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BEHAVIOR OF INVARIANT METRICS NEAR CONVEXIFIABLE BOUNDARY POINTS

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Abstract. The behaviour of the Carathéodory, Kobayashi and Azukawa metrics near convex boundary points of domains in $\mathbb{C}^n$ is studied.

Keywords: Carathéodory metric, Kobayashi metric, Azukawa metric, convexifiable point

MSC 2000: 32F45

1. Introduction

Let D be a domain in $\mathbb{C}^n$. Denote by $\mathcal{O}(D,\Delta)$ and $\mathcal{O}(\Delta,D)$ the spaces of all holomorphic mappings from D into the unit disc $\Delta\subset\mathbb{C}$ and from Δ to D, respectively. Let $z\in D$ and $X\in\mathbb{C}^n$. The Carathéodory and Kobayashi metrics are defined by

$$C_D(z,X)=\sup\{|(Xf)(z)|\colon f\in\mathcal{O}(D,\Delta)\},$$
$$K_D(z,X)=\inf\{|r|^{-1}\colon \exists f\in\mathcal{O}(\Delta,D),\ f(0)=z,\ f'(0)=rX\}.$$

Denote by $\mathrm{PSH}(D,\mathbb{R}^-)$ the space of all negative plurisubharmonic functions on D. The pluricomplex Green function [5] and the Azukawa metric [1] are defined by

$$g_D(z,w)=\sup\{u(w)\colon u\in\mathrm{PSH}(D,\mathbb{R}^-),\ u(\cdot)\leqslant\log\|\cdot-z\|+O_u(1)\},$$
$$A_D(z,X)=\limsup_{\lambda\not\to 0}\frac{\exp g(z,z+\lambda X)}{|\lambda|}.$$

It is clear that $C_D(z,X)\leqslant A_D(z,X)\leqslant K_D(z,X)$.

Let z_0 be a C^1-smooth boundary point of D and X a continuous $(1,0)$ vector field at z_0. Denote by X_N the projection of X_{z_0} on the complex normal to ∂D at

z_0 and set $d(z) = \operatorname{dist}(z, \partial D)$. Graham [4] showed that if D is a bounded strongly pseudoconvex domain then

$$\lim_{z\to z_0} C_D(z, X_z)d(z) = \lim_{z\to z_0} K_D(z, X_z)d(z) = \tfrac{1}{2}\|X_N\|.$$

The main purpose of this note is to extend the Graham result for a convex boundary points.

Theorem 1. *Let z_0 be a convex C^1-smooth boundary point of a domain $D \subset \mathbb{C}^n$. Assume that ∂D does not contain any germ of complex line through z_0. Then*

$$\lim_{z\to z_0} K_D(z, X_z)d(z) = \lim_{z\to z_0} A_D(z, X_z)d(z) = \tfrac{1}{2}\|X_N\|. \tag{1}$$

Theorem 2. *Let z_0 be a convex boundary point of a smooth bounded pseudoconvex domain $D \subset \mathbb{C}^n$. Assume that ∂D does not contain any segment with endpoint at z_0. Then*

$$\lim_{z\to z_0} C_D(z, X_z)d(z) = \tfrac{1}{2}\|X_N\|.$$

Remark. If the boundary of a bounded domain is real-analytic, then it does not contain any real segment.

Note that, by the Lempert theorem [7], the Carathéodory and Kobayashi metrics of a convex domain coincide. This, together with the arguments given in the proof of Theorem 1, shows that

$$\lim_{z\to z_0} C_D(z, X_z)d(z) = \lim_{z\to z_0} K_D(z, X_z)d(z) = \tfrac{1}{2}\|X_N\|$$

for any C^1-smooth boundary point z_0 of such a domain.

On the other hand, the following examples show that, in general, the condition for nonexistence of nontrivial holomorphic curves in Theorem 1 is essential.

Proposition 3.

(a) *If G is a Cartesian product of n compact plane sets, then $K_{\mathbb{C}^n\setminus G} \equiv 0$.*

(b) *If $D = \Delta^2 \setminus \{z \in \mathbb{C}^2 : \operatorname{Re} z_1 \leqslant 0,\ |z_2| \leqslant \frac{1}{4}\}$, then*

$$\tfrac{1}{8}\|X_N\| \leqslant \liminf_{z\to 0} A_D(z, X_z)d(z) \leqslant \limsup_{z\to 0} K_D(z, X_z)d(z) \leqslant \tfrac{3}{8}\|X_N\|.$$

2. Proofs

P r o o f of Theorem 1. First, we shall prove that

$$\limsup_{z\to z_0} K_D(z,X_z)d(z) \leqslant \tfrac{1}{2}\|X_N\| \tag{2}$$

for any C^1-smooth boundary point z_0 of an arbitrary domain $D \Subset \mathbb{C}^n$.

