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DOMINATION IN BIPARTITE GRAPHS AND IN THEIR COMPLEMENTS

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Abstract. The domatic numbers of a graph G and of its complement \overline{G} were studied by J. E. Dunbar, T. W. Haynes and M. A. Henning. They suggested four open problems. We will solve the following ones:

Characterize bipartite graphs G having $d(G) = d(\overline{G})$.

Further, we will present a partial solution to the problem:

Is it true that if G is a graph satisfying $d(G) = d(\overline{G})$, then $\gamma(G) = \gamma(\overline{G})$?

Finally, we prove an existence theorem concerning the total domatic number of a graph and of its complement.

Keywords: bipartite graph, complement of a graph, domatic number

MSC 2000: 05C69

We consider finite undirected graphs without loops and multiple edges. Mostly we treat bipartite graphs. The bipartition classes of such a graph will be denoted by P and Q and their cardinalities by p and q respectively; the notation will be chosen so that $p \ge q$. By $N_G(x)$ we denote the open neighbourhood of a vertex x in a graph G, i.e. the set of all vertices which are adjacent to x in G.

A subset D of the vertex set V(G) of a graph G is called dominating (or total dominating) in G, if for each $x \in V(G) - D$ (or for each $x \in V(G)$, respectively) there exists $y \in D$ adjacent to x. A domatic (or total domatic) partition of G is a partition of V(G), all of whose classes are dominating (or total dominating, respectively) sets in G. The domination number (or total domination number) of G is the minimum number of vertices of a dominating (or total dominating, respectively) set in G. The domatic [1] (or total domatic [2]) number of G is the maximum number of classes of a domatic, respectively) partition of G. The domination number

of G is denoted by $\gamma(G)$, its total domination number by $\gamma_t(G)$, its domatic number by d(G), its total domatic number by $d_t(G)$.

Before solving the first mentioned problem we exclude certain cases.

Lemma 1. Let G be a graph with an isolated vertex. Then $d(G) \neq d(\overline{G})$.

Proof. Let v be an isolated vertex in G. It is contained in all dominating sets in G and thus no two of them may be disjoint and d(G) = 1. In \overline{G} there exists the domatic partition $\{\{v\}, V(G) - \{v\}\}$ and thus $d(\overline{G}) = 2$.

If q = 1 for a bipartite graph G, then either G or \overline{G} has an isolated vertex. Therefore the following proposition holds.

Proposition. Let G be a bipartite graph in which one bipartition class consists of one element. Then $d(G) \neq d(\overline{G})$.

Lemma 2. Let G be a bipartite graph with bipartition classes P, Q, let p = |p|, q = |Q|, $p \ge q \ge 2$. Then $d(G) \le q \le d(\overline{G})$.

Proof. No proper subset of P or of Q is dominating in G. Therefore if D is a dominating set in G, then either D = P, or D = Q, or $D \cap P \neq 0$ and $D \cap Q \neq 0$. A domatic partition of G is either $\{P,Q\}$, therefore with two classes, or has the property that each of its classes has a non-empty intersection with Q and thus it has at most q classes; this implies $d(G) \leq q$. In the complement \overline{G} the sets P, Q induce complete subgraphs and therefore each union of a non-empty subset of P and a non-empty subset of Q is dominating in \overline{G} . We have $p \geq q$ and therefore there exists a partition $\{M_1, \ldots, M_q\}$ of P with Q classes. If $Q = \{y_1, \ldots, y_q\}$, we may take the partition $\{M_1 \cup \{y_1\}, \ldots, M_q \cup \{y_q\}\}$ of V(G) and this is a domatic partition of G. Therefore $q \leq d(\overline{G})$.

Now we prove a theorem.

Theorem 1. Let G be a bipartite graph without isolated vertices and with bipartition classes P, Q, let p = |P|, q = |Q|, $p \ge q \ge 2$. The equality $d(G) = d(\overline{G})$ holds if and only if the following conditions are satisfied:

- (i) The degree of each vertex of P in G is at least q 1.
- (ii) The number of vertices of P of degree q is greater than or equal to the number of vertices of Q of degree p.
- (iii) Either $p \leq 2q 1$, or there exists at least one vertex of Q of degree p.

Proof. Let the conditions (i), (ii), (iii) hold. Let y_1, \ldots, y_q be the vertices of Q. Let $M_0 = \{x \in P \mid N_G(x) = Q\}$ and $M_i = \{x \in P \mid y_i \notin N_G(x)\}$

for $i = 1, \ldots, q$. The condition (i) implies that the sets M_0, M_1, \ldots, M_q are pairwise disjoint; some of them may be empty. Let $J_0 = \{i \in \{1, \ldots, q\} \mid M_i = 0\},\$ $J_1 = \{i \in \{1, \ldots, q\} \mid M_i \neq 0\}$. For $i \in J_0$ the vertex x_i is adjacent to all vertices of P and its degree is p. By (ii) we have $|M_0| \ge |J_0|$ and thus there exists a partition $\{L_i \mid i \in J_0\}$ of M_0 . Now define sets D_i for $i = 1, \ldots, q$. If $i \in J_0$, then $D_i = L_i \cup \{y_i\}$. If $i \in J_1$, then $D_i = M_i \cup \{y_i\}$. The partition $\mathcal{D} = \{D_1, \ldots, D_q\}$ is a domatic partition of G and thus $d(G) \ge q$ and, by Lemma 2, d(G) = q. The partition \mathcal{D} is also a domatic partition of \overline{G} and thus $d(\overline{G}) \ge q$. Suppose that $d(\overline{G}) \ge q+1$ and let \mathcal{D}' be the corresponding domatic partition of \overline{G} . At most q classes of \mathcal{D}' may have non-empty intersections with Q and therefore there exists a class D' of \mathcal{D}' which is a subset of P. Each vertex of Q is adjacent in \overline{G} and thus non-adjacent in G to a vertex of D'. If there exists a vertex of Q of degree p (condition (iii)), then this vertex is adjacent in G to all vertices of P and thus also to all of D', which is a contradiction. If such a vertex does not exist, then $p \leq 2q - 1$ by (iii). By (i) each vertex of D' is adjacent in G to at most one vertex of Q (to exactly one, if D' is minimal with respect to inclusion), therefore $|D'| \leq q$. No proper subset of Q is dominating in G, because for each vertex of Q there exists a vertex of D' adjacent in G only to it. Hence each class of \mathcal{D}' has a non-empty intersection with P. As D' contains at least q vertices of P, the number of all other classes of \mathcal{D}' is at most p-q and $|\mathcal{D}'| \leq p-q+1$. By (iii) then $|\mathcal{D}'| \leq q$, which is a contradiction. Therefore $d(\overline{G}) = q$ and $d(G) = d(\overline{G})$.

Now suppose that (i) does not hold. There exists a vertex $x_0 \in P$ whose degree is at most q-2 and therefore there exist vertices $y_1 \in Q$, $y_2 \in Q$ which are not adjacent to x_0 . Suppose that d(G) = q and let $\mathcal{D} = D_1, \ldots, D_q$ be the corresponding domatic partition. Each class of \mathcal{D} has exactly one element in common with Q; without loss of generality let $D_1 \cap Q = y_1, D_2 \cap Q = y_2$. But then both D_1, D_2 must contain x_0 , which is a contradiction. Therefore $d(G) < q \leq d(\overline{G})$.

Suppose that (ii) does not hold; by our notation this means $|M_0| < |J_0|$. Suppose that d(G) = q and let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be the corresponding partition. We use the notation $Q = \{y_1, \ldots, y_q\}$ and without loss of generality we suppose that $D_i \cap Q =$ $\{y_i\}$ for $i = 1, \ldots, q$. If $i \in J_1$, then $M_i \subseteq D_i - \{y_i\}$. Therefore if $i \in J_0$, then $D_i \cap P \subseteq M_0$. As $|M_0| < |J_0|$ and all these intersections must be non-empty and pairwise disjoint, we have a contradiction. Therefore again $d(G) < q \leq d(\overline{G})$.

Now suppose that (iii) does not hold; therefore $p \ge 2q$ and $J_0 = \emptyset$, which means $M_i \ne \emptyset$ for each $i \in \{1, \ldots, q\}$. In each M_i we choose a vertex x_i and denote $A = \{x_1, \ldots, x_q\}$. In G the vertices x_i, y_i are adjacent for each $i \in \{1, \ldots, q\}$, therefore A is a dominating set in G. As $p \ge 2q$, the set P - A has at least q elements and we may choose a partition $\{S_1, \ldots, S_q\}$ of P - A with q classes. Evidently $S_i \cup \{y_i\}$

is a dominating set in G for each $i \in \{1, \ldots, q\}$ and $\{A, S_1 \cup \{y_1\}, \ldots, S_q \cup \{y_q\}\}$ is a domatic partition of G. We have $d(\overline{G}) \ge q+1 > q \ge d(G)$.

The problem whether $d(G) = d(\overline{G})$ implies $\gamma(G) = \gamma(\overline{G})$ will be solved only for bipartite graphs.

Theorem 2. Let G be a bipartite graph such that $d(G) = d(\overline{G})$. Then $\gamma(G) = \gamma(\overline{G})$.

Proof. Again we may restrict our considerations to graphs with $q \ge 2$ and without isolated vertices. According to Theorem 1 the equality $d(G) = d(\overline{G})$ implies the validity of the conditions (i), (ii), (iii) and $d(G) = d(\overline{G}) = q$. If there exists at least one vertex $y \in Q$ of degree p, then by (ii) there exists at least one vertex $x \in P$ of degree q. The set $\{x, y\}$ is dominating in G. We have $q \ge 2$ and therefore no one-element set may be dominating in G and $\gamma(G) = 2$. If vertex y exists, then $p \le 2q - 1$ must hold by (iii). We use the notation from the proof of Theorem 1. We have $M_i \ne \emptyset$ for all $i \in \{1, \ldots, q\}$. As the sets M_i are pairwise disjoint subsets of Pand $p \le 2q - 1$, there exists some $j \in \{1, \ldots, q\}$ such that $|M_j| = 1$. Let $M_j = \{x\}$. The set $\{x, y_j\}$ is dominating in G and $\gamma(G) = 2$. In the graph \overline{G} each two-element set consisting of a vertex of P and a vertex of Q is dominating, because P and Qinduce complete subgraphs of \overline{G} . No vertex is adjacent in \overline{G} to all others, because such a vertex would be isolated in G. Therefore $\gamma(\overline{G}) = 2 = \gamma(G)$.

In the case of the total domatic number the situation is more complicated. We will give a full characterization only for the case q = 2; for a general case we will prove only an existence theorem. From our considerations we must exclude graphs with isolated vertices, because for them the total domatic number is not well-defined. In particular, for bipartite graphs we exclude the case q = 1, because in this case the complement contains an isolated vertex.

For q = 2 we can give a full characterization.

Theorem 3. Let G be a bipartite graph without isolated vertices and with bipartition classes P, Q, let p = |P|, q = |Q| = 2, $p \ge 2$. The equality $d_t(G) = d_t(\overline{G})$ holds if and only if exactly one vertex of Q has degree p.

Proof. Let $Q = \{y_1, y_2\}$. Suppose (without loss of generality) that y_1 has degree p, while y_2 has not. Then there exists a vertex $x \in P$ non-adjacent to y_2 . Its degree in G is 1. In [2] it is stated that $d_t(G)$ cannot exceed the minimum degree of a vertex in G and therefore $d_t(G) = 1$. In G the vertex y_1