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# HOMOMORPHISMS OF ALGEBRAS

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*Dedicated to the memory of Otakar Borůvka on the occasion of his centenary.*

*Abstract.* A construction of all homomorphisms of an algebra with a finite number of operations into an algebra of the same type is presented that consists in replacing algebras by suitable mono-unary algebras (possibly with some nullary operations) and their homomorphisms by suitable homomorphisms of the corresponding mono-unary algebras. Since a construction of all homomorphisms between two mono-unary algebras is known (see, e.g., [6], [7], [8]), a construction of all homomorphisms of an arbitrary algebra with a finite number of operations into an algebra of the same type can be described.

*Keywords*: algebra, mono-unary algebra, homomorphism of algebras, $m$-decomposable mapping, mono-unary algebra with one acceptable and several nullary operations, mono-unary algebra with one binding and several nullary operations

*MSC 2000*: 08A60, 08A62, 18B99

## 1. Introduction

About 1950, Otakar Borůvka presented a problem whose generalization may be formulated as follows. Let $(A, o)$, $(A', o')$ be mono-unary algebras. Construct all homomorphisms of the algebra $(A, o)$ into $(A', o')$. By a mono-unary algebra $(A, o)$, we understand a set $A$ with a mapping $o$ of the set $A$ into itself.

The solution of this problem appeared in [6]; see also [7], [8]. We shall need some results of these papers in our examples; these results will be mentioned in due course.

Using this construction, we presented a construction of all homomorphisms of a groupoid into a groupoid and of an algebra with one $n$-ary operation into an algebra of the same type; see [12], [13]. In the present paper we prove that also a construction of all homomorphisms of an arbitrary algebra with a finite number of operations into an algebra of the same type may be reduced to a construction of some suitable homomorphisms between suitable mono-unary algebras. A previous version of these results was published in [15] without proofs.

## 2. Preliminaries

Fundamental information on algebras can be found, e.g., in [1], [5]. It is advantageous to separate nullary operations from operations with positive arities. Therefore, an algebra with a finite number of operations will be denoted by $\big(A, (a_i)_{1\leqslant i\leqslant p}, (o_i)_{1\leqslant i\leqslant m}\big)$ where $A$ is a nonempty set, $p \geqslant 0$, $m \geqslant 0$ are integers, $a_i$ are constants (i.e. nullary operations) for $1 \leqslant i \leqslant p$, and $o_i$ are operations of positive arities for any $i$ with $1 \leqslant i \leqslant m$. We will suppose that $p+m > 0$. Operations of positive arities will be said to be *nonnullary.* If $p = 1$ we write $\big(A, a_1, (o_i)_{1\leqslant i\leqslant m}\big)$ for $\big(A, (a_i)_{1\leqslant i\leqslant p}, (o_i)_{1\leqslant i\leqslant m}\big)$; similarly, a simpler symbol $\big(A, (a_i)_{1\leqslant i\leqslant p}, o_1\big)$ replaces $\big(A, (a_i)_{1\leqslant i\leqslant p}, (o_i)_{1\leqslant i\leqslant m}\big)$ if $m = 1$.

Let $\big(A, (a_i)_{1\leqslant i\leqslant p}, (o_i)_{1\leqslant i\leqslant m}\big)$, $\big(A', (a'_i)_{1\leqslant i\leqslant p}, (o'_i)_{1\leqslant i\leqslant m}\big)$ be algebras. They are said to be *similar* or to be *of the same type*, if the arity of $o_i$ equals the arity of $o'_i$ for any $i$ with $1 \leqslant i \leqslant m$. The arity of the operation $o_i$ will be denoted by $r(i)$. A mapping $h$ of $A$ into $A'$ is called a *homomorphism* of the first algebra into the latter if $h(a_i) = a'_i$ for any $i$ with $1 \leqslant i \leqslant p$ and if for any $i$ with $1 \leqslant i \leqslant m$ the mapping $h$ satisfies the condition $h\big(o_i(x_1, \ldots, x_{r(i)})\big) = o'_i\big(h(x_1), \ldots, h(x_{r(i)})\big)$ for any $x_1, \ldots, x_{r(i)}$ in $A$. If $r(i) = 1$, then we interprete the symbol $o_i(x_1, \ldots, x_{r(i)})$ as $o_i(x_1)$.

**Example 1.** (a) Suppose $p > 0$, $m = 0$, i.e., $\big(A, (a_i)_{1\leqslant i\leqslant p}\big)$ is an algebra with $p$ nullary operations. If $\big(A', (a'_i)_{1\leqslant i\leqslant p}\big)$ is a similar algebra, then any homomorphism $h$ of the first algebra into the latter may be obtained in the following way: We take an arbitrary mapping $g$ of the set $A - \{a_i\,;\ 1 \leqslant i \leqslant p\}$ into $A'$ and define $h(x) = g(x)$ for any $x \in A - \{a_i\,;\ 1 \leqslant i \leqslant p\}$, $h(a_i) = a'_i$ for any $i$ with $1 \leqslant i \leqslant p$.

(b) Suppose $p > 0$, $m = 1$ where arity of $o_1 = o$ equals 1. Thus, $\big(A, (a_i)_{1\leqslant i\leqslant p}, o\big)$ is an algebra with $p$ nullary and one unary operation. If $\big(A', (a'_i)_{1\leqslant i\leqslant p}, o'\big)$ is a similar algebra, then any homomorphism of the first algebra into the latter may be obtained as follows. We construct all homomorphisms of the mono-unary algebra $(A, o)$ into $(A', o')$ according to [6]; any such homomorphism $h$ must be tested whether the condition $h(a_i) = a'_i$ is satisfied for any $i$ with $1 \leqslant i \leqslant p$. If some of these equations is violated, $h$ must be rejected.

These conditions can be respected during the construction of a homomorphism $h$ of $(A, o)$ into $(A', o')$. If $h(a_i)$ is to be constructed for some $i$ with $1 \leqslant i \leqslant p$, then $a'_i$ must be chosen for $h(a_i)$. If the construction described in [6] does not allow it, we stop the construction of $h$ and start the construction of the next homomorphism.

In this way, all homomorphisms of the mono-unary algebra $(A, o)$ into $(A', o')$ are started by [6]; only those are finished which satisfy the above mentioned conditions.

In what follows, we will investigate the case where $m > 0$; we admit both the cases $p = 0$ and $p > 0$. As we have said we denote by $r(i)$ the arity of the operation $o_i$. Without loss of generality, we may suppose that $r(1) \leqslant \ldots \leqslant r(m)$. Furthermore, put $n = r(m)$ and for any $i$ with $1 \leqslant i \leqslant n$ denote by $k(i)$ the number of operations $o_j$ having the arity $i$. It is easy to see that for any $j$ with $1 \leqslant j \leqslant k(1)$ the operation $o_j$ has arity 1, i.e., $r(j) = 1$, and for any $j$ with $k(1)+\ldots+k(i-1) < j \leqslant k(1)+\ldots+k(i)$ the operation $o_j$ has arity $i$, i.e., $r(j) = i$. Furthermore, we have $k(1)+\ldots+k(n) = m$, $k(n) > 0$.

Two cases may occur. Either $n \leqslant m$ or $0 < m < n$. The first case means that no arity exceeds the number of nonnullary operations, the latter admits at least one arity exceeding the number of nonnullary operations. These cases require different methods.

In the constructions that will be described, some operations with sets and mappings will appear. We mention them now.

Let $A$, $A'$ be sets, $m \geqslant 2$ an integer. We denote by $A^m$ the set of all ordered $m$-tuples $(x_1, \ldots, x_m)$ where $x_i \in A$ for any $i$ with $1 \leqslant i \leqslant m$. A mapping $f$ of the set $A^m$ into $(A')^m$ is said to be *$m$-decomposable* if there exists a mapping $h$ of the set $A$ into $A'$ such that $f(x_1, \ldots, x_m) = \big(h(x_1), \ldots, h(x_m)\big)$ for any $(x_1, \ldots, x_m) \in A^m$. Then we put $f = h^m$. The $m$-decomposable mappings have the following properties.

**Lemma 1.** *Let $A$, $A'$ be sets, $m \geqslant 2$ an integer, $f$ an $m$-decomposable mapping of the set $A^m$ into $(A')^m$. If $h^m = f = g^m$, then $h = g$.*

**Lemma 2.** *Let $A$, $A'$, $A''$ be sets, $m \geqslant 2$ an integer, $h$ a mapping of $A$ into $A'$, $h'$ a mapping of $A'$ into $A''$. Then $(h')^m \circ h^m = (h' \circ h)^m$.*

The proofs may be obtained by a slight generalization of the proofs of Lemma 1 and Lemma 3 in [12].

## 3. Category $\mathbf{MAA}k(0)\ldots k(n)$

We now investigate the first case where no arity exceeds the number of nonnullary operations, i.e., $n \leqslant m$.

Our algebra has the form $\big(A, (a_i)_{1\leqslant i\leqslant p}, (o_i)_{1\leqslant i\leqslant m}\big)$. The case $m = 1$ implies $1 \leqslant n \leqslant m = 1$ and, therefore, $n = 1$, i.e., the algebra has one unary and $p$ nullary operations. Since such algebras have been investigated in Example 1(b), we may suppose $m \geqslant 2$. We define an algebra with $p$ nullary and one unary operation as follows.

The carrier of the algebra is the set $A^m$. For any $i$, $1 \leqslant i \leqslant p$, we take the constant $(a_i, \ldots, a_i) \in A^m$. The unary operation $o$ will be defined by

$$o(x_1, \ldots, x_m) = \big(o_1(x_1, \ldots, x_{r(1)}), \ldots, o_m(x_1, \ldots, x_{r(m)})\big) \quad \text{for any } (x_1, \ldots, x_m) \in A^m.$$

We put $o = \mathbf{um}[o_1, \ldots, o_m]$. Then $\big(A^m, ((a_i, \ldots, a_i))_{1\leqslant i\leqslant p}, \mathbf{um}[o_1, \ldots, o_m]\big)$ is the resulting algebra.