It is well-known that for any point z close to z_0 there exists a point $\pi(z) \in \partial D$ such that $\lim_{z\to z_0} \pi(z) = z_0$, $\|z-\pi(z)\| = d(z)$ and z belongs to the real normal to ∂D at $\pi(z)$. Moreover, we may find orthonormal transformations Ψ_z for which:

(i) $\lim_{z\to z_0} \Psi_z = \Psi_{z_0}$;

(ii) the first coordinate v_1 of $\Phi_z(\cdot) = \Psi_z(\cdot - \pi(z))$ is the complex normal to the boundary of the domain $G_z = \Phi_z(D)$ at the point 0;

(iii) the ray $\operatorname{Re} v_1$ coincides with the interior normal to G_z at 0.

For any $\varepsilon > 0$, set

$$E_\varepsilon = \{v \in \mathbb{C}^n :\ \operatorname{Re} v_1 + \varepsilon\|v\| < 0\}.$$

Note that there are neighbourhoods U of z_0 and V of 0 such that $E_\varepsilon \cap V \subset G_z$ for any $z \in U$. Let $V_z = \{v \in \mathbb{C}^n :\ vd(z) \in V\}$, $v(z) = \Psi_z(z)$ and $Y_z = (\Psi_z)_* X_z$. Then

$$K_D(z,X_z) \leqslant K_{G_z}(v(z),Y_z) \leqslant K_{E_\varepsilon\cap V}(v(z),Y_z) = \frac{K_{E_\varepsilon\cap V_z}(-1,Y_z)}{d(z)}.$$

It is not difficult to prove that

$$\lim_{\varepsilon\to 0+}\lim_{z\to z_0} K_{E_\varepsilon\cap V_z}(-1,Y_z) = K_{E_0}(-1,Y_{z_0}) = \tfrac{1}{2}\|X_N\|$$

which implies (2).

Let now z_0 be a convex boundary point of a domain D such that ∂D does not contain any nontrivial holomorphic curve through z_0. Then there exists a bounded neihgbourhood U of z_0, for which the domain $F = D \cap U$ is convex. Using ideas from the proofs of Theorem 1 and Corollary 4 in [2], and Lemma 2.1.1 in [3], we shall prove that

$$\lim_{z\to z_0} \frac{A_D(z,X_z)}{A_F(z,X_z)} = 1 \tag{3}$$

which completes the proof of (1). Indeed, for $z \in F$ close to z_0 denote by H_z the half-space whose boundary is the real tangent hyperplane to F at $\pi(z)$ and which contains F. If $(X_z)_N$ is the projection of X_z on the complex normal to D at $\pi(z)$, then by (3) we have

$$\begin{aligned}\liminf_{z\to z_0} A_D(z,X_z)d(z) &= \liminf_{z\to z_0} A_F(z,X_z)d(z) \geqslant \lim_{z\to z_0} A_{H_z}(z,X_z)d(z)\\ &= \lim_{z\to z_0} \tfrac{1}{2}\|(X_z)_N\| = \tfrac{1}{2}\|X_n\|.\end{aligned}$$

To prove (3), note that, by Lempert's theorem [7], we have

$$g_F(z,w) = \inf\{\ln|\alpha| \colon \exists f \in \mathcal{O}(\Delta, F),\ \ f(0)=z,\ \ f(\alpha)=w\}.$$

Since F is a bounded convex domain whose bounadry does not contain any germ of complex line through z_0, it follows that z_0 is a peak point for F [9]. Although the statement is not explicitely stated in [9], the method of the proof of Proposition 2.4 in [9] gives this result. Then normal family arguments and the maximum principle imply that [3, 8]

$$\lim_{z\to z_0,\, w\in F\setminus V} g_F(z,w) = 0 \tag{4}$$

for any neighbourhood $V \subset U$ of z_0.

