any $p, q \in X$ and $\{x\} = f^{-}(b) = f^{-}(f(x))$. Thus (1) holds.

(2): Suppose $(X, \leqslant)$, X^{xy}, X_{xy}, Y are defined as in the first part of the proof and $x, y \in X$ are incomparable elements. Let $g(x) = a$ and $g(y) = b$, which means $g(x) < g(y)$. Furthermore, let $g(t) = a$ for any $t \in X_{xy}$, $g(s) = c$ for any $s \in X^{xy}$ and $g(r) = c$ for any $r \in Y$. Now $g(u) = g(v)$ for any pair $(u, v) \in X^{xy} \times X^{xy}$, $(u, v) \in X_{xy} \times X_{xy}$, $(u, v) \in Y \times Y$ and $g(u) < g(v)$ for any pair $(u, v) \in X_{xy} \times X^{xy}$, which implies g is isotone, because $p \leqslant q$ implies $g(p) \leqslant g(q)$ for any $p, q \in X$ and similarly as above $\{y\} = f^{-}(b) = f^{-}(f(y))$. Thus (2) holds. □

The following simple example shows that the condition of the existence of at least a three-element chain in Lemma 1 can not be replaced by the supposition of the existence of at least a two-element chain.

Example. Let $(X, \leqslant)$ be an ordered set, $X = \{a, b, c\}$ where $a < b$, $c < b$ and $a \parallel c$. None of the below written isotone mappings f_i, $i = \{1, 2, \ldots, 6\}$ satisfies the condition $f(a) < f(c)$ and $\{c\} = f^{-}(f(c))$ for incomparable elements $a, c \in X$. We do not include constant mappings and automorphisms, because they clearly do not satisfy the above condition.

$$f_1 = \begin{pmatrix} a & b & c \\ a & b & b \end{pmatrix}, \quad f_2 = \begin{pmatrix} a & b & c \\ c & b & b \end{pmatrix}, \quad f_3 = \begin{pmatrix} a & b & c \\ a & b & a \end{pmatrix},$$

$$f_4 = \begin{pmatrix} a & b & c \\ c & b & c \end{pmatrix}, \quad f_5 = \begin{pmatrix} a & b & c \\ b & b & a \end{pmatrix}, \quad f_6 = \begin{pmatrix} a & b & c \\ b & b & c \end{pmatrix}.$$

Lemma 2. *Let $f\colon X_1 \longrightarrow X_2$ be a mapping of an ordered set $(X_1, \leqslant)$ into another one $(X_2, \leqslant)$. The following conditions are equivalent:*
(1) *f is an isotone mapping,*
(2) $f([x)_{\leqslant}) \subseteq [f(x))_{\leqslant}$,
(3) $[f^{-}(f(x)))_{\leqslant} \subseteq f^{-}([f(x))_{\leqslant})$ *for any $x \in X_1$.*

P r o o f. (1) $\Rightarrow$ (2): Let $x \in X_1$ be an arbitrary element and suppose $y \in f([x)_{\leqslant})$. Then there exists $z \in [x)_{\leqslant}$, which means $x \leqslant z$, such that $y = f(z)$. Since f is isotone, we have $f(x) \leqslant f(z)$, which implies $y \in [f(x))_{\leqslant}$. Thus $f([x)_{\leqslant}) \subseteq [f(x))_{\leqslant}$.

(2) $\Rightarrow$ (3): Let $x_0 \in X_1$ be an arbitrary element. Suppose $y \in [f^{-}(f(x_0)))_{\leqslant}$. Then there exists $z \in f^{-}(f(x_0))$, i.e. $f(z) = f(x_0)$, such that $z \leqslant y$, which means $y \in [z)_{\leqslant}$. Then $f(y) \in f([z)_{\leqslant}) \subseteq [f(z))_{\leqslant} = [f(x_0))_{\leqslant}$ (in virtue of (2)), hence $y \in f^{-}(f(y)) \subseteq f^{-}([f(x_0))_{\leqslant})$. Consequently $[f^{-}(f(x_0)))_{\leqslant} \subseteq f^{-}([f(x_0))_{\leqslant})$.

(3) $\Rightarrow$ (1): Suppose (3) and $x, y \in X_1$, $x \leqslant y$. Since $x \in f^{-}(f(x))$ we have $y \in [x)_{\leqslant} \subseteq [f^{-}(f(x)))_{\leqslant} \subseteq f^{-}([f(x))_{\leqslant})$, hence $f(y) \in [f(x))_{\leqslant}$. Consequently $f(x) \leqslant f(y)$, therefore (1) holds. $\square$

Now we can enunciate a theorem characterizing totally ordered sets by a condition which is motivated by the theory of morphisms of hyperstructures.