**Lemma 3.** *Suppose that $n \geqslant 1$, $k(0), \ldots, k(n)$ are nonnegative integers, $k(n) > 0$, put $m = k(1) + \ldots + k(n)$ and suppose $m \geqslant 2$, $n \leqslant m$. Let $\big(A, (a_i)_{1\leqslant i\leqslant k(0)}, (o_i)_{1\leqslant i\leqslant m}\big)$, $\big(A', (a'_i)_{1\leqslant i\leqslant k(0)}, (o'_i)_{1\leqslant i\leqslant m}\big)$ be similar algebras with $k(1)$ unary, $\ldots$, $k(n)$ $n$-ary operations. Then the following assertions hold:*

(i) *If $h$ is a homomorphism of $\big(A, (a_i)_{1\leqslant i\leqslant k(0)}, (o_i)_{1\leqslant i\leqslant m}\big)$ into $\big(A', (a'_i)_{1\leqslant i\leqslant k(0)}, (o'_i)_{1\leqslant i\leqslant m}\big)$, then $h^m$ is an $m$-decomposable homomorphism of $\big(A^m, ((a_i, \ldots, a_i))_{1\leqslant i\leqslant k(0)}, \mathbf{um}[o_1, \ldots, o_m]\big)$ into $\big((A')^m, ((a'_i, \ldots, a'_i))_{1\leqslant i\leqslant k(0)}, \mathbf{um}[o'_1, \ldots, o'_m]\big)$.*

(ii) *If $f$ is an $m$-decomposable homomorphism of $\big(A^m, ((a_i, \ldots, a_i))_{1\leqslant i\leqslant k(0)}, \mathbf{um}[o_1, \ldots, o_m]\big)$ into $\big((A')^m, ((a'_i, \ldots, a'_i))_{1\leqslant i\leqslant k(0)}, \mathbf{um}[o'_1, \ldots, o'_m]\big)$, then there exists a homomorphism $h$ of $\big(A, (a_i)_{1\leqslant i\leqslant k(0)}, (o_i)_{1\leqslant i\leqslant m}\big)$ into $\big(A', (a'_i)_{1\leqslant i\leqslant k(0)}, (o'_i)_{1\leqslant i\leqslant m}\big)$ such that $h^m = f$.*

Proof. (1) If $h$ is a homomorphism of $\big(A, (a_i)_{1\leqslant i\leqslant p}, (o_i)_{1\leqslant i\leqslant m}\big)$ into $\big(A', (a'_i)_{1\leqslant i\leqslant p}, (o'_i)_{1\leqslant i\leqslant m}\big)$ and $(x_1, \ldots, x_m) \in A^m$ is arbitrary, then $\mathbf{um}[o_1, \ldots, o_m](x_1, \ldots, x_m) = \big(o_1(x_1, \ldots, x_{r(1)}), \ldots, o_m(x_1, \ldots, x_{r(m)})\big)$, which implies that

$$\begin{aligned} h^m\big(\mathbf{um}[o_1, \ldots, o_m](x_1, \ldots, x_m)\big) &= \big(h(o_1(x_1, \ldots, x_{r(1)})), \ldots, h(o_m(x_1, \ldots, x_{r(m)}))\big) \\ &= \big(o'_1(h(x_1), \ldots, h(x_{r(1)})), \ldots, o'_m(h(x_1), \ldots, h(x_{r(m)}))\big) \\ &= \mathbf{um}[o'_1, \ldots, o'_m](h(x_1), \ldots, h(x_m) \\ &= \mathbf{um}[o'_1, \ldots, o'_m](h^m(x_1, \ldots, x_m)). \end{aligned}$$

Consequently, $h^m$ is a homomorphism of $\big(A^m, \mathbf{um}[o_1, \ldots, o_m]\big)$ into $\big((A')^m, \mathbf{um}[o'_1, \ldots, o'_m]\big)$. Since $h(a_i) = a'_i$, we have $h^m(a_i, \ldots, a_i) = (a'_i, \ldots, a'_i) \in (A')^m$ for any $i$ with $1 \leqslant i \leqslant k(0)$. Thus (i) holds.

(2) If $f$ is an $m$-decomposable homomorphism of the algebra $\big(A^m, ((a_i, \ldots, a_i))_{1\leqslant i\leqslant k(0)}, \mathbf{um}[o_1, \ldots, o_m]\big)$ into $\big((A')^m, ((a'_i, \ldots, a'_i))_{1\leqslant i\leqslant k(0)}, \mathbf{um}[o'_1, \ldots, o'_m]\big)$, then there exists a mapping $h$ of $A$ into $A'$ such that $f = h^m$. Since $h^m(a_i, \ldots, a_i) = (a'_i, \ldots, a'_i)$, we obtain $h(a_i) = a'_i$ for any $i$ with $1 \leqslant i \leqslant k(0)$.

For any $(x_1, \ldots, x_m) \in A^m$ we have

$$\mathbf{um}[o_1, \ldots, o_m](x_1, \ldots, x_m) = \big(o_1(x_1, \ldots, x_{r(1)}), \ldots, o_m(x_1, \ldots, x_{r(m)})\big),$$

which implies

$$\begin{aligned}
&\big(h(o_1(x_1,\ldots,x_{r(1)})),\ldots,h(o_m(x_1,\ldots,x_{r(m)}))\big)\\
&\quad= h^m\big(o_1(x_1,\ldots,x_{r(1)}),\ldots,o_m(x_1,\ldots,x_{r(m)})\big)\\
&\quad= f\big(\mathbf{um}[o_1,\ldots,o_m](x_1,\ldots,x_m)\big)\\
&\quad= \mathbf{um}[o'_1,\ldots,o'_m](f(x_1,\ldots,x_m))\\
&\quad= \mathbf{um}[o'_1,\ldots,o'_m]\big(h(x_1),\ldots,h(x_m)\big)\\
&\quad= \big(o'_1(h(x_1),\ldots h(x_{r(1)})),\ldots,o'_m(h(x_1),\ldots,h(x_{r(m)}))\big)
\end{aligned}$$

and, therefore, $h\big(o_i(x_1,\ldots,x_{r(i)})\big)=o'_i\big(h(x_1),\ldots,h(x_{r(i)})\big)$ for any $i$ with $1\leqslant i\leqslant m$.

Thus $h$ is a homomorphism of $\big(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\big)$ into $\big(A',(a'_i)_{1\leqslant i\leqslant k(0)},(o'_i)_{1\leqslant i\leqslant m}\big)$ and (ii) holds. □

Let $n\geqslant 1$ be an integer, $k(0),\ldots,k(n)$ nonnegative integers such that $k(n)\geqslant 1$. Put $k(1)+\ldots+k(n)=m$ and suppose that $m\geqslant 2$, $n\leqslant m$. Let $A\neq\emptyset$ be a set. A unary operation $o$ on the set $A^m$ is said to be *acceptable with respect to the sequence* $(k(1),\ldots,k(n))$ if it has the following property: If $(x_1,\ldots,x_m)\in A^m$, $(y_1,\ldots,y_m)\in A^m$, $o(x_1,\ldots,x_m)=(x'_1,\ldots,x'_m)$, $o(y_1,\ldots,y_m)=(y'_1,\ldots,y'_m)$, $1\leqslant i\leqslant n$ and $x_j=y_j$ for any $j$ with $1\leqslant j\leqslant i$, then $x'_j=y'_j$ for any $j$ with $1\leqslant j\leqslant k(1)+\ldots+k(i)$.

**Lemma 4.** *Let $n\geqslant 1$ be an integer, $k(1),\ldots,k(n)$ nonnegative integers such that $k(n)\geqslant 1$. Put $k(1)+\ldots+k(n)=m$ and suppose that $m\geqslant 2$, $n\leqslant m$. Let $A\neq\emptyset$ be a set. Then the following assertions hold:*

(i) *Let $o_1,\ldots,o_m$ be nonnullary operations on $A$ such that for any $j$ with $1\leqslant j\leqslant m$ the arity $r(j)$ of the operation $o_j$ is the least integer $i$ satisfying the condition $1\leqslant j\leqslant k(1)+\ldots+k(i)$. Then $\mathbf{um}[o_1,\ldots,o_m]$ is a unary operation on the set $A^m$ acceptable with respect to the sequence $(k(1),\ldots,k(n))$.*

(ii) *If $o$ is a unary operation on the set $A^m$ acceptable with respect to the sequence $(k(1),\ldots,k(n))$, then there exist nonnullary operations $o_1,\ldots,o_m$ on $A$ such that for any $j$ with $1\leqslant j\leqslant m$ the arity $r(j)$ of the operation $o_j$ equals the least integer $i$ satisfying $1\leqslant j\leqslant k(1)+\ldots+k(i)$. Furthermore, $\mathbf{um}[o_1,\ldots,o_m]=o$ holds.*

Proof. (1) If $o_1,\ldots,o_m$ are nonnullary operations on $A$ with the properties presented in (i), then $\mathbf{um}[o_1,\ldots,o_m](x_1,\ldots,x_m)=\big(o_1(x_1,\ldots,x_{r(1)}),\ldots,o_m(x_1,\ldots,x_{r(m)})\big)$ for any $(x_1,\ldots,x_m)\in A^m$.

Suppose $(x_1,\ldots,x_m)\in A^m$, $(y_1,\ldots,y_m)\in A^m$, let $i$ be an integer such that $1\leqslant i\leqslant n$ and $x_j=y_j$ holds for any integer $j$ with $1\leqslant j\leqslant i$. Then $r(j)\leqslant i$ for any $j$ with $j\leqslant k(1)+\ldots+k(i)$. It follows that $(x_1,\ldots,x_{r(j)})=(y_1,\ldots,y_{r(j)})$ for any $j$ with $j\leqslant k(1)+\ldots+k(i)$, which entails $o_j(x_1,\ldots,x_{r(j)})=o_j(y_1,\ldots,y_{r(j)})$. Thus, if we put $\mathbf{um}[o_1,\dots,o_m](x_1,\dots,x_m) = (x'_1,\dots,x'_m)$, $\mathbf{um}[o_1,\dots,o_m](y_1,\dots,y_m) = (y'_1,\dots,y'_m)$, then for any $j$ with $1 \leqslant j \leqslant k(1)+\ldots+k(i)$ we have $x'_j = o_j(x_1,\dots,x_{r(j)}) = o_j(y_1,\dots,y_{r(j)}) = y'_j$. Therefore the operation $\mathbf{um}[o_1,\dots,o_m]$ is acceptable with respect to the sequence $(k(1),\dots,k(n))$. Thus (i) holds.