Shrinking V (if necessary), we may choose a positive number $\varepsilon > 0$ and another neighbourhood $W \subset V$ of z_0 such that if $\psi(w) = \varphi(w) + \log\|w - z_0\|$, $C = \sup\limits_{D\cap\partial U} \psi$, $c = 1 + \sup\limits_{D\cap\partial W} \psi$, then $\inf\limits_{D\cap\partial V} \psi \geqslant \max\{C, c\}$. Fix $z \in H = D \cap W$ and set $u(z) = \inf\limits_{w\in D\cap\partial W} g_F(z,w)$. It is easy to see that the function

$$v(z,w) = \begin{cases} g_F(z,w), & w \in H,\\ \max\{g_F(z,w), (c-\psi(w))u(z)\}, & w \in D\cap V\setminus W,\\ \max\{(c-\psi(w))d(z), (c-C)u(z)\}, & w\in F\setminus V,\\ (c-C)u(z), & w \in D\setminus U\end{cases}$$

is plurisubharmonic function in the second variable with logarithmic pole at z. We may assume that $\operatorname{diam} U \leqslant 1$. Then $v(z,w) < cu(z)$ and hence $g_D(z,w) \geqslant v(z,w) - cu(z)$. It follows from (4) that $\lim\limits_{z\to z_0} u(z) = 0$. Now, the equality $v(z,w) = g_F(z,w)$ for $w \in H$ shows that

$$\lim_{z\to z_0} \inf_{w\in H} (g_D(z,w) - g_F(z,w)) = 0$$

which implies (3). □

P r o o f of Theorem 2. In view of Theorem 1, it suffices to prove only the inequality

$$\liminf_{z\to z_0} C_D(z,X_z)d(z) \geqslant \tfrac{1}{2}\|X_N\|. \tag{5}$$

Let U be a neighbourhood of z_0, for which $G = D\cap U$ is a convex domain whose boundary does not contain any segment with endpoint at z_0. Then we may find a number $C_1 > 0$ and neighbourhoods $W \subset V \subset\subset U$ such that $\operatorname{dist}(G\setminus V, H_{\pi(z)}) > C_1$ for any $z \in D\cap W$, where $H_{\pi(z)}$ denotes the real tangent hyperplane to ∂D at $\pi(z)$. Let $p = \exp((\Phi_{z_0})_1)$, $f_z = ((\Phi_z)_1 + d(z))/((\Phi_z)_1 - d(z))$, and χ be a smooth cut-off function χ with $\chi\equiv 1$ on V and $\chi \equiv 0$ on $\mathbb{C}^n\setminus U$. For any $m\in\mathbb{N}$, set $g_{z,m} = \overline{\partial}(\chi f_z p^m)$ and extend trivially $g_{z,m}$ as a smooth $\overline{\partial}$-closed $(0,1)$ form on $\overline{D}$. By [6], there exists a smooth function $h_{z,m}$ on D with $\overline{\partial}h_{z,m} = g_{zm}$ and $\|h_{z,m}\|_{C^1(D)} \leqslant C_2\|g_{z,m}\|_{C^{n+1}(D)}$ for some constant $C_2 > 0$ which depends only on D.

Using the Leibniz formula, we obtain

$$\|g_{z,m}\|_{C^{n+1}(D)} \leqslant 4^{n+1}\|\overline{\partial}\chi\|_{C^{n+1}(\mathbb{C}^n)}\,\|f_z\|_{C^{n+1}(G\setminus V)}\|p^m\|_{C^{n+1}(G\setminus V)}.$$

The Cauchy inequalities show that

$$\|f_z\|_{C^{n+1}(G\setminus V)} \leqslant \frac{(n+1)!}{C_1^{n+1}}.$$

On the other hand, it is easy to see that

$$\|p^m\|_{C^{n+1}(G\setminus V)} \leqslant C_3 m^{n+1}\sup_{G\setminus V}|p|^m.$$

Since p is a peak function for G at z_0, it follows from the last four inequalities that for any $\varepsilon > 0$ there exists $m\in\mathbb{N}$ with $\|h_{z,m}\|_{C^1(D)} \leqslant \varepsilon$ if $z\in D\cap W$.

Then $\tilde{f}_z = \chi f_z p^m - h_{z,m}$ is a holomorphic function on D and $\sup\limits_D|\tilde{f}_z| \leqslant 1+\varepsilon$. Using that $f_z(z) = 0$ and $\chi\equiv 1$ on $V\ni z$, we get

$$(1+\varepsilon)C_D(z,X_z) \geqslant |X_z\tilde{f}_z| \geqslant \frac{|p(z)|^m\|X(z)_N\|}{2d(z)} - \varepsilon\|X_z\|.$$

Since $\lim\limits_{z\to z_0} p(z) = 1$, letting $z\to z_0$ and $\varepsilon\to 0+$, we obtain (5). □

P r o o f of Proposition 3. (a) For simplicity of the notations, we will consider only the case $n = 2$. The proof in the general case is analogous.

