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CONVEX CHAINS IN A PSEUDO MV-ALGEBRA

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Abstract. For a pseudo MV-algebra $\mathscr{A}$ we denote by $\ell(\mathscr{A})$ the underlying lattice of $\mathscr{A}$. In the present paper we investigate the algebraic properties of maximal convex chains in $\ell(\mathscr{A})$ containing the element 0. We generalize a result of Dvurečenskij and Pulmannová.

Keywords: pseudo MV-algebra, convex chain, Archimedean property, direct product decomposition

MSC 2000: 06D35

1. Introduction

Convex chains in MV-algebras have been investigated in [8]; the results concerned the relations between convex chains in an MV-algebra $\mathscr{A}$ and direct product decompositions of $\mathscr{A}$.

The notion of a pseudo MV-algebra was introduced by Georgescu and Iorgulescu [5], [4], and by Rachůnek [9] (who applied the term 'non-commutative MV-algebra'); cf. also the forthcoming monograph [3] by Dvurečenskij and Pulmannová. We apply the terminology and the notation from [3] and [5].

To each pseudo MV-algebra $\mathscr{A}$ there corresponds a distributive lattice $\ell(\mathscr{A})$ such that the underlying sets of $\mathscr{A}$ and of $\ell(\mathscr{A})$ coincide.

In the present paper we prove that Theorem 2.4 of [8] on convex chains remains valid for pseudo MV-algebras. In the proof we apply a theorem from [7] dealing with direct product decompositions of pseudo MV-algebras.

The main result of Section 6.4.3 in [3] is the following theorem:

(A) Let $\mathscr{A}$ be a pseudo MV-algebra such that

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(i) the lattice $\ell(\mathscr{A})$ is a chain;

(ii) $\mathscr{A}$ is Archimedean.

Then $\mathscr{A}$ is an MV-algebra.

By applying [7], we sharpen Theorem (A) in proving that the condition (i) can be replaced by the weaker condition

(i_1) the lattice $\ell(\mathscr{A})$ is a direct product ot chains.

The substance of the assertion (A) consists in the fact that the operation $\oplus$ in $\mathscr{A}$ is commutative.

We prove the following result (for the terminology, cf. Section 2):

(B) Let X be a maximal convex chain in a pseudo MV-algebra with $0 \in X$. Suppose that each pair of nonzero elements of X is archimedean. Then

$$x_1 \oplus x_2 = x_2 \oplus x_1 \text{ for each } x_1, x_2 \in X.$$

If X is a maximal convex chain in a pseudo MV-algebra $\mathscr{A}$, then either (i) X is an underlying lattice of a pseudo MV-algebra, or (ii) X is a positive cone of a lattice ordered group.

2. Preliminaries

We recall some basic definitions and facts concerning pseudo MV-algebras (cf. [3], and also [5] (Chapter 6) or [7]).

Let A be a nonempty set and let $\mathscr{A} = (A, \oplus, ^-, ^\sim, 0, 1)$ be a structure of type (2,1,1,0 0). For each $x, y \in A$ we put

$$y \odot x = (x^- \oplus y^-)^\sim.$$

Assume that for each $x, y, z \in A$ the following axioms are satisfied:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(A2) $x \oplus 0 = 0 \oplus x = x$;

(A3) $x \oplus 1 = 1 \oplus x = 1$;

(A4) $1^\sim = 0$; $1^- = 0$;

(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;

(A6) $x \oplus x^\sim \odot y = y \oplus y^\sim \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x$;

(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;

(A8) $(x^-)^\sim = x$.

Then the structure $\mathscr{A}$ is defined to be a pseudo MV-algebra.

With respect to (A6) we remark that the expression

$$x \oplus x^\sim \odot y$$

is to be understood in the sense that it is equal to

$$x \oplus (x^{\sim} \odot y),$$

and similarly in further analogous situations.

In what follows we assume that $\mathscr{A}$ is a pseudo MV-algebra.

For $x, y \in A$ we put $x \leqslant y$ if $x^{-} \oplus y = 1$. Then the structure $\ell(\mathscr{A}) = (A; \leqslant)$ is a distributive lattice with the least element 0 and with the greatest element 1.