(2) Let $o$ be a unary operation on the set $A^m$ acceptable with respect to the sequence $(k(1),\dots,k(n))$, $j$ an arbitrary integer with $1 \leqslant j \leqslant m$.

There exists the least integer $i$ such that $j \leqslant k(1)+\ldots+k(i)$. Let $x_1,\dots,x_i$ be arbitrary elements in $A$. We choose $x_{i+1},\dots,x_m$ in $A$ arbitrarily and define $(x'_1,\dots,x'_m) = o(x_1,\dots,x_m)$. Finally, we put $o_j(x_1,\dots,x_i) = x'_j$. The acceptability of the operation $o$ with respect to the sequence $(k(1),\dots,k(n))$ implies that $o_j(x_1,\dots,x_i)$ is defined correctly.

Indeed, let $y_{i+1},\dots,y_m$ in $A$ be arbitrary elements and $(y'_1,\dots,y'_m) = o(y_1,\dots,y_i, y_{i+1},\dots,y_m)$ where $y_1 = x_1,\dots,y_i = x_i$, i.e., $x_t = y_t$ for any integer $t$ with $1 \leqslant t \leqslant i$. The acceptability of $o$ implies that $x'_t = y'_t$ for any $t$ with $1 \leqslant t \leqslant k(1)+\ldots+k(i)$. In particular, $x'_j = y'_j$, hence $x'_j$ does not depend on the elements $x_{i+1},\dots,x_m$.

If $i$ is the least integer with $j \leqslant k(1)+\ldots+k(i)$ then $r(j) = i$ by our definition. Let $(x_1,\dots,x_m) \in A^m$ be arbitrary and $o(x_1,\dots,x_m) = (x'_1,\dots,x'_m)$. Then $\mathbf{um}[o_1,\dots,o_m](x_1,\dots,x_m) = \big(o_1(x_1,\dots,x_{r(1)}),\dots,o_m(x_1,\dots,x_{r(m)})\big) = (x'_1,\dots,x'_m) = o(x_1,\dots,x_m)$ by our definition of the operations $o_1,\dots,o_m$. Hence (ii) holds. □

We now introduce some categories of algebras. More details concerning categories can be found, e.g., in [1], [4], [16].

We denote by $\mathbf{MAA}k(0)\dots k(n)$ the category which is defined as follows (category of Mono-unary Algebras with one Acceptable and several nullary operations).

An object of the category is an algebra whose carrier is the set $A^m$ which has $k(0)$ nullary operations of the form $(a_i,\dots,a_i) \in A^m$ for any $i$ with $1 \leqslant i \leqslant k(0)$ and a unary operation that is acceptable with respect to the sequence $(k(1),\dots,k(n))$. Thus an object of this category is of the form $\big(A^m, ((a_i,\dots,a_i))_{1\leqslant i\leqslant k(0)}, o\big)$.

Morphisms of this category are $m$-decomposable homomorphisms of these algebras.

It is easy to see that $\mathbf{MAA}k(0)\dots k(n)$ is a category.

Furthermore, we denote by $\mathbf{ALG}k(0)\dots k(n)$ the category whose objects are algebras with $k(0)$ nullary, …, $k(n)$ $n$-ary operations; its morphisms are homomorphisms of these algebras. We mention that the conditions $n \leqslant m = k(1)+\ldots+k(n)$, $m \geqslant 2$ are satisfied.

We now define a functor $F$ from $\mathbf{ALG}k(0)\dots k(n)$ to $\mathbf{MAA}k(0)\dots k(n)$ by presenting the object mapping $Fo$ and the morphism mapping $Fr$.

For any object $\big(A, (a_i)_{1\leqslant i\leqslant k(0)}, (o_i)_{1\leqslant i\leqslant m}\big)$ in $\mathbf{ALG}k(0)\dots k(n)$ we define

$$Fo\big(A, (a_i)_{1\leqslant i\leqslant k(0)}, (o_i)_{1\leqslant i\leqslant m}\big) = \big(A^m, ((a_i,\dots,a_i))_{1\leqslant i\leqslant k(0)}, \mathbf{um}[o_1,\dots,o_m]\big).$$

If $\big(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\big)$, $\big(A',(a'_i)_{1\leqslant i\leqslant k(0)},(o'_i)_{1\leqslant i\leqslant m}\big)$ are objects in the category $\mathbf{ALG}k(0)\dots k(n)$ and $h$ is a homomorphism of $\big(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\big)$ into $\big(A',(a'_i)_{1\leqslant i\leqslant k(0)},(o'_i)_{1\leqslant i\leqslant m}\big)$, we put

$$Fr(h)=h^m.$$

**Theorem 1.** *Let $n\geqslant 1$, let $k(0),\dots,k(n)$ be nonnegative integers such that $k(n)\geqslant 1$, $k(1)+\dots+k(n)\geqslant 2$, $k(1)+\dots+k(n)\geqslant n$. Then $F$ is an isomorphism of the category $\mathbf{ALG}k(0)\dots k(n)$ onto $\mathbf{MAA}k(0)\dots k(n)$.*

Proof. Put $m=k(1)+\dots+k(n)$.

If $\big(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\big)$ is an arbitrary object of $\mathbf{ALG}k(0)\dots k(n)$, then $Fo\big(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\big)$ has nullary operations $(a_i,\dots,a_i)\in A^m$ for any $i$ with $1\leqslant i\leqslant k(0)$ and the unary operation $\mathbf{um}[o_1,\dots,o_m]$ that is acceptable with respect to the sequence $(k(1),\dots,k(n))$ by Lemma 4. It follows that $Fo$ maps the class of all objects in $\mathbf{ALG}k(0)\dots k(n)$ into the class of all objects in $\mathbf{MAA}k(0)\dots k(n)$. It is easy to see that $Fo$ is an injective mapping.

If $\big(A^m,((a_i,\dots,a_i))_{1\leqslant i\leqslant k(0)},o\big)$ is an object of the category $\mathbf{MAA}k(0)\dots k(n)$, then $a_i$ is a nullary operation on $A$ for any $i$ with $1\leqslant i\leqslant k(0)$. By Lemma 4, there exist nonnullary operations $o_1,\dots,o_m$ on $A$ such that $\mathbf{um}[o_1,\dots,o_m]=o$ and that for any $j$ with $1\leqslant j\leqslant m$ the arity $r(j)$ of the operation $o_j$ equals the least integer $i$ satisfying $1\leqslant j\leqslant k(1)+\dots+k(i)$. Hence

$$Fo\big(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\big)=\big(A^m,((a_i,\dots,a_i))_{1\leqslant i\leqslant k(0)},o\big).$$

It follows that $Fo$ is a bijection.

By Lemma 1 and Lemma 3, $Fr$ is a bijection of the class of all morphisms of $\mathbf{ALG}k(0)\dots k(n)$ onto the class of all morphisms of the category $\mathbf{MAA}k(0)\dots k(n)$. Consider an arbitrary object $\big(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\big)$ in $\mathbf{ALG}k(0)\dots k(n)$. Then $Fr(\mathrm{id}_A)=(\mathrm{id}_A)^m=\mathrm{id}_{A^m}$ and, hence, $Fr(1_{(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m})})=1_{Fo(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m})}$. Furthermore, if $\big(A',(a'_i)_{1\leqslant i\leqslant k(0)},(o'_i)_{1\leqslant i\leqslant m}\big)$, $\big(A'',(a''_i)_{1\leqslant i\leqslant k(0)},(o''_i)_{1\leqslant i\leqslant m}\big)$ are objects in $\mathbf{ALG}k(0)\dots k(n)$ and $h$ is a homomorphism of the algebra $\big(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\big)$ into $\big(A',(a'_i)_{1\leqslant i\leqslant k(0)},(o'_i)_{1\leqslant i\leqslant m}\big)$ and $k$ is a homomorphism of $\big(A',(a'_i)_{1\leqslant i\leqslant k(0)},(o'_i)_{1\leqslant i\leqslant m}\big)$ into $\big(A'',(a''_i)_{1\leqslant i\leqslant k(0)},(o''_i)_{1\leqslant i\leqslant m}\big)$, then $k\circ h$ is a homomorphism of the first algebra into the third and $Fr(k\circ h)=(k\circ h)^m=k^m\circ h^m=Fr(k)\circ Fr(h)$ by Lemma 2. Thus, $Fr$ satisfies the conditions characterizing the morphism mappings of a functor.

Hence $F$ is a functor and, regarding the bijectivity of $Fo$ and $Fr$, it is an isomorphism of the category $\mathbf{ALG}k(0)\dots k(n)$ onto $\mathbf{MAA}k(0)\dots k(n)$. □

**Construction 1.** *Suppose that $n\geqslant 1$, $k(0),\dots,k(n)$ are nonnegative integers, $k(n)>0$, put $m=k(1)+\dots+k(n)$ and suppose $m\geqslant 2$, $n\leqslant m$.*

*Let* $\big(A, (a_i)_{1\leqslant i\leqslant k(0)}, (o_i)_{1\leqslant i\leqslant m}\big)$, $\big(A', (a'_i)_{1\leqslant i\leqslant k(0)}, (o'_i)_{1\leqslant i\leqslant m}\big)$ *be similar algebras with* $k(1)$ *unary, …,* $k(n)$ $n$*-ary operations.*

*Construct the algebras* $\big(A^m, ((a_i,\dots,a_i))_{1\leqslant i\leqslant k(0)}, \mathbf{um}[o_1,\dots,o_m]\big)$, $\big((A')^m, ((a'_i,\dots,a'_i)), \mathbf{um}[o'_1,\dots,o'_m]\big)$.