Proposition 1. *Let $(X, \leqslant)$ be an ordered set containing at least a three-element chain. Then $(X, \leqslant)$ is a totally ordered set if and only if any isotone selfmapping f of the ordered set $(X, \leqslant)$ satisfies the following condition:*

$$(*) \qquad f^{-}(f([x)_{\leqslant})) = f^{-}([f(x))_{\leqslant}) \text{ for any } x \in X.$$

P r o o f. $\Rightarrow$: Let $(X, \leqslant)$ be a totally ordered set and $f\colon (X, \leqslant) \longrightarrow (X, \leqslant)$ an isotone mapping. Suppose $y \in f^{-}([f(x))_{\leqslant})$, which means $f(y) \in [f(x))_{\leqslant}$, i.e. $f(x) \leqslant f(y)$. If $f(x) = f(y)$ then $y \in f^{-}(f(x))$ and as $f^{-}(f(x)) \subseteq f^{-}(f([x)_{\leqslant}))$ we have $y \in f^{-}(f([x)_{\leqslant}))$. If $f(x) < f(y)$ then $x \leqslant y$ (since the mapping f is isotone and $(X, \leqslant)$ is totally ordered), i.e. $y \in [x)_{\leqslant}$, which implies $f(y) \in f([x)_{\leqslant})$. This implies $y \in f^-(f([x)_\leqslant))$ and consequently $f^-([f(x))_\leqslant) \subseteq f^-(f([x)_\leqslant))$. The set inclusion $f([x)_\leqslant) \subseteq [f(x))_\leqslant$, which was established in Lemma 2, implies $f^-(f([x)_\leqslant)) \subseteq f^-([f(x))_\leqslant)$ and consequently $f^-(f([x)_\leqslant)) = f^-([f(x))_\leqslant)$.

$\Leftarrow$: Let $(X, \leqslant)$ be an ordered set containing at least a three-element chain. Let $x, y \in X$ be incomparable ($x \parallel y$) and suppose $f^-(f[x)_\leqslant) = f^-([f(x))_\leqslant)$ holds for any isotone mapping $f\colon (X, \leqslant) \longrightarrow (X, \leqslant)$. Let g_0 be a mapping from Lemma 1 (2), i.e. $g_0(x) < g_0(y)$ and $\{y\} = g_0^-(g_0(y))$. Since $g_0(x) < g_0(y)$ then $g_0(y) \in [g_0(x))_\leqslant$, which implies $y \in g_0^-([g_0(x))_\leqslant)$. Now $y \in g_0^-(g_0([x)_\leqslant))$ by the assumption $(*)$, which implies $g_0(y) \in g_0([x)_\leqslant)$. Therefore there exists $z \in [x)_\leqslant$ such that $g_0(z) = g_0(y)$. Then $z \in g_0^-(g_0(z)) = g_0^-(g_0(y)) = \{y\}$ and consequently we have $z = y$, thus $y \in [x)_\leqslant$ which means $x \leqslant y$. This is a contradiction to the assumption of incomparability of x and y. Thus $(X, \leqslant)$ is a totally ordered set. $\square$

Remark. It can be easily proved that the condition $(*)$ can be replaced by the dual one:

$$f^-(f((x]_\leqslant)) = f^-((f(x)]_\leqslant) \text{ for any } x \in X.$$

In the proof it is useful to apply such an isotone mapping that $f(x) < f(y)$ and $\{x\} = f^-(f(x))$, whose existence was stated in Lemma 1 (1).

Proposition 2. *Let $(X, \leqslant)$ be an ordered set containing at least a three-element chain. Then $(X, \leqslant)$ is a totally ordered set if and only if any isotone selfmapping f of the ordered set $(X, \leqslant)$ satisfies the following condition:*

$$(**) \qquad [f^-(f(x)))_\leqslant = f^-([f(x))_\leqslant) \text{ for any } x \in X.$$