We consider a partial binary operation $+$ on A which is defined as follows (cf. [3], p. 427):

2.1. Definition Let $x, y \in A$**.** Then $x + y$ is defined if and only if $x \leqslant y^{-}$ and in this case we put

$$x + y = x \oplus y.$$

For lattice ordered groups we apply the notation as in [2]. In particular, the group operation in a lattice ordered group is denoted by the symbol $+$, though it is not assumed to be commutative.

The underlying lattice of a lattice ordered group G will be denoted by $\ell(G)$.

Suppose that G is a lattice ordered group with a strong unit u. Consider the interval $[0, u]$ of G. We denote $[0, u] = A_1$ and for $x, y \in A_1$ we put

$$x \oplus y = (x + y) \wedge u, \quad x^{-} = u - x, \quad x^{\sim} = -x + u, \quad 1 = u.$$

The algebraic structure $(A_1; \oplus, ^{-}, ^{\sim}, 0, 1)$ will be denoted by $\Gamma(G, u)$.

2.2. Proposition (cf. [5])**.** *If G is a lattice ordered group with a strong unit u, then $\Gamma(G, u)$ is a pseudo MV-algebra.*

We have now the operation $+$ in G; to avoid a confusion in the notation, let us denote the binary operation from 2.1 by the symbol $+_p$ instead of $+$.

The notion of a subalgebra of a pseudo MV-algebra is defined in the usual way.

2.3. Proposition (cf. [3], p. 443, Exercise 7)**.** *Let $\mathscr{A}$ be a pseudo MV-algebra. Then there exists a lattice ordered group G with a strong unit u such that $\mathscr{A}$ is a subalgebra of $\Gamma(G, u)$. Moreover, we have*

(i) *the lattice $\ell(\mathscr{A})$ is a sublattice of the lattice $\ell(G)$;*

(ii) *let $x, y \in A$; then $x +_p y$ exists iff $x + y \leqslant u$, and in this case $x +_p y = x + y$.*

Below we again apply the phrase "$x + y$ is defined in A (or exists in A)" meaning that $x + y \leqslant u$. Further, G is always as in 2.3.

Let $g \in G$, $n \in \mathbb{N}$, $g_i = g$ for $i = 1, 2, \ldots, n$. We denote

$$ng = g_1 + g_2 + \ldots + g_n.$$

If $a \in A$ and if $na \in A$ (i.e., $na \leqslant u$), then we say that na exists in $\mathscr{A}$.

A pseudo MV-algebra $\mathscr{A}$ is called Archimedean if, whenever $a \in A$ and na exists for each $n \in \mathbb{N}$, then $a = 0$.

A pair (g, g') of elements of A is called Archimedean if, whenever for each $n \in \mathbb{N}$ the element ng exists and $ng \leqslant g'$, then $g = 0$.

It is easy to verify that the following conditions for $\mathscr{A}$ are equivalent:

(i) $\mathscr{A}$ is Archimedean.

(ii) Each pair of nonzero elements of A is Archimedean.

3. Auxiliary results

In this section we apply the notation as in Section 2 with one distinction. Namely, for $x, y \in A$ with $x \leqslant y$ we put

$$[x, y] = \{z \in A \colon\ x \leqslant z \leqslant y\}.$$

In view of [3], p. 427 we have

3.1. Lemma. *Let $x, y \in A$. Then $x + y$ is defined iff $x \leqslant y^-$ iff $y \leqslant x^\sim$.*

3.2. Lemma (cf. [3], 6.4.5). *Let $x, y \in A$. Then $x \leqslant y$ if and only if there is an element $b \in A$ with $x + b = y$. In that case, b is uniquely determined.*

3.2.1. Lemma. *Let $x, y \in A$. Then $x \leqslant y$ if and only if there is an element $b_1 \in A$ with $b_1 + x = y$. In that case, b_1 is uniquely determined.*

Proof. The assertion 'if' is obvious. For proving the converse assertion it suffices to use the method of the proof of 6.4.5 in [3] and to apply the Axiom (A6) and 3.1. □