*Construct all homomorphisms of* $\big(A^m, ((a_i,\dots,a_i))_{1\leqslant i\leqslant k(0)}, \mathbf{um}[o_1,\dots,o_m]\big)$ *into* $\big((A')^m, ((a'_i,\dots,a'_i))_{1\leqslant i\leqslant k(0)}, \mathbf{um}[o'_1,\dots,o'_m]\big)$ *using Example* 1(b).

*Test these homomorphisms and reject all of them that are not* $m$*-decomposable.*

*For any* $m$*-decomposable homomorphism* $f$ *construct the mapping* $h$ *such that* $f = h^m$.

*Any constructed mapping is a homomorphism of the algebra* $\big(A, (a_i)_{1\leqslant i\leqslant k(0)}, (o_i)_{1\leqslant i\leqslant m}\big)$ *into* $\big(A', (a'_i)_{1\leqslant i\leqslant k(0)}, (o'_i)_{1\leqslant i\leqslant m}\big)$ *and any homomorphism of the first algebra into the latter can be constructed in this way.*

**Example 2.** Let $A = \{a,b,c\}$, suppose that one nullary, one unary and two binary operations are defined on $A$. The value of the nullary operation is $a$, the unary operation $o_1$ and the two binary operations $o_2$, $o_3$ have the following tables:

| $o_1$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| | $b$ | $c$ | $a$ |

| $o_2$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $b$ | $c$ |
| $b$ | $b$ | $b$ | $a$ |
| $c$ | $c$ | $a$ | $c$ |

| $o_3$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $b$ | $c$ |
| $b$ | $c$ | $a$ | $b$ |
| $c$ | $b$ | $c$ | $a$ |

Thus $k(0) = 1$, $k(1) = 1$, $k(2) = 2$, $m = k(1) + k(2) = 3 > 2 = n$. Hence we construct the mono-unary algebra $(A^3, \mathbf{um}[o_1,o_2,o_3])$ where the table of the operation $\mathbf{um}[o_1,o_2,o_3]$ is as follows (we write $xyz$ for $(x,y,z)$ for the sake of brevity):

| $xyz$ | $aaa$ | $aab$ | $aac$ | $aba$ | $abb$ | $abc$ | $aca$ | $acb$ | $acc$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{um}[o_1,o_2,o_3](xyz)$ | $baa$ | $baa$ | $baa$ | $bbb$ | $bbb$ | $bbb$ | $bcc$ | $bcc$ | $bcc$ |

| $xyz$ | $baa$ | $bab$ | $bac$ | $bba$ | $bbb$ | $bbc$ | $bca$ | $bcb$ | $bcc$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{um}[o_1,o_2,o_3](xyz)$ | $cbc$ | $cbc$ | $cbc$ | $cba$ | $cba$ | $cba$ | $cab$ | $cab$ | $cab$ |

| $xyz$ | $caa$ | $cab$ | $cac$ | $cba$ | $cbb$ | $cbc$ | $cca$ | $ccb$ | $ccc$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{um}[o_1,o_2,o_3](xyz)$ | $acb$ | $acb$ | $acb$ | $aac$ | $aac$ | $aac$ | $aca$ | $aca$ | $aca$ |

Thus $Fo\big(A, a, (o_i)_{1\leqslant i\leqslant 3}\big) = (A^3, aaa, \mathbf{um}[o_1,o_2,o_3])$. The algebra $(A^3, \mathbf{um}[o_1,o_2,o_3])$ has the graph from Fig. 1.

We construct all endomorphisms of $(A, a, (o_i)_{1\leqslant i\leqslant 3})$. Hence, we need all 3-decomposable endomorphisms of $(A^3, aaa, \mathbf{um}[o_1,o_2,o_3])$. Regarding the fact that $aaa$ is a nullary operation, we obtain $f(aaa) = aaa$ for any endomorphism $f$ of

$\big(A^3, aaa, \mathbf{um}[o_1, o_2, o_3]\big)$. It follows that

$$\begin{aligned} f(baa) &= f(\mathbf{um}[o_1, o_2, o_3](aaa)) = \mathbf{um}[o_1, o_2, o_3](f(aaa)) \\ &= \mathbf{um}[o_1, o_2, o_3](aaa) = baa, \\ f(cbc) &= f(\mathbf{um}[o_1, o_2, o_3](baa)) = \mathbf{um}[o_1, o_2, o_3](f(baa)) \\ &= \mathbf{um}[o_1, o_2, o_3](baa) = cbc. \end{aligned}$$

Thus, if $f = h^3$, then $h(a) = a$, $h(b) = b$, $h(c) = c$. It follows that $h = \mathrm{id}_A$ is the only endomorphism of $(A, a, (o_i)_{1 \leqslant i \leqslant 3})$.

We have used here only the fundamental property of a homomorphism of the algebra $(A^3, \mathbf{um}[o_1, o_2, o_3])$; the details of the construction of homomorphisms were not necessary in this simple example.

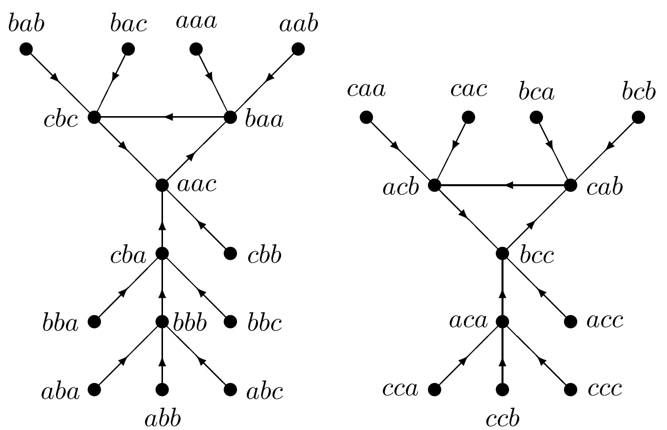


Fig. 1.

## 4. Category $\mathbf{MAB}k(0)\ldots k(n)$

We now investigate the remaining case where at least one arity exceeds the number of nonnullary operations, i.e. $0 < m < n$.

Our algebra is of the form $\big(A, (a_i)_{1 \leqslant i \leqslant p}, (o_i)_{1 \leqslant i \leqslant m}\big)$. We define an algebra with $p$ nullary and one unary operation as follows. The carrier of the algebra is $A^n$. For any $i$, $1 \leqslant i \leqslant p$, we take the constant $(a_i, \ldots, a_i) \in A^n$. The unary operation $o$ will be defined by $o(x_1, \ldots, x_n) = \big(x_{m+1}, \ldots, x_n, o_1(x_1, \ldots, x_{r(1)}), \ldots, o_m(x_1, \ldots, x_{r(m)})\big)$ for any $(x_1, \ldots, x_n) \in A^n$. We put $o = \mathbf{un}[o_1, \ldots, o_n]$.

**Lemma 5.** *Suppose that $n \geqslant 1$, $k(0), \ldots, k(n)$ are nonnegative integers such that $k(n) > 0$. Put $m = k(1) + \ldots + k(n)$ and suppose $0 < m < n$. Let*

$\big(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\big)$, $\big(A',(a'_i)_{1\leqslant i\leqslant k(0)},(o'_i)_{1\leqslant i\leqslant m}\big)$ *be similar algebras with* $k(1)$ *unary, ..., $k(n)$ $n$-ary operations. Then the following assertions hold.*

(i) *If $h$ is a homomorphism of $\big(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\big)$ into $\big(A',(a'_i)_{1\leqslant i\leqslant k(0)},(o'_i)_{1\leqslant i\leqslant m}\big)$, then $h^n$ is an $n$-decomposable homomorphism of $\big(A^n,(a_i,\dots,a_i)_{1\leqslant i\leqslant k(0)},\mathbf{un}[o_1,\dots,o_m]\big)$ into $\big((A')^n,(a'_i,\dots,a'_i)_{1\leqslant i\leqslant k(0)},\mathbf{un}[o'_1,\dots,o'_m]\big)$.*

(ii) *If $f$ is an $n$-decomposable homomorphism of $\big(A^n,((a_i,\dots,a_i))_{1\leqslant i\leqslant k(0)},\mathbf{un}[o_1,\dots,o_m]\big)$ into $\big((A')^n,((a'_i,\dots,a'_i))_{1\leqslant i\leqslant k(0)},\mathbf{un}[o'_1,\dots,o'_m]\big)$ then there exists a homomorphism $h$ of $\big(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\big)$ into $\big(A',(a'_i)_{1\leqslant i\leqslant k(0)},(o'_i)_{1\leqslant i\leqslant m}\big)$ such that $h^n=f$.*

Proof. (1) If $h$ is a homomorphism of $\big(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\big)$ into $\big(A',(a'_i)_{1\leqslant i\leqslant k(0)},(o'_i)_{1\leqslant i\leqslant m}\big)$ and $(x_1,\dots,x_n)\in A^n$ is arbitrary, then $\mathbf{un}[o_1,\dots,o_m](x_1,\dots,x_n) = \big(x_{m+1},\dots,x_n,o_1(x_1,\dots,x_{r(1)}),\dots,o_m(x_1,\dots,x_{r(m)})\big)$, which implies that

$$\begin{aligned}
&h^n(\mathbf{un}[o_1,\dots,o_m](x_1,\dots,x_n))\\
&\quad=\big(h(x_{m+1}),\dots,h(x_n),h(o_1(x_1,\dots,x_{r(1)})),\dots,h(o_m(x_1,\dots,x_{r(m)}))\big)\\
&\quad=\big(h(x_{m+1}),\dots,h(x_n),o'_1(h(x_1),\dots,h(x_{r(1)})),\dots,o'_m(h(x_1),\dots,h(x_{r(m)}))\big)\\
&\quad=\mathbf{un}[o'_1,\dots,o'_m](h(x_1),\dots,h(x_n))\\
&\quad=\mathbf{un}[o'_1,\dots,o'_m](h^n(x_1,\dots,x_n)).
\end{aligned}$$

Thus, $h^n$ is a homomorphism of $(A^n,\mathbf{un}[o_1,\dots,o_m])$ into $((A')^n,\mathbf{un}[o'_1,\dots,o'_m])$. Since $h(a_i)=a'_i$ we obtain $h^n(a_i,\dots,a_i)=(a'_i,\dots,a'_i)\in(A')^n$ for any $i$ with $1\leqslant i\leqslant k(0)$. Thus (i) holds.