Proof. $\Rightarrow$: Let $(X, \leqslant)$ be a totally ordered set and $f\colon X \longrightarrow X$ an isotone mapping. Suppose $y \in f^-([f(x))_\leqslant)$, which means $f(y) \in [f(x))_\leqslant$, i.e. $f(x) \leqslant f(y)$. If $f(x) = f(y)$ then $y \in f^-(f(x))$ and we have $y \in [f^-(f(x))_\leqslant)$. If $f(x) < f(y)$ then $x \leqslant y$ (since the mapping f is isotone and $(X, \leqslant)$ is totally ordered). Further, it follows from $[f^-(f(x)))_\leqslant = \{t \mid \exists u \in X\colon f(u) = f(x),\ u \leqslant t\}$ that $y \in [f^-(f(x)))_\leqslant$, consequently $f^-([f(x))_\leqslant) \subseteq [f^-(f(x)))_\leqslant$. By virtue of the set inclusion $[f^-(f(x)))_\leqslant \subseteq f^-([f(x))_\leqslant)$, which was established in Lemma 2, we get $f^-([f(x))_\leqslant) = [f^-(f(x)))_\leqslant$.

$\Leftarrow$: Let $(X, \leqslant)$ be an ordered set containing at least a three-element chain, let $x, y \in X$ be incomparable ($x \parallel y$) and suppose $[f^-(f(x)))_\leqslant = f^-([f(x))_\leqslant)$ for any isotone mapping $f\colon (X, \leqslant) \longrightarrow (X, \leqslant)$. Let f_α be a mapping from Lemma 1 (1), i.e. $f_\alpha(x) < f_\alpha(y)$ and $\{x\} = f_\alpha^-(f_\alpha(x))$. Since $f_\alpha(x) < f_\alpha(y)$ then $f_\alpha(y) \in [f_\alpha(x))_\leqslant$, which implies $y \in f_\alpha^-([f_\alpha(x))_\leqslant)$. Now $y \in [f_\alpha^-(f_\alpha(x)))_\leqslant$ by the assumption $(**)$, which implies that there exists $z \in f_\alpha^-(f_\alpha(x))$ such that $z \leqslant y$. Since $f_\alpha^-(f_\alpha(x)) =$

$\{x\}$ then $z = x$. This implies $x \leqslant y$, which is a contradiction to the assumption of incomparability of x and y. Thus $(X, \leqslant)$ is a totally ordered set. $\square$

Remark. It can be easily proved as above that the condition $(**)$ can be replaced by the dual one:

$$(f^{-}(f(x))]_{\leqslant} = f^{-}((f(x)]_{\leqslant}) \text{ for any } x \in X.$$

In the proof it is again useful to consider such an isotone mapping that $f(x) < f(y)$ and $\{y\} = f^{-}(f(y))$, whose existence was obtained in Lemma 1 (2).

Conditions $(*)$ and $(**)$ from Propositions 1 and 2 are motivated by the so called 2-, 3-, 4-homomorphisms studied in the hypergroup theory [6, 7, 9].

Definition 1. Let $(X, \leqslant)$ be an ordered set. A mapping $f\colon (X, \leqslant) \longrightarrow (X, \leqslant)$ is called:

$(\overline{*})$ a 2-endomorphism if it satisfies the condition

$$f^{-}(f([x)_{\leqslant})) = f^{-}([f(x))_{\leqslant}) \text{ for any } x \in X,$$

$(\overline{**})$ a 3-endomorphism if it satisfies the condition

$$[f^{-}(f(x)))_{\leqslant} = f^{-}([f(x))_{\leqslant}) \text{ for any } x \in X,$$

$(\overline{***})$ a 4-endomorphism if both the conditions for 2- and 3-endomorphisms are satisfied:

$$[f^{-}(f(x)))_{\leqslant} = f^{-}([f(x))_{\leqslant}) = f^{-}(f([x)_{\leqslant})) \text{ for any } x \in X.$$

The set of all k-endomorphisms $f\colon (X, \leqslant) \longrightarrow (X, \leqslant)$ where $k = 2, 3, 4$ will be denoted by $k - \mathrm{End}(X, \leqslant)$.

Lemma 3. *Let $(X, \leqslant)$ be an ordered set. Then*

$$k - \mathrm{End}(X, \leqslant) \subseteq \mathrm{End}(X, \leqslant)$$

for any $k \in \{2, 3, 4\}$.