Under the notation as in G, we have

(1) $$b = -x + y, \quad b_1 = y - x.$$

3.3. Lemma. *Let $a \in A$. There exists $c \in A$ such that $c \leqslant a$ and $a + c = a \oplus a$.*

Proof. Since $a \oplus a = (a + a) \wedge u$, the relation $a \leqslant a \oplus a$ is valid in G. Hence according to 2.3, this relation holds in the lattice $\ell(\mathscr{A})$ as well. Then 3.2 yields that there is $c \in A$ such that $a + c = a \oplus a$. Thus in G we have $a + c \leqslant a + a$, whence $c \leqslant a$, and this inequality holds also in $\ell(\mathscr{A})$. □

3.4. Lemma. *Let a and c be as in 3.3. Further, let $t \in [0, c]$. Then $a + t$ exists in $\mathscr{A}$ and $a + t \in [a, a + c]$.*

Proof. Since $t \leqslant c$ and $a + c$ exists in $\mathscr{A}$, in view of 2.3 we conclude that $a + t$ also exists in $\mathscr{A}$. Further, we obviously have $a + t \leqslant a + c$. □

For each $t \in [0, c]$ we put $\varphi(t) = a + t$.

3.5. Lemma. *φ is an isomorphism of the lattice $[0, c]$ onto the lattice $[a, a + c]$.*

Proof. It is obvious that φ is a mapping of the set $[0, c]$ into the set $[a, a + c]$. If $t_1, t_2 \in [0, c]$ then

$$t_1 \leqslant t_2 \Leftrightarrow \varphi(t_1) \leqslant \varphi(t_2).$$

Let $z \in [a, a + c]$. Hence in view of 3.2 there exists $b \in A$ with $a + b = z$. Thus $a + b \leqslant a + c$, whence $b \leqslant c$. Then $\varphi(b) = z$ and so φ is an epimorphism. Therefore φ is an isomorphism of $[0, c]$ onto $[a, a + c]$. □

3.6. Corollary. *If an interval $[0, a]$ is a chain, then the interval $[a, a \oplus a]$ is a chain as well.*

3.7. Lemma. *If an interval $[0, a]$ is a chain, then the interval $[0, a \oplus a]$ is a chain as well.*

Proof. Assume that $[0, a]$ is a chain. By way of contradiction, suppose that the interval $[0, a \oplus a]$ fails to be a chain. Then in view of 3.6 there exists an element $b \in A$ such that a and b are incomparable and $b \leqslant a + c$. Put

$$a \wedge b = u_1, \quad a \vee b = v. \tag{2}$$

Hence $v \in [a, a + c]$. In view of 3.5 there exists $t \in A$ with $t \leqslant c$ such that

$$v = a + t \leqslant a + c.$$

Hence $t = -a + v$. The relations (2) yield

$$-a + v = -u_1 + b,$$

thus $b = u_1 + t$.

Since $t \leqslant a$, according to 3.2.1 there exists $u_2 \in A$ with $a = u_2 + t$.

Now, from the fact that a and b are incomparable we conclude that u_1 and u_2 are incomparable. Both u_1 and u_2 belong to the interval $[0, a]$, which is a chain; so we have arrived at a contradiction. □

3.8. Lemma. *Let $a, b \in A$. Suppose that both $[0, a]$ and $[0, b]$ are chains. Then either* (i) $a \wedge b = 0$, *or* (ii) *a and b are comparable.*

P r o o f. Assume that $a \wedge b = u_1 > 0$. In view of 3.2 there exist $x, y \in A$ such that

$$u_1 + x = a, \quad u_1 + y = b.$$

Hence $x = -u_1 + a$, $y = -u_1 + b$ and thus

$$x \wedge y = 0.$$

We have $u_1, x \in [0, a]$, hence

$$\text{either} \quad x < u_1, \quad \text{or} \quad u_1 \leqslant x.$$

Similarly we obtain that

$$\text{either} \quad y < u_1, \quad \text{or} \quad u_1 \leqslant y.$$

a) If $u_1 \leqslant x$ and $u_1 \leqslant y$, then $u_1 \leqslant x \wedge y = 0$, which is a contradiction.