(2) If $f$ is an $n$-decomposable homomorphism of the algebra $\big(A^n,((a_i,\dots,a_i))_{1\leqslant i\leqslant k(0)},\mathbf{un}[o_1,\dots,o_m]\big)$ into $\big((A')^n,((a'_i,\dots,a'_i))_{1\leqslant i\leqslant k(0)},\mathbf{un}[o'_1,\dots,o'_m]\big)$, then there exists a mapping $h$ of $A$ into $A'$ such that $f=h^n$. Since $h^n(a_i,\dots,a_i)=(a'_i,\dots,a'_i)$, we obtain $h(a_i)=a'_i$ for any $i$ with $1\leqslant i\leqslant k(0)$.

For any $(x_1,\dots,x_n)\in A^n$ we have

$$\begin{aligned}
&\mathbf{un}[o_1,\dots,o_m](x_1,\dots,x_n)\\
&\qquad=\big(x_{m+1},\dots,x_n,o_1(x_1,\dots,x_{r(1)}),\dots,o_m(x_1,\dots,x_{r(m)})\big),
\end{aligned}$$

which implies

$$\begin{aligned}
&\big(h(x_{m+1}),\dots,h(x_n),h(o_1(x_1,\dots,x_{r(1)})),\dots,h(o_m(x_1,\dots,x_{r(m)}))\big)\\
&\quad=h^n\big(x_{m+1},\dots,x_n,o_1(x_1,\dots,x_{r(1)}),\dots,o_m(x_1,\dots,x_{r(m)})\big)\\
&\quad=f(\mathbf{un}[o_1,\dots,o_m](x_1,\dots,x_n))\\
&\quad=\mathbf{un}[o'_1,\dots,o'_m](f(x_1,\dots,x_n))\\
&\quad=\mathbf{un}[o'_1,\dots,o'_m](h(x_1),\dots,h(x_n))\\
&\quad=\big(h(x_{m+1}),\dots,h(x_n),o'_1(h(x_1),\dots,h(x_{r(1)})),\dots,o'_m(h(x_1),\dots,h(x_{r(m)}))\big)
\end{aligned}$$

and, therefore, $h(o_i(x_1,\dots,x_{r(i)})) = o'_i(h(x_1),\dots,h(x_{r(i)}))$ for any $i$ with $1 \leqslant i \leqslant m$.

Thus, $h$ is a homomorphism of $\bigl(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\bigr)$ into $\bigl(A',(a'_i)_{1\leqslant i\leqslant k(0)},(o'_i)_{1\leqslant i\leqslant m}\bigr)$ and (ii) holds. □

Let $n \geqslant 1$ be an integer, $k(0),\dots,k(n)$ nonnegative integers such that $k(n) \geqslant 1$. Put $k(1)+\ldots+k(n) = m$ and suppose $m < n$. Let $A \neq \emptyset$ be a set. A unary operation $o$ on the set $A^n$ is said to be *binding with respect to the sequence* $(k(1),\dots,k(n))$ if it has the following properties.

(i) If $(x_1,\dots,x_n) \in A^n$ and $o(x_1,\dots,x_n) = (x'_1,\dots,x'_n)$, then $x'_1 = x_{m+1},\dots,x'_{n-m} = x_n$.

(ii) Let $(x_1,\dots,x_n)$, $(y_1,\dots,y_n)$ be in $A^n$ and $o(x_1,\dots,x_n) = (x'_1,\dots,x'_n)$, $o(y_1,\dots,y_n) = (y'_1,\dots,y'_n)$, $1 \leqslant i \leqslant n$ and $x_j = y_j$ for any integer $j$ with $1 \leqslant j \leqslant i$, then $x'_{n-m+j} = y'_{n-m+j}$ for any $j$ with $1 \leqslant j \leqslant k(1)+\ldots+k(i)$.

**Lemma 6.** *Let $n \geqslant 1$ be an integer, $k(1),\dots,k(n)$ nonnegative integers such that $k(n) \geqslant 1$. Put $k(1)+\ldots+k(n) = m$ and suppose $m < n$. Let $A \neq \emptyset$ be a set. Then the following assertions hold.*

(i) *Let $o_1,\dots,o_m$ be nonnullary operations on $A$ such that for any integer $j$ with $1 \leqslant j \leqslant n$ the arity of $o_j$ is the least integer $i$ satisfying $1 \leqslant j \leqslant k(1)+\ldots+k(i)$. Then $\mathbf{un}[o_1,\dots,o_m]$ is a unary operation on the set $A^n$ binding with respect to the sequence $(k(1),\dots,k(n))$.*

(ii) *If $o$ is a unary operation on the set $A^n$ that is binding with respect to the sequence $(k(1),\dots,k(n))$, then there exist nonnullary operations $o_1,\dots,o_m$ on the set $A$ such that for any $j$ with $1 \leqslant j \leqslant m$ the arity $r(j)$ of the operation $o_j$ equals the least $i$ satisfying $1 \leqslant j \leqslant k(1)+\ldots+k(i)$. Furthermore, $\mathbf{un}[o_1,\dots,o_m] = o$.*

P r o o f. (1) If $o_1,\dots,o_m$ are nonnullary operations on $A$ with the properties presented in (i), then $\mathbf{un}[o_1,\dots,o_m](x_1,\dots,x_n) = \bigl(x_{m+1},\dots,x_n,o_1(x_1,\dots,x_{r(1)}),\dots,o_m(x_1,\dots,x_{r(m)})\bigr)$ for any $(x_1,\dots,x_n) \in A^n$. Hence, if $\mathbf{un}[o_1,\dots,o_m](x_1,\dots,x_n) = (x'_1,\dots,x'_n)$, then $x'_1 = x_{m+1},\dots,x'_{n-m} = x_n$.

Suppose $(x_1,\dots,x_n) \in A^n$, $(y_1,\dots,y_n) \in A^n$, $\mathbf{un}[o_1,\dots,o_m](x_1,\dots x_n) = (x'_1,\dots,x'_n)$, $\mathbf{un}[o_1,\dots,o_m](y_1,\dots,y_n) = (y'_1,\dots,y'_n)$. If $1 \leqslant i \leqslant n$ and $x_t = y_t$ for any $t$ with $1 \leqslant t \leqslant i$, then $x'_{n-m+j} = o_j(x_1,\dots,x_{r(j)}) = o_j(y_1,\dots,y_{r(j)}) = y'_{n-m+j}$ for any $j$ with $j \leqslant k(1)+\ldots+k(i)$ because $r(j) \leqslant i$.

We have proved that $\mathbf{un}[o_1,\dots,o_m]$ is a binding operation with respect to the sequence $(k(1),\dots,k(n))$. Thus (i) holds.

(2) Let $o$ be a unary operation on the set $A^n$ that is binding with respect to the sequence $(k(1),\dots,k(n))$, $j$ an arbitrary integer with $1 \leqslant j \leqslant m$.

There exists the least integer $i$ such that $j \leqslant k(1)+\ldots+k(i)$. Let $x_1,\dots x_i$ be arbitrary elements in $A$. We choose $x_{i+1},\dots,x_n$ in $A$ arbitrarily and define

$(x'_1, \dots, x'_n) = o(x_1, \dots, x_n)$. Finally, we put $o_j(x_1, \dots, x_i) = x'_{n-m+j}$. Since $o$ is binding with respect to the sequence $(k(1), \dots, k(n))$, the operation $o_j$ is defined correctly.

Indeed, let $y_{i+1}, \dots, y_n$ in $A$ be arbitrary elements and $(y'_1, \dots, y'_n) = o(y_1, \dots, y_i, y_{i+1}, \dots y_n)$ where $y_1 = x_1, \dots, y_i = x_i$, i.e., $y_t = x_t$ for any $t$ with $1 \leqslant t \leqslant i$. Since $o$ is binding with respect to $(k(1), \dots, k(n))$ we obtain $o_t(x_1, \dots, x_{r(t)}) = x'_{n-m+t} = y'_{n-m+t} = o_t(y_1, \dots, y_{r(t)})$ for any $t$ with $1 \leqslant t \leqslant k(1) + \dots + k(i)$. In particular, $x'_{n-m+j} = y'_{n-m+j}$, hence $x'_{n-m+j}$ does not depend on elements $x_{i+1}, \dots, x_n$.

If $i$ is the least integer with $j \leqslant k(1) + \dots + k(i)$, then $r(j) = i$ by our definition of $o_j$.

Let $(x_1, \dots, x_n) \in A^n$ be arbitrary. Put $o(x_1, \dots, x_n) = (x'_1, \dots, x'_n)$. We obtain $\mathbf{un}[o_1, \dots, o_m](x_1, \dots, x_n) = \big(x_{m+1}, \dots, x_n, o_1(x_1, \dots, x_{r(1)}), \dots, o_m(x_1, \dots, x_{r(m)})\big) = (x'_1, \dots, x'_n) = o(x_1, \dots, x_n)$ by our definition of the operations $o_1, \dots, o_m$. Hence (ii) holds. $\square$

We denote by $\mathbf{MAB}k(0)\dots k(n)$ the category which is defined as follows (category of Mono-unary Algebras with one Binding and several nullary operations).