P r o o f. Suppose $k = 2$, thus $f \in 2 - \mathrm{End}(X, \leqslant)$ is an arbitrary mapping and suppose $x, y \in X$, $x \leqslant y$ are arbitrary elements. Then $y \in [x)_{\leqslant}$, which implies $f(y) \in f([x)_{\leqslant})$ and $y \in f^{-}(f(y)) \subseteq f^{-}(f([x)_{\leqslant})) = f^{-}([f(x))_{\leqslant})$, thus $f(y) \in [f(x))_{\leqslant}$, which means $f(x) \leqslant f(y)$. Now suppose $f \in 3 - \mathrm{End}(X, \leqslant)$. Then the mapping f satisfies the condition (3) from Lemma 2, thus $f\colon (X, \leqslant) \longrightarrow (X, \leqslant)$ is an isotone mapping. Finally, $4 - \mathrm{End}(X, \leqslant) = [2 - \mathrm{End}(X, \leqslant)] \cap [3 - \mathrm{End}(X, \leqslant)] \subseteq \mathrm{End}(X, \leqslant)$. $\square$

The above stated results can be summarized in the following theorem:

Theorem 1. *Let $(X, \leqslant)$ be an ordered set containing at least a three-element chain. Then the following conditions are equivalent:*
(1) $(X, \leqslant)$ *is a totally ordered set,*
(2) $\mathrm{End}(X, \leqslant) \subseteq k - \mathrm{End}(X, \leqslant)$,
(3) $\mathrm{End}(X, \leqslant) = k - \mathrm{End}(X, \leqslant)$,
for any $k \in \{2, 3, 4\}$.

Proof. Proof follows immediately from Propositions 1, 2 and Lemma 3. $\square$

Remark. It is easy to find an example of an ordered set which is not a chain with isotone transformations which are neither 2-endomorphisms nor 3-endomorphisms. Indeed, consider $X = \{a, b, c\}$ and suppose $a < b$, $a < c$, $b \parallel c$. Define $f\colon X \longrightarrow X$, $f(a) = f(c) = a$, $f(b) = b$. Then

$$f^{-}(f([c)_{\leqslant})) = f^{-}(a) = \{a, c\} \subsetneqq X = f^{-}(X) = f^{-}([f(c))_{\leqslant}).$$

For the ordered set $(X, \geqslant)$, where $\geqslant$ denotes the order dual to $\leqslant$, we have

$$[f^{-}(f(b)))_{\geqslant} = [\{b\})_{\geqslant} = \{a, b\} \subsetneqq X = f^{-}(\{a, b\}) = f^{-}([f(b))_{\geqslant}),$$

hence $f \in \mathrm{End}(X, \leqslant) \cap \mathrm{End}(X, \geqslant)$; however, f is neither a 2-endomorphism of $(X, \leqslant)$ nor a 3-endomorphism of $(X, \geqslant)$ (see Fig. 1).

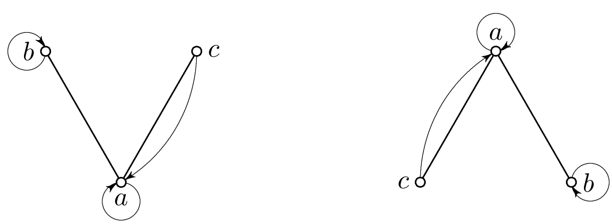

Fig. 1

Now we will consider total orders in connection with the regularity of endomorphism monoids or the corresponding ordered sets. First, we summarize the results of the second part of the paper [2] (Theorem 2.6) in the following characterization theorem. In another terminology the same result can be obtained from [1]. See also [5], Theorem 2.1. Recall that the endomorphism monoid is said to be regular if any isotone mapping $f \in \mathrm{End}(X, \leqslant)$ is a regular element of $\mathrm{End}(X, \leqslant)$, i.e. for any $f \in \mathrm{End}(X, \leqslant)$ there exists $g \in \mathrm{End}(X, \leqslant)$ such that $f \circ g \circ f = f$. See e.g. [8], p. 44, 48–55, [1, 2, 11].

Theorem 2. *Let $(X, \leqslant)$ be a nonempty ordered set which is neither a chain nor an antichain. Then the monoid* $\mathrm{End}(X, \leqslant)$ *is regular if and only if $(X, \leqslant)$ satisfies at least one of the following conditions:*