b) Assume that $x < u_1$ and $u_1 \leqslant y$. Then both x and y belong to the interval $[0, b]$. Hence they cannot be incomparable. Therefore some of them is equal to 0. Then either $a = u_1$ or $b = u_1$. This yields that a and b are comparable.

c) The case $y < u_1$ and $u_1 \leqslant x$ is analogous to the case b).

d) Suppose that $x < u_1$ and $y < u_1$. Since $[0, u_1]$ is a chain, we conclude that x and y are comparable. Then some of these elements must be equal to 0. Hence, similarly as in b), the elements a and b are comparable. □

A chain X in $\ell(\mathscr{A})$ is convex if, whenever $x_1, x_2 \in X$, $y \in A$ and $x_1 \leqslant y \leqslant x_2$, then $y \in X$. A convex chain X in $\ell(\mathscr{A})$ is called maximal convex if, whenever Y is a convex chain in $\ell(\mathscr{A})$ with $X \subseteq Y$, then $X = Y$.

From Axiom of Choice we conclude that for each convex chain X in $\ell(\mathscr{A})$ there exists a maximal convex chain Y in $\ell(\mathscr{A})$ with $X \subseteq Y$. From this and from 3.8 we infer

3.9. Lemma. *Let X be a chain in $\ell(\mathscr{A})$ with $0 \in X$. Then there exists a unique maximal convex chain Y in $\ell(\mathscr{A})$ such that $X \subseteq Y$.*

3.10. Lemma. *Let X be a maximal convex chain in $\ell(\mathscr{A})$ with $0 \in X$. Then X is closed with respect to the operation $\oplus$.*

P r o o f. Let $a, b \in X$. Without loss of generality we can suppose that $b \leqslant a$. In view of 3.6 and 3.9 we obtain $a \oplus a \in X$. Further, $a \oplus b \leqslant a \oplus a$, thus $a \oplus b \in X$. □

We denote by $\mathscr{C}_m$ the set of all maximal convex chains in $\ell(\mathscr{A})$ containing the element 0.

3.11. Lemma. *Let $X \in \mathscr{C}_m$ and let x_1 be the greatest element of X. Then $x_1 \oplus x_1 = x_1$.*

Proof. We have $x_1 \leqslant x_1 \oplus x_1$. According to 3.10, $x_1 \oplus x_1 \in X$, hence $x_1 \oplus x_1 \leqslant x_1$. Thus $x_1 \oplus x_1 = x_1$. □

4. Direct product decompositions

The notion of the direct product of pseudo MV-algebras is defined in the usual way (cf., e.g., [6]). We apply the standard notation

$$\mathscr{A}_1 \times \mathscr{A}_2 \times \ldots \times \mathscr{A}_n \quad \text{or} \quad \prod_{i\in I} \mathscr{A}_i.$$

If φ is an isomorphism of a pseudo MV-algebra $\mathscr{A}$ onto $\prod_{i\in I} \mathscr{A}_i$, then we say that the relation

$$\varphi\colon\ \mathscr{A} \to \prod_{i\in I} \mathscr{A}_i \tag{1}$$

is a direct product decomposition of $\mathscr{A}$.

An analogous terminology will be used for lattices.

Suppose that L is a distributive lattice with the least element 0 and the greatest element u. Let a and b be complementary elements of L, i.e.,

$$a \wedge b = 0, \quad x \vee b = u.$$

The following assertion is well-known.

4.1. Lemma. *For each $x \in L$ let $\varphi(x) = (x \wedge a, x \wedge b)$. Then*

$$\varphi\colon\ L \to [0,a] \times [0,b]$$

is a direct product decomposition of the lattice L.

Again, let $\mathscr{A}$ and G be as in the previous sections.