Objects of this category are algebras whose carrier has the form $A^n$ where $A$ is a set. Any algebra has $k(0)$ constants of the form $(a_i, \dots, a_i) \in A^n$ for any $i$ with $1 \leqslant i \leqslant k(0)$. Furthermore, it has a unary operation $o$ that is binding with respect to the sequence $(k(1), \dots, k(n))$. Thus, an object of $\mathbf{MAB}k(0)\dots k(n)$ is of the form $\big(A^n, ((a_i, \dots, a_i))_{1 \leqslant i \leqslant k(0)}, o\big)$. Morphisms of this category are $n$-decomposable homomorphisms of these algebras. It is easy to see that $\mathbf{MAB}k(0)\dots k(n)$ is a category.

We now define a functor $G$ from $\mathbf{ALG}k(0)\dots k(n)$ to $\mathbf{MAB}k(0)\dots k(n)$ by presenting the object mapping $Go$ and the morphism mapping $Gr$.

For any object $\big(A, (a_i)_{1 \leqslant i \leqslant k(0)}, (o_i)_{1 \leqslant i \leqslant m}\big)$ in $\mathbf{ALG}k(0)\dots k(n)$ we define

$$Go\big(A, (a_i)_{1 \leqslant i \leqslant k(0)}, (o_i)_{1 \leqslant i \leqslant m}\big) = \big(A^n, ((a_i, \dots, a_i))_{1 \leqslant i \leqslant k(0)}, \mathbf{un}[o_1, \dots, o_m]\big).$$

If $\big(A, (a_i)_{1 \leqslant i \leqslant k(0)}, (o_i)_{1 \leqslant i \leqslant m}\big)$, $\big(A', ((a'_i)_{1 \leqslant i \leqslant k(0)}, (o'_i)_{1 \leqslant i \leqslant m}\big)$ are objects in the category $\mathbf{ALG}k(0)\dots k(n)$ and $h$ is a homomorphism of $\big(A, (a_i)_{1 \leqslant i \leqslant k(0)}, (o_i)_{1 \leqslant i \leqslant m}\big)$ into $\big(A', (a'_i)_{1 \leqslant i \leqslant k(0)}, (o'_i)_{1 \leqslant i \leqslant m}\big)$, then we put

$$Gr(h) = h^n.$$

**Theorem 2.** *Let $n \geqslant 1$, $k(0), \dots, k(n)$ be nonnegative integers such that $k(n) \geqslant 1$, $k(1) + \dots + k(n) < n$. Then $G$ is an isomorphism of the category $\mathbf{ALG}k(0)\dots k(n)$ onto $\mathbf{MAB}k(0)\dots k(n)$.*

Proof. Put $m = k(1) + \dots + k(n)$.

If $\big(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\big)$ is an object of the category $\mathbf{ALG}k(0)\ldots k(n)$, then $Go\big(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\big)$ has nullary operations $(a_i,\ldots,a_i)\in A^n$ for any $i$ with $1\leqslant i\leqslant k(0)$ and the unary operation $\mathbf{un}[o_1,\ldots,o_m]$ that is binding with respect to the sequence $(k(1),\ldots,k(n))$ by Lemma 6. It follows that $Go$ maps the class of all objects of the category $\mathbf{ALG}k(0)\ldots k(n)$ into the class of all objects of the category $\mathbf{MAB}k(0)\ldots k(n)$. It is easy to see that $Go$ is an injective mapping.

If $\big(A^n,((a_i,\ldots,a_i))_{1\leqslant i\leqslant k(0)},o\big)$ is an object of the category $\mathbf{MAB}k(0)\ldots k(n)$, then $a_i$ is a nullary operation on $A$ for any $i$ with $1\leqslant i\leqslant k(0)$. By Lemma 6 there exist nonnullary operations $o_1,\ldots,o_m$ on $A$ such that $\mathbf{un}[o_1,\ldots,o_m]=o$ and for any $j$ with $1\leqslant j\leqslant m$ the arity $r(j)$ of the operation $o_j$ equals the least integer $i$ satisfying $1\leqslant j\leqslant k(1)+\ldots+k(i)$. Hence $Go\big(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\big)=\big(A^n,((a_i,\ldots,a_i))_{1\leqslant i\leqslant k(0)},o\big)$. It follows that $Go$ is a bijection.

If $h$ is a homomorphism of $\big(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\big)$ into $\big(A',(a'_i)_{1\leqslant i\leqslant k(0)},(o'_i)_{1\leqslant i\leqslant m}\big)$ where these algebras are objects in $\mathbf{ALG}k(0)\ldots k(n)$, then $Gr(h)$ is an $n$-decomposable homomorphism of $Go\big(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\big)$ into $\big(Go(A',(a'_i)_{1\leqslant i\leqslant k(0)},(o'_i)_{1\leqslant i\leqslant m}\big)$ and $Gr$ is a bijection of the class of all morphisms in $\mathbf{ALG}k(0)\ldots k(n)$ onto the class of all morphisms in $\mathbf{MAB}k(0)\ldots k(n)$ by Lemma 1 and Lemma 5. Similarly as in Theorem 1 we prove that the condition

$$Gr(1_{(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m})})=1_{Go(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m})}$$

is satisfied for any object $\big(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\big)$ in $\mathbf{ALG}k(0)\ldots k(n)$. Finally, if $\big(A,(a_i)_{1\leqslant i\leqslant k(0)},(o_i)_{1\leqslant i\leqslant m}\big)$, $\big(A',(a'_i)_{1\leqslant i\leqslant k(0)},(o'_i)_{1\leqslant i\leqslant m}\big)$, $\big(A'',(a''_i)_{1\leqslant i\leqslant k(0)},(o''_i)_{1\leqslant i\leqslant m}\big)$ are objects in $\mathbf{ALG}k(0)\ldots k(n)$, if $h$ is a homomorphism of the first algebra into the second and $k$ a homomorphism of the second algebra into the third, then $k\circ h$ is a homomorphism of the first algebra into the third and $Gr(k\circ h)=Gr(k)\circ Gr(h)$ holds.

It follows that $G$ is a functor and, therefore, an isomorphism. $\square$

**Remark 1.** It is possible to present a construction similar to Construction 1 where the category $\mathbf{MAB}k(0)\ldots k(n)$ substitutes $\mathbf{MAA}k(0)\ldots k(n)$. The details are left to the reader.

**Example 3.** Let $\big(A,a_1,(o_i)_{1\leqslant i\leqslant 2}\big)$ be an algebra where $A=\{a,b,c\}$, $a_1=a$, $o_1$ is a binary and $o_2$ a ternary operation given by the following tables:

| $o_1$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $c$ | $b$ |
| $b$ | $b$ | $b$ | $a$ |
| $c$ | $c$ | $c$ | $c$ |

| $xyz$ | $aaa$ | $aab$ | $aac$ | $aba$ | $abb$ | $abc$ | $aca$ | $acb$ | $acc$ |
|---|---|---|---|---|---|---|---|---|---|
| $o_2(xyz)$ | $a$ | $b$ | $c$ | $a$ | $b$ | $c$ | $a$ | $b$ | $c$ |

| $xyz$ | $baa$ | $bab$ | $bac$ | $bba$ | $bbb$ | $bbc$ | $bca$ | $bcb$ | $bcc$ |
|---|---|---|---|---|---|---|---|---|---|
| $o_2(xyz)$ | $c$ | $b$ | $a$ | $c$ | $b$ | $a$ | $c$ | $b$ | $a$ |

| $xyz$ | $caa$ | $cab$ | $cac$ | $cba$ | $cbb$ | $cbc$ | $cca$ | $ccb$ | $ccc$ |
|---|---|---|---|---|---|---|---|---|---|
| $o_2(xyz)$ | $c$ | $c$ | $c$ | $c$ | $c$ | $c$ | $c$ | $c$ | $c$ |

Thus $k(0) = 1$, $k(1) = 0$, $k(2) = 1$, $k(3) = 1$, $m = k(1) + k(2) + k(3) = 2 < 3 = n$. We construct the mono-unary algebra $(A^3, \mathbf{un}[o_1, o_2])$. The operation $\mathbf{un}[o_1, o_2]$ has the following table:

| $xyz$ | $aaa$ | $aab$ | $aac$ | $aba$ | $abb$ | $abc$ | $aca$ | $acb$ | $acc$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{un}[o_1, o_2](xyz)$ | $aaa$ | $bab$ | $cac$ | $aca$ | $bcb$ | $ccc$ | $aba$ | $bbb$ | $cbc$ |

| $xyz$ | $baa$ | $bab$ | $bac$ | $bba$ | $bbb$ | $bbc$ | $bca$ | $bcb$ | $bcc$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{un}[o_1, o_2](xyz)$ | $abc$ | $bbb$ | $cba$ | $abc$ | $bbb$ | $cba$ | $aac$ | $bab$ | $caa$ |

| $xyz$ | $caa$ | $cab$ | $cac$ | $cba$ | $cbb$ | $cbc$ | $cca$ | $ccb$ | $ccc$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{un}[o_1, o_2](xyz)$ | $acc$ | $bcc$ | $ccc$ | $acc$ | $bcc$ | $ccc$ | $acc$ | $bcc$ | $ccc$ |

This algebra can be represented by the graph in Fig. 2.

We construct all endomorphisms of the algebra $(A, a_1, (o_i)_{1 \leqslant i \leqslant 2})$. If $h$ is an arbitrary endomorphism of this algebra then $h(a) = a$.

Let $f = h^3$ be a 3-decomposable endomorphism of the algebra $(A^3, aaa, \mathbf{un}[o_1, o_2])$. Since $f(aaa) = aaa$, we have the following possibilities for $bbb$:

(a) $f(bbb) = aaa$; (b) $f(bbb) = bbb$; (c) $f(bbb) = ccc$.

If (a) occurs, then $h(b) = a$. Furthermore, $\mathbf{un}[o_1, o_2](acb) = bbb$ implies that $aaa = f(bbb) = f(\mathbf{un}[o_1, o_2](acb)) = \mathbf{un}[o_1, o_2](f(acb))$, which implies that $f(acb) = aaa$ and, hence, $h(c) = a$. Clearly, the mapping $h$ defined by $h(x) = a$ for any $x \in A$ is an endomorphism of $(A, a_1, (o_i)_{1 \leqslant i \leqslant 2})$.