(1) *There exists a pair of antichains X_1, X_2 such that $(X, \leqslant) = X_1 \oplus X_2$ (the ordinal sum).*
(2) *There exists a two-element decomposition $\{X_1, X_2\}$ of the set X and there also exists a pair of elements $(a, b) \in X_1 \times X_2$ such that for $x, y \in X$ we have $x < y$ if and only if either $x = a$, $y \in X_2$ or $x \in X_1$ and $y = b$.*
(3) *There exists a pair of different elements $a, b \in X$ such that $X_0 = X \setminus \{a, b\}$ is a nonempty antichain and $(X, \leqslant) = \{a\} \oplus X_0 \oplus \{b\}$.*
(4) *We have $X = \{a_1, a_2, a_3, b_1, b_2, b_3\}$, where $a_1 < b_1$, $a_2 < b_1$, $a_1 < b_2$, $a_2 < b_3$, $a_3 < b_2, a_3 < b_3$ (i.e. $(X, \leqslant)$ is a six-crown).*

Definition 2. An ordered set $(X, \leqslant)$ is said to be of a locally finite height if for any pair a, b of elements such that $a < b$ every chain $K \subseteq X$ with the least element a and the greatest element b is finite.

The following auxiliary assertions are useful for the next proposition. The first lemma concerns—in fact—the existence of a retract of totally ordered sets with respect to morphisms-isotone mappings.

Lemma 4. *Let $(X, \leqslant)$, $(Y, \preceq)$ be totally ordered sets and $f\colon (X, \leqslant) \longrightarrow (Y, \preceq)$ an isotone mapping. Then there exist a subset $R \subseteq X$ and an isomorphism h of the ordered set $(f(X), \preceq)$ onto $(R, \leqslant)$ such that $(f \circ g \circ f)(x) = f(x)$ for any $x \in X$.*

P r o o f. Let $X/(f \circ f^-)$ be the decomposition of the set X canonical to the mapping $f\colon X \longrightarrow Y$. Let $R = \{r_B \mid B \in X/(f \circ f^-)\}$ be the set of all representatives of blocks $B \in X/(f \circ f^-)$, where $r_B \in B$ is a unique representative chosen arbitrarily. Define a mapping $h\colon f(X) \longrightarrow R$ as the inverse mapping for the restriction $f \restriction R$. Now, for an arbitrary $x \in X$ we have $x \in B$ for some $B \in X/(f \circ f^-)$; if r is the representative of a block B then $f(x) = f(r)$ and $(f \circ h \circ f)(x) = f(r) = f(x)$. $\square$

Lemma 5. *Let $(Y, \preceq)$, $(X, \leqslant)$ be totally ordered sets where $(Y, \preceq)$ is of a locally finite height. If $S \subseteq Y$ and h is an isotone mapping of $(S, \preceq)$ into $(X, \leqslant)$, then there exists an extension g of the mapping h that is an isotone mapping of $(Y, \preceq)$ into $(X, \leqslant)$.*

P r o o f. We define $g(y) = h(y)$ for any element $y \in S$. If $y \in Y \setminus S$ and there exists $y' \in S$ such that $y' \prec y$ we choose the greatest element y' with this property and put $g(y) = g(y')$. If such y' does not exist, then for some $y'' \in S$ we have $y \prec y''$. Now we find the least y'' with the mentioned property and put $g(y) = g(y'')$. Then

$g\colon Y \longrightarrow X$ is well-defined and it is an isotone mapping of the ordered set $(Y, \preceq)$ into $(X, \leqslant)$. $\square$

Corollary. *Let $(X, \leqslant)$, $(Y, \preceq)$ be totally ordered sets where $(Y, \preceq)$ is of a locally finite height. Then for any isotone mapping $f\colon (X, \leqslant) \longrightarrow (Y, \preceq)$ there exists an isotone mapping $f\colon (Y, \preceq) \longrightarrow (X, \leqslant)$ such that $f = f \circ g \circ f$.*

Indeed, we take the mapping $h\colon f(X) \longrightarrow X$ from Lemma 4 and construct its extension $g\colon Y \longrightarrow X$ by Lemma 5. Then g is isotone and $(f \circ g \circ f)(x) = (f \circ h \circ f)(x) = f(x)$ by Lemma 4.

Proposition 3. *Let $(X, \leqslant)$ be an ordered set of a locally finite height which contains at least one four-element chain. Then $(X, \leqslant)$ is a totally ordered set if and only if the monoid* $\mathrm{End}(X, \leqslant)$ *is regular.*

P r o o f. Suppose $(X, \leqslant)$ is an ordered set of a localy finite height containing at least one four-element chain such that its endomorphism monoid $\mathrm{End}(X, \leqslant)$, i.e. the monoid of all isotone selfmappings of $(X, \leqslant)$, is regular. Since none of the ordered sets described by conditions (1)...(4) contains any four-element chain we conclude by Theorem 2 that the poset $(X, \leqslant)$ cannot contain incomparable elements, i.e. it is a totally ordered set.