4.2. Lemma (cf. [6], 4.2). *Assume that e is an element of A which has a complement in the lattice $\ell(\mathscr{A})$. Put $A_e=[0,e]$ and*

$$x^{-e}=-x+e, \quad x^{\sim e}=e-x.$$

Then the algebraic structure $\mathscr{A}_e=(A_e;\oplus,{}^{-e},{}^{\sim e},0,e)$ is a pseudo MV-algebra.

4.3. Corollary. *Let e be as in 4.2. Then $e\oplus e=e$.*

4.4. Lemma. *Let e be an element of A such that $e\oplus e=e$. Then e has a complement in the lattice $\ell(\mathscr{A})$.*

Proof. Denote $e\wedge e^-=p$. Thus $p\leqslant e^-$ and hence

$$e+p=e\oplus p\leqslant e\oplus e=e.$$

Therefore $p=0$. From the relation $e\wedge e^-=0$ and from [3], Proposition 1.20 we infer that

$$u=e\oplus e^-=e\vee e^-.$$

Hence e^- is a complement of the element e in $\ell(\mathscr{A})$. □

Let a,b and φ be as in 4.1. Further, let $\mathscr{A}_a$ and $\mathscr{A}_b$ be defined analogously to $\mathscr{A}_e$ in 4.2. Then we have

4.5. Proposition (cf. [7], 4.3). *The relation*

$$\varphi\colon \mathscr{A}\to\mathscr{A}_a\times\mathscr{A}_b$$

expresses a direct product decomposition of the pseudo MV-algebra $\mathscr{A}$.

If $\mathscr{A}_a$ is as in 4.5, then it is called a direct factor of $\mathscr{A}$.

Now let $\mathscr{C}_m$ be as in Section 3 and let $X\in\mathscr{C}_m$. Assume that X has a greatest element a. Then according to 3.11 and 4.4, the element a has a complement in the lattice $\ell(\mathscr{A})$. Hence in view of 4.2 we can construct the pseudo MV-algebra $\mathscr{A}_a$. If we put $\mathscr{A}_a=X_a$, then X is the underlying lattice of the pseudo MV-algebra X_a.

From 4.5 we conclude

4.6. Theorem. *Let $X\in\mathscr{C}_m$ such that X has the greatest element a. Then the pseudo MV-algebra X_a is a direct factor of the pseudo MV-algebra $\mathscr{A}$.*

It is obvious that each direct factor of a pseudo MV-algebra must have a greatest element. Therefore 4.6 is a generalization of Theorem 2.4 in [8] concerning direct product decompositions of MV-algebras.

Now let us suppose that the lattice $\ell(\mathscr{A})$ is a direct product of chains. It means that there are linearly ordered sets L_i $(i \in I)$ and a direct product decomposition

$$\varphi_1\colon\ L \to \prod_{i\in I} L_i, \tag{1}$$

where $L = \ell(\mathscr{A})$. Since the lattice L is bounded, all L_i must be bounded; let us denote by 0^i and 1^i the least element or the greatest element of L_i, respectively.

For $i \in I$ we denote by u^i the element of L such that

$$\varphi_1(u^i)_i = 1^i \quad \text{and} \quad \varphi_1(u^i)_j = 0^j \quad \text{if} \quad j \in I,\ \ j \neq i.$$

Further, put $A_i = [0, u^i]$. Then A_i is a lattice under the partial order induced by that from L. It is obvious that A_i is isomorphic to L_i.

Let $x \in L$. We put

$$\varphi(x) = (x \wedge u^i)_{i\in I}.$$

Applying (1) we obtain by simple steps

4.7. Lemma. *The relation*

$$\varphi\colon\ L \to \prod_{i\in I} A_i \tag{2}$$

is a direct product decomposition of the lattice L.

For each $i \in I$, the element u^i has a complement in the lattice L. Hence we can construct the pseudo MV-algebra $\mathscr{A}_{u^i}$ as in 4.2; the underlying lattice of $\mathscr{A}_{u^i}$ is equal to A_i.

Let $x, y, z \in A_i$. Then the validity of the relation $x \leqslant y$ in A_i is equivalent to the validity of this relation in $\ell(\mathscr{A})$. Similarly, $x \oplus y = z$ holds in $\mathscr{A}_{u^i}$ iff this equality holds in $\mathscr{A}$.