If (b) occurs, we have $h(a) = a$, $h(b) = b$. If $h(c) = a$, then $f(acb) = h(a)h(c)h(b) = aab$ and, therefore, $bab = \mathbf{un}[o_1, o_2](aab) = \mathbf{un}[o_1, o_2](f(acb)) = f(\mathbf{un}[o_1, o_2](acb)) = f(bbb) = bbb$, which is a contradiction. Suppose $h(c) = b$. Then $f(acb) = abb$ and, consequently, $bcb = \mathbf{un}[o_1, o_2](abb) = \mathbf{un}[o_1, o_2](f(acb)) = f(\mathbf{un}[o_1, o_2](acb)) = f(bbb) = bbb$, which is a contradiction. Thus the only possible case is $h(c) = c$ and $h = \mathrm{id}_A$ which is, clearly, an endomorphism of the given algebra.

If (c) occurs, we have $h(a) = a$, $h(b) = c$. It follows that $f(bba) = cca$ and hence $acc = \mathbf{un}[o_1, o_2](cca) = \mathbf{un}[o_1, o_2](f(bba)) = f(\mathbf{un}[o_1, o_2](bba)) = f(abc) = ach(c)$, which implies that $c = h(c)$. Therefore $ccc = f(bcb)$ and, hence, $ccc = \mathbf{un}[o_1, o_2](ccc) = \mathbf{un}[o_1, o_2](f(bcb)) = f(\mathbf{un}[o_1, o_2](bcb)) = f(bab) = h(b)h(a)h(b) = cac$, which is a contradiction.

Hence, our algebra $(A, a_1, (o_i)_{1 \leqslant i \leqslant 2})$ has only two endomorphisms: $h_1(x) = a$ for any $x \in A$ and $h_2 = \mathrm{id}_A$.

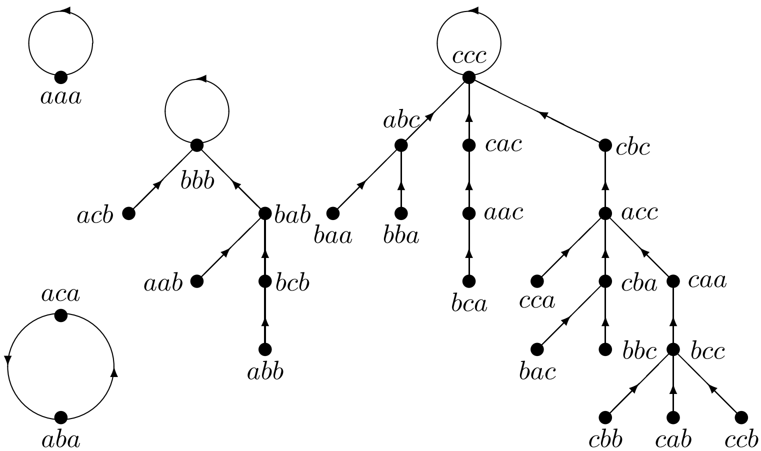


Fig. 2.

In order to demonstrate the use of the construction described in Example 1(b) we proceed as follows. From this construction it follows that a cycle must be assigned to the cycle formed of elements $aca$, $aba$ by a homomorphism $f$.

If the assigned cycle is formed of the only element $aaa$, then, clearly, $f = h^3$ entails $h(a)h(c)h(a) = f(aca) = aaa$ and, therefore, $h(a) = a$, $h(c) = a$. Similarly, $h(a)h(b)h(a) = f(aba) = aaa$ implies $h(b) = a$. It is easy to see that $f(xyz) = aaa$ for any $xyz \in A^3$ is a 3-decomposable endomorphism of the algebra $(A^3, aaa, \mathbf{un}[o_1, o_2])$. Clearly, the mapping $h(x) = a$ for any $x \in A$ is an endomorphism of the algebra $(A, a_1, (o_i)_{1\leqslant i\leqslant 2})$.

Let the assigned cycle be formed by the element $bbb$. Then, similarly as above, we obtain $h(x) = b$ for any $x \in A$ and, therefore, $f(xyz) = bbb$ for any $xyz \in A^3$. Clearly, $f$ is not an endomorphism of the algebra $(A^3, aaa, \mathbf{un}[o_1, o_2])$ because the condition $f(aaa) = aaa$ is violated. Similarly, we obtain no endomorphism of the algebra $(A^3, aaa, \mathbf{un}[o_1, o_2])$ if the cycle containing the only element $ccc$ is assigned to the cycle $\{aba, aca\}$.

The last possibility is to assign the cycle $\{aba, aca\}$ to $\{aba, aca\}$. Putting $f(aba) = aca$, $f(aca) = aba$, we have $h(a) = a$, $h(b) = c$, $h(c) = b$. Clearly, the elements $bba$, $cac$ are in the same component of the algebra $(A^3, \mathbf{un}[o_1, o_2])$ while $f(bba) = cca$, $f(cac) = bab$ are in different components. It follows that $f$ is no endomorphism of the algebra $(A^3, \mathbf{un}[o_1, o_2])$ and—a fortiori—no endomorphism of $(A^3, aaa, \mathbf{un}[o_1, o_2])$. Thus, we must define $f(aba) = aba$, $f(aca) = aca$, which leads us to $h = \mathrm{id}_A$.

**Remark 2.** Constructions presented in [12] and [13] may be regarded as particular cases of the construction described in this section. Taking $k(0) = k(1) = 0$, $k(2) = 1$ we obtain $m = 1$, $n = 2$ and the category **ALG**001 is the category of all groupoids.

It is easy to see that the functor $G$ described here is not the only isomorphism of $\mathbf{ALG}001$ onto a category of mono-unary algebras. Thus, our constructions present only one case of several possibilities.

## 5. Applications

Homomorphisms of algebras play an important role in various problems. We present such a problem in computer science.

Let $S$, $V$ be sets such that $S \cap V = \emptyset$ where $V$ is finite. Suppose that $o_a$ is a unary operation on the set $S$ for any $a \in V$. For any $s \in S$ and any $a \in V$ define $f(s,a) = o_a(s)$. Furthermore, let $s_0 \in S$ be a fixed element and $F \subseteq S$ a set. Then the ordered quintuple $(S, V, f, s_0, F)$ is said to be an *acceptor*; elements in $S$ are called *states*, $s_0$ is the *initial state*, $F$ is the set of *final states*. The set $V$ is said to be an *alphabet*, its elements are referred to as *symbols*, $f$ appears under the name of *transition function*. Clearly, $(S, s_0, (o_a)_{a\in V})$ is an algebra where $s_0$ is a nullary operation and the numbering of nonnullary operations is replaced by indexing using symbols as indices. It is easy to see that the quadruple $(S, V, f, s_0)$ makes it possible to construct the algebra $(S, s_0, (o_a)_{a\in V})$ and vice versa.

Finite sequences of elements in $V$ are called *strings*. Thus, if $x$ is a nonempty string over $V$, there exist an integer $n \geqslant 1$ and elements $a_1, \dots, a_n$ in $V$ such that $x = (a_1, \dots, a_n)$; it is usual to write strings without parantheses and commas, hence $x = a_1 \dots a_n$. This string can be identified with a mapping of the set $\{i \in \mathbb{N};\ 1 \leqslant i \leqslant n\}$ into $V$ where $\mathbb{N}$ denotes the set of all nonnegative integers. Putting $n = 0$ we obtain a mapping of the empty set $\{i \in \mathbb{N};\ 1 \leqslant i \leqslant 0\}$ into $V$ which is the empty set. The string identified with $\emptyset$ is denoted by $\Lambda$ and is referred to as the *empty string*. The set of all strings over $V$ is denoted by $V^*$.

An acceptor $(S, V, f, s_0, F)$ is a device that either accepts or rejects any string over $V$, i.e., for any $x \in V^*$, the acceptor decides whether $x$ is accepted or not. The work of this acceptor is as follows. Suppose that $x \in V^*$ is given. Then $x$ is *accepted* by $(S, V, f, s_0, F)$ if one of the following cases occurs:

(1) $x = \Lambda$, $s_0 \in F$.

(2) $x = a_1 \dots a_n$ where $n \geqslant 1$, $a_1, \dots, a_n \in V$. Defining $s_i = o_{a_i}(s_{i-1})$ by induction for any $i$ with $1 \leqslant i \leqslant n$ we obtain $s_n \in F$.

The set of all strings accepted by an acceptor $A$ is referred to as the *language accepted by* $A$; it will be denoted by $\mathbf{L}(A)$. More about acceptors and their languages may be found, e.g., in [17], [3], [2].

As was said above, homomorphisms of algebras play an important role in a problem concerning languages accepted by acceptors. More exactly, let $A = (S, V, f, s_0, F)$, $A' = (S', V, f', s'_0, F')$ be acceptors. A mapping $h$ of $S$ into $S'$ is said to be a *homomorphism* of the acceptor $A$ into $A'$ if it is a homomorphism of the algebra $(S, s_0, (o_a)_{a\in V})$ into $(S', s'_0, (o'_a)_{a\in V})$ and $h^{-1}(F') = F$. Clearly, $h(F) = h(h^{-1}(F')) \subseteq F'$.

**Theorem 3.** *Let $A = (S, V, f, s_0, F)$, $A' = (S', V, f', s'_0, F')$ be acceptors, $h$ a homomorphism of $A$ into $A'$. Then both the acceptors accept the same language.*

Proof. (1) Let a string $x \in V^*$ be accepted by the first acceptor $A$.

If $x = \Lambda$ then $s_0 \in F$, which implies $s'_0 = h(s_0) \in h(F) \subseteq F'$ and, therefore, $\Lambda$ is accepted by $A'$.