Let $G = G_1\times G_2$, $z = (z_1,z_2)\in\mathbb{C}^2\setminus G$ and $X = (X_1,X_2)\in\mathbb{C}^2$. We may assume that $z_1\in\mathbb{C}\setminus G_1$. Let $M = \max\limits_{t\in G_2}|t|$ and $\varepsilon > 0$ be such that $U := z_1 +$

$\varepsilon\Delta \in \mathbb{C} \setminus G_1$. Set $f(t) = (z_1 + trX_1, z_2 + trX_2 + t^2r^3)$ for $r > |X_1|s/(2\varepsilon)$ where $s = |X_2| + \sqrt{|X_2| + 4r(|z_2| + M)}$. Then

$$|f_2(t)| \leqslant M \Rightarrow r|tr|^2 - |X_2| \cdot |tr| \leqslant |z_2| + M \Rightarrow |tr| \leqslant \frac{s}{2r} \Rightarrow |trX_1| < \varepsilon$$

which shows $f_1(t) \in U$ and hence $f \in \mathcal{O}(\mathbb{C}, \mathbb{C}^2 \setminus G)$. It follows that $K_{\mathbb{C}^2\setminus G} \leqslant 1/r$ and, letting $r \to \infty$, we are done.

(b) To prove that

$$\tfrac{1}{8}\|X_N\| \leqslant \liminf_{z\to z_0} A_D(z, X_z)d(z), \tag{6}$$

let $z \in D$, $|z_2| \leqslant \frac{1}{4}$ and

$$f(z,w) = \frac{1}{1+|z_2|}\begin{cases} \max\{|w_2 - z_2|, (\frac{1}{4} - |z_2|)|(w_1 - z_1)/(w_1 + \overline{z}_1)|\}, & |w_2| \leqslant \frac{1}{4}, \\ |w_2 - z_2|, & \frac{1}{4} < |w_2| < 1. \end{cases}$$

Then $\log f$ is a negative plurisubharmonic function on D, with logarithmic pole at z and

$$\tfrac{1}{8}\|X_N\| = \lim_{z\to z_0}\Big(\operatorname{Re} z_1 \lim_{\lambda\not\to 0} \frac{f(z, z+\lambda X_z)}{|\lambda|}\Big)$$

if $X_N \neq 0$, which implies (6).

Finally, we will prove that

$$\limsup_{z\to z_0} K_D(z, X_z)d(z) \leqslant \tfrac{3}{8}\|X_N\|. \tag{7}$$

In view of Theorem 1, it suffices to consider the case when $X_N \neq 0$ and hence we may assume that $X_z = (1, X_z')$. Let $a > 1$, $0 \leqslant b < 2(2a-1)/(2a+1)$, $z \in D$ and $x := \operatorname{Re} z_1 > 0$. We have that $1 > B := |z_2| + x|X_z'|(2/a + b)$ for $\|z\| \ll 1$. Set

$$f(t) = \Big(z_1 + x\Big(\frac{a+t}{a-t} - 1 + bt\Big), z_2 + txX_z'\Big(\frac{2}{a} + b\Big) + t^2(1-B)\Big)$$

and $A = \frac{1}{2}\sqrt{(1+4B)/(1-B)}$. We shall verify that $f \in \mathcal{O}(\Delta, D)$. It is clear that $|f_2(t)| < 1$ for $t \in \Delta$. On the other hand, if $|z_1| \ll 1$, then

$$\frac{1 - |\operatorname{Im} z_1|}{x} \geqslant \frac{a+1}{a-1} + b = \sup_{t\in\Delta}\Big|\frac{a+t}{a-t} + bt\Big|$$

which implies $|f_1(t)| < 1$ for $t \in \Delta$. Since $\lim_{z\to 0} A = \frac{1}{2}$, we may assume that $bA < (a-A)/(a+A)$. Now, the equality $\inf_{|t|\leqslant A} \operatorname{Re}(a+t)/(a-t) = (a-A)/(a+A)$ shows that

$$|f_2(t)| \leqslant \frac{1}{4} \Rightarrow |t| \leqslant A \Rightarrow \operatorname{Re} f_1(t) > 0$$

which completes our verification that $f \in \mathcal{O}(\Delta, D)$. Since $f'(0) = x(2/a + b)X_z$, it follows that

$$\limsup_{z \to z_0} K_D(z, X_z)d(z) \leqslant \frac{a}{2 + ab}.$$

Letting $b \to 2(2a-1)/(2a+1)$ and $a \to 1$, we obtain (7). □

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