In view of the definition of the Archimedean pseudo MV-algebra we conclude

4.8. Lemma. *Suppose that the pseudo MV-algebra $\mathscr{A}$ is Archimedean. Then all $\mathscr{A}_{u^i}$ are Archimedean.*

4.9. Lemma. *Suppose that $\mathscr{A}$ is Archimedean. Then all $\mathscr{A}_{u^i}$ are commutative.*

P r o o f. This is a consequence of Theorem (A) (cf. Introduction) and of 4.8. □

Further, from 4.7 and Theorem 6.4 of [7] we obtain

4.10. Lemma. *The relation*

$$\psi\colon \mathscr{A} \to \prod_{i\in I} \mathscr{A}_{u^i}$$

is a direct product decomposition of the pseudo MV-algebra $\mathscr{A}$.

Summarizing, 4.9 and 4.10 yield

4.11. Theorem. *Let* $\mathscr{A}$ *be a pseudo MV-algebra such that*
(i_1) *the lattice* $\ell(\mathscr{A})$ *is a direct product of chains;*
(ii_1) $\mathscr{A}$ *is archimedean. Then* $\mathscr{A}$ *is commutative (i.e., it is an MV-algebra).*

This generalizes Theorem 6.4.3 in [3].

5. An alternative for elements of $\mathscr{C}_m$

Let $\mathscr{C}_m$ be as in Section 3 and let $X \in \mathscr{C}_m$. The investigation of the present section would be trivial in the case $X = \{0\}$; thus let us suppose that X has more than one element. Consider the following condition for X:
(α) There exists $a \in X$ and a positive integer n such that na is not defined in A.

We will deal separately with the case when α is valid and with the case when (α) does not hold.

a) First suppose that the condition (α) is satisfied. Then there exists the least positive integer n such that $n > 1$ and na is not defined in A for some $a \in X$.

Hence $(n-1)a$ is defined in A; denote $(n-1)a = b$. Then in G we have

$$b \leqslant b \oplus a < na.$$

Put

$$c_1 = -b + (b \oplus a), \quad c_2 = -(b \oplus a) + na.$$

Then $c_1 \geqslant 0$, $c_2 > 0$ and $c_1 \in A$. Further,

$$c_1 < -b + na = a,$$
$$c_1 + c_2 = a.$$

From these relations we obtain that c_2 belongs to A as well. Denote

$$z = (b \oplus a)^-, \quad p = (b \oplus a) \wedge z.$$

Suppose that $p > 0$. Put $q = p \wedge c_2$. Hence either $q = p$ or $q = c_2$ and thus $0 < q \leqslant c_2$. Moreover, $q \leqslant z$ and hence the element

$$(b \oplus a) + q$$

is defined in $\mathscr{A}$.

Since

$$(b \oplus a) + q \leqslant (b \oplus a) + c_2 = na,$$

we obtain

$$b \oplus a < (b \oplus a) + q \leqslant na \wedge u = ((n-1)a + a) \wedge u = (b + a) \wedge u = b \oplus a,$$

which is a contradiction.

Therefore we must have $p = 0$, hence

$$(b \oplus a) \wedge z = 0.$$

Then according to [5], Proposition 1.20 we have

$$(b \oplus a) \vee z = (b \oplus a) + z = (b \oplus a) \oplus z = u.$$

If $r \in A$, $b \oplus a < r$, then the distributivity of $\ell(\mathscr{A})$ yields that the elements

$$b \oplus a, \quad r \wedge z$$

are incomparable. Hence r cannot belong to X. We have proved

5.1. Lemma. *Let (α) be valid and let n, a be as in the condition (α). Then the element $x_0 = ((n-1)a) \oplus a$ is the greatest element of X.*