Suppose that $x = a_1 \dots a_n$ where $n \geqslant 1$ and $a_1, \dots, a_n \in V$. Then we have defined $s_j = o_{a_j}(s_{j-1})$ for any $j$ with $1 \leqslant j \leqslant n$ and we have obtained $s_n \in F$. Similarly, $s'_j = o'_{a_j}(s'_{j-1})$ holds for any $j$ satisfying $1 \leqslant j \leqslant n$. Since $h$ is a homomorphism, we obtain $s'_0 = h(s_0)$. If we suppose $j > 0$, $s'_{j-1} = h(s_{j-1})$, then $s'_j = o'_{a_j}(s'_{j-1}) = o'_{a_j}(h(s_{j-1})) = h(o_{a_j}(s_{j-1})) = h(s_j)$. Hence $s'_j = h(s_j)$ for any $j$ with $0 \leqslant j \leqslant n$ follows by induction. In particular, we obtain $s'_n = h(s_n) \in h(F) \subseteq F'$. Thus, $A'$ accepts $x$.

(2) Suppose that $x \in V^*$ is accepted by $A'$.

If $x = \Lambda$, then $s'_0 \in F'$ and, therefore, $h(s_0) = s'_0 \in F'$, which means $s_0 \in h^{-1}(F') = F$. Consequently, $A$ accepts $\Lambda$.

Suppose that $x = a_1 \dots a_n$ is accepted by $A'$ where $n \geqslant 1$ and $a_1, \dots, a_n \in V$. We have $s_j = o_{a_j}(s_{j-1})$, $s'_j = o'_{a_j}(s'_{j-1})$ for any $j$ with $1 \leqslant j \leqslant n$. Similarly as in (1), we prove that $h(s_j) = s'_j$ holds for any $j$ with $1 \leqslant j \leqslant n$. This yields, in particular, $h(s_n) = s'_n \in F'$ and, hence, $s_n \in h^{-1}(F') = F$. Thus, $A$ accepts $x$. □

**Example 4.** Let $A = (S, V, f, s_0, F)$, $A' = (S', V, f', s'_0, F')$ be acceptors where $S = \{1, 2, 3\}$, $S' = \{1', 2', 3', 4', 5'\}$, $V = \{a, b\}$, $s_0 = 1$, $s'_0 = 1'$, $F = \{1\}$, $F' = \{1'\}$ and the functions $f$, $f'$ are defined by the following tables:

| $f$ | $a$ | $b$ |
|---|---|---|
| 1 | 2 | 3 |
| 2 | 3 | 1 |
| 3 | 3 | 3 |

| $f'$ | $a$ | $b$ |
|---|---|---|
| $1'$ | $2'$ | $3'$ |
| $2'$ | $3'$ | $1'$ |
| $3'$ | $3'$ | $3'$ |
| $4'$ | $3'$ | $5'$ |
| $5'$ | $4'$ | $3'$ |

Our problem may be formulated as follows: Do the acceptors $A$, $A'$ accept the same language? In order to answer this question we try to construct a homomorphism $g$ of $A$ into $A'$. Using Construction 1, we construct the algebras $\big(S^2, (s_0, s_0), \mathbf{um}[o_a, o_b]\big)$ and $\big((S')^2, (s'_0, s'_0), \mathbf{um}[o'_a, o'_b]\big)$. The operations of these algebras are described by the following tables (we write again $xy$ for $(x, y)$):

| $xy$ | 11 | 12 | 13 | 21 | 22 | 23 | 31 | 32 | 33 |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{um}[o_a, o_b](xy)$ | 23 | 23 | 23 | 31 | 31 | 31 | 33 | 33 | 33 |

| $xy$ | $1'1'$ | $1'2'$ | $1'3'$ | $1'4'$ | $1'5'$ | $2'1'$ | $2'2'$ | $2'3'$ | $2'4'$ | $2'5'$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{um}[o'_a, o'_b](xy)$ | $2'3'$ | $2'3'$ | $2'3'$ | $2'3'$ | $2'3'$ | $3'1'$ | $3'1'$ | $3'1'$ | $3'1'$ | $3'1'$ |

| $xy$ | $3'1'$ | $3'2'$ | $3'3'$ | $3'4'$ | $3'5'$ | $4'1'$ | $4'2'$ | $4'3'$ | $4'4'$ | $4'5'$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{um}[o'_a, o'_b](xy)$ | $3'3'$ | $3'3'$ | $3'3'$ | $3'3'$ | $3'3'$ | $3'5'$ | $3'5'$ | $3'5'$ | $3'5'$ | $3'5'$ |

| $xy$ | $5'1'$ | $5'2'$ | $5'3'$ | $5'4'$ | $5'5'$ |
|---|---|---|---|---|---|
| $\mathbf{um}[o'_a, o'_b](xy)$ | $4'3'$ | $4'3'$ | $4'3'$ | $4'3'$ | $4'3'$ |

The mono-unary algebras $(S^2, \mathbf{um}[o_a, o_b])$ and $((S')^2, \mathbf{um}[o'_a, o'_b])$ are represented by Fig. 3 and Fig. 4.

Let $h$ be a 2-decomposable homomorphism of the algebra $(S^2, \mathbf{um}[o_a, o_b])$ into $((S')^2, \mathbf{um}[o'_a, o'_b])$. Then there exists a mapping $g$ of $S$ into $S'$ such that $h = g^2$. Using the construction described in Example 1(b) we obtain that the cycle having the element $3'3'$ is assigned to the cycle having the element 33 by any homomorphism $h$ of the algebra $(S^2, \mathbf{um}[o_a, o_b])$ into $((S')^2, \mathbf{um}[o'_a, o'_b])$. Thus, we have $h(\mathbf{um}[o_a, o_b])(3, 3) = (3', 3')$ and hence, $g(3) = 3'$. Since $g(1) = g(s_0) = s'_0 = 1'$, the only problem is to define $g(2)$. We have $h(3, 1) = (g(3), g(1)) = (3', 1')$, $\mathbf{um}[o_a, o_b](2, 2) = (3, 1)$. It follows that $\mathbf{um}[o'_a, o'_b](h(2, 2)) = h(\mathbf{um}[o_a, o_b])(2, 2) = h(3, 1) = (3', 1')$. Consequently, $h(2, 2) = (x, x) \in (S')^2$, $\mathbf{um}[o'_a, o'_b](x, x) = (3', 1')$. The only element $x \in S'$ with this property is $x = 2'$. Hence $h(2, 2) = (2', 2')$ and, therefore, $g(2) = 2'$.

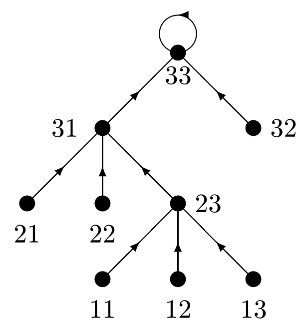


Fig. 3.

It follows that $g$ is a homomorphism of the algebra $(S, s_0, (o_x)_{x \in V})$ into the algebra $(S', s'_0, (o'_x)_{x \in V})$. Furthermore, $g^{-1}(F') = g^{-1}(\{1'\}) = \{1\}$ and thus, $g$ is a homomorphism of $A$ into $A'$. By Theorem 3, the acceptors $A$ and $A'$ accept the same language. Clearly, $\mathbf{L}(A') = \mathbf{L}(A) = \{(ab)^n;\ n \in \mathbb{N}\}$.

**Remark 3.** We present some other applications.

(a) If $\big(A, (a_i)_{1 \leqslant i \leqslant p}, (o_i)_{1 \leqslant i \leqslant m}\big)$, $(A', (a'_i)_{1 \leqslant i \leqslant p}, (o'_i)_{1 \leqslant i \leqslant m})$ are algebras, the above explained methods enable us to state whether there exists an isomorphism of the

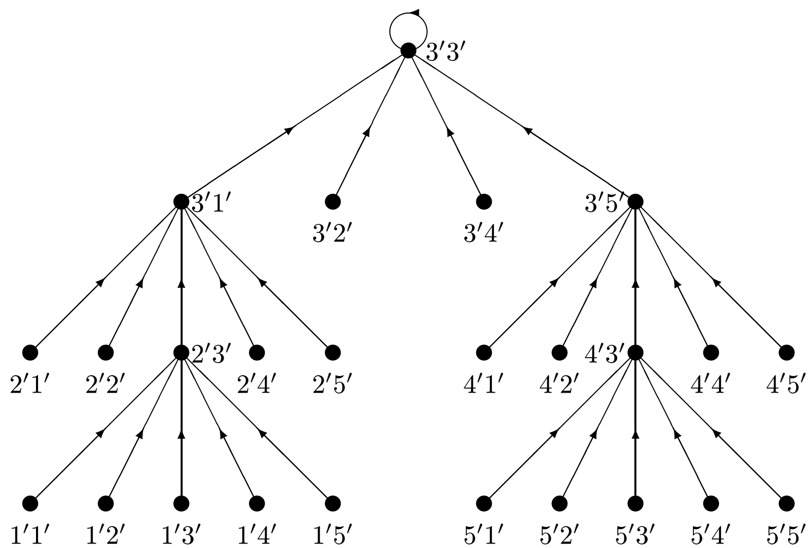


Fig. 4.

first algebra into the latter, i.e., whether the first algebra may be embedded into the other. As an example we may choose the algebras $(S, s_0, (o_a)_{a\in V})$, $(S', s'_0, (o'_a)_{a\in V})$ and the isomorphism $g$ constructed in Example 4.

(b) In [11] a construction of all strong homomorphisms of a relational structure with one relation into a relational structure of the same type is described. The fundamental step of this construction consists in lifting the relations to the power sets which creates operations on power sets. The strong homomorphisms can be obtained from homomorphisms between algebras defined on power sets.

Repeating this procedure for a relational structure with more than one relation, we obtain an algebra with more than one operation. Hence, we need homomorphisms between such algebras, which is ensured by methods explained in the present paper.

## 6. Concluding remarks

The problem formulated by O. Borůvka about 1950 stimulated investigation of mono-unary algebras (cf. a survey of results in [8]). Constructions of homomorphisms between mono-unary algebras made it possible to construct homomorphisms of relational structures, see [9], [10], [11], [14]. Since relational structures are in a close connection to hyperstructures, applications of mono-unary algebras also in this region can be expected. All these papers together with [12], [13], and the present one prove that mono-unary algebras are very useful.

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