Now suppose that $(X, \leqslant)$ is a totally ordered set of cardinality at least four. Then by Lemmas 4, 5 we have that for any isotone mapping $f\colon (X, \leqslant) \longrightarrow (X, \leqslant)$ there exists an isotone mapping $g\colon (X, \leqslant) \longrightarrow (X, \leqslant)$ such that $f \circ g \circ f = f$; if f is constant we can consider $g = f$. Therefore the monoid $\mathrm{End}(X, \leqslant)$ is regular. $\square$

Theorem 3. *Let $(X, \leqslant)$ be an ordered set of a locally finite height which contains at least a four-element chain. Then the following conditions are equivalent:*

(1) *$(X, \leqslant)$ is a totally ordered set.*
(2) *The monoid* $\mathrm{End}(X, \leqslant)$ *is regular.*
(3) *For any $k \in \{2, 3, 4\}$ each element of* $\mathrm{End}(X, \leqslant)$ *is a k-endomorphism.*

P r o o f. Proof follows immediately from Theorem 1 and Proposition 3. $\square$

Remark. A general characterization of totally ordered sets with regular monoids of isotone selfmappings is not expressed in explicit form involving simple descriptions of such chains. Some results in this direction have been obtained in [2] and in more elegant form in [1]. The above considered class of ordered sets includes also ordered sets with saturated chains, graded ordered sets in the sense of [3] chapt. I., locally finite trees etc. The last concept yields the simple characterization of ω-chains stated below.

A locally finite or discrete rooted tree ([5]) is an ordered set $(T, \leqslant)$ with the least element $r \in T$ (called the root of T) such that any principal beginning $(x]_{\leqslant}$ of $(T, \leqslant)$ is a finite chain (with the least element r). From the above results (Theorems 1, 2 and Proposition 3) we get the following characterization of the first infinite ordinal ω or ω-chains.

Corollary. *Let $(T, \leqslant)$ be a discrete rooted tree containing at least a four-element chain. Then the following conditions are equivalent:*
(1) *$(T, \leqslant)$ is a totally ordered set of an ordinal type at most ω.*
(2) $\mathrm{End}(T, \leqslant)$ *is regular.*
(3) *The tree $(T, \leqslant)$ satisfies at least one of conditions* (2) *and* (3) *from Theorem* 1.

Remark. In connection with the above investigations it is to be noted that the implication contained in Lemma 1 and a certain modification of the converse implication to the one just mentioned yield a characterization of the class of ordered sets containing at least three-element chains, as follows:

Let $(X, \leqslant)$ be an ordered set without isolated elements which is not totally ordered. Then the following conditions are equivalent:
(A) The ordered set $(X, \leqslant)$ contains at least a three-element chain.
(B) For any ordered pair $(x, y) \in X \times X$ of $\leqslant$-incomparable elements there exists a pair of isotone mappings $f, g \colon (X, \leqslant) \longrightarrow (X, \leqslant)$ such that
    (1) $f(x) < f(y)$ and $\{x\} = f^{-}(f(x))$,
    (2) $g(x) < g(y)$ and $\{y\} = g^{-}(g(y))$.

Indeed, the implication (A) $\Rightarrow$ (B) follows directly from Lemma 1. We will verify the converse implication (B) $\Rightarrow$ (A). Suppose the condition (B) holds. Then evidently $(X, \leqslant)$ cannot be an antichain. Assume $(X, \leqslant)$ contains at most two-element chains. Let $a, b \in X$ be a pair of incomparable elements. If a is a minimal element of a component K of $(X, \leqslant)$ we denote $x = b$, $y = a$. Then $g(b) < g(a)$ by (2) and for any $c \in K$ such that $a < c$ we have $g(y) = g(a) \leqslant g(c) = g(y)$, thus $g^{-}(g(y)) \neq \{y\}$, which is a contradiction. If a is a maximal element of $(X, \leqslant)$ then we put $x = a$, $y = b$ and by (B) (1) we have $f(a) < f(b)$. For any $d \in X$, $d < a$ then $f(d) \leqslant f(a) = f(x) = f(d)$, thus $\{x\} \neq f^{-}(f(x))$ in this case, which is a contradiction again. Therefore $(X, \leqslant)$ contains at least a three-element chain, i.e. (A) is satisfied.

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