By applying 5.1 and 4.6 we conclude

5.2. Theorem. *Let (α) be valid and let x_0 be as in* 5.1. *Then X is the underlying lattice of the pseudo MV-algebra X_{x_0} and this is a direct factor of the pseudo MV-algebra $\mathscr{A}$.*

b) Now let us suppose that the condition (α) fails to be valid for the chain X. Hence for each $a \in X$ and each $n \in \mathbb{N}$ the element na is defined in A. Then we have

$$2a = a + a = a \oplus a, \quad 3a = a + a + a = a \oplus a \oplus a, \ldots$$

Thus from 3.10 we infer

5.3. Lemma. *Assume that* (α) *does not hold for* X. *Then for* $a \in X$, *all elements* na *belong to* X.

5.4. Lemma. *Assume that* (α) *does not hold for* X. *Let* $x_1, x_2 \in X$. *Then* $x_1 + x_2$ *is defined in* A *and* $x_1 + x_2 \in X$.

P r o o f. Without loss of generality we can suppose that $x_1 \leqslant x_2$. Then in view of 5.3, $2x_2$ exists in A and $2x_2 \in X$. Since $x_1 + x_2 \leqslant 2x_2 \leqslant u$, according to 2.3 we get $x_1 + x_2 \in A$ and $x_1 \oplus x_2 = x_1 + x_2$. We know that X is closed with respect to the operation $\oplus$; therefore $x_1 + x_2 \in X$. $\square$

5.5. Corollary. *Assume that* (α) *does not hold. Then* X *is a subsemigroup of the group* G.

Denote $Y = X \cup (-X)$. The set Y is partially ordered by the relation of partial order induced from G. Then Y is linearly ordered. Applying 5.5, by simple calculation we can verify that Y is closed with respect to the operation $+$. Thus we have

5.6. Lemma. Y *is an* ℓ*-subgroup of the lattice ordered group* G.

Summarizing, we obtain

5.7. Theorem. *Assume that the condition* (α) *does not hold for* X. *Then there exist an* ℓ*-subgroup* Y *of* G *such that* Y *is linearly ordered and* $Y^+ = X$.

The following example shows that there exist a pseudo MV-algebra $\mathscr{A}$ and a chain $X \in \mathscr{C}_m$ such that X does not satisfy the condition (α).

Example. Let X_1 be the additive group of all reals with the natural linear order and $Y_1 = X_1$. Put $G = X_1 \circ Y_1$, where $\circ$ denotes the operation of lexicographic product. Put $u = (1, 0)$. Then u is a strong unit in G and hence we can construct the pseudo MV-algebra $\mathscr{A} = \Gamma(G, u)$; in fact, $\mathscr{A}$ is an MV-algebra. Put

$$X = \{(0, y) \colon\ 0 \leqslant y \in Y\}.$$

Then X is a maximal convex chain in $\ell(\mathscr{A})$ with $0 \in X$, and X does not satisfy the condition (α).

Let (B) be as in Section 1.

P r o o f of (B). Let X be as in the assumption of (B). We distinguish two cases.

a) Suppose that X satisfies the condition (α). We apply 5.2. Under the notation from 5.2, X has the greatest element x_0. Since $\ell(X_{x_0}) = X$, in view of the remark at the end of Section 2 we conclude that the pseudo MV-algebra $\mathscr{A}$ is Archimedean. Hence (A) yields that the operation $\oplus$ in X is Abelian.

b) Suppose that X does not satisfy the condition (α). Then we can apply 5.7. The assumption of (B) implies that the linearly ordered group Y is Archimedean. It is well-known that each Archimedean lattice ordered group is Abelian. Hence for each $x_1, x_2 \in X$ we have

$$x_1 \oplus x_2 = x_1 + x_2 = x_2 + x_1 = x_2 \oplus x_1.$$

□

Added in Proof. This is a correction concerning Section 3 of the author's paper *State homomorphisms on MV-algebras.* Czechoslovak Math. J. *51(126)* (2001), 609–616. Lemma 3.2 of this paper is not correct; the author is indebted to A. Di Nola and M. Navara for this observation. In Section 3 it should be added the assumption that the state homomorphism m is, at the same time, an MV-homomorphism of $\mathscr{A}$ into $[0,1]$, and that $\mathscr{S}$ is the set of all morphisms with the mentioned properties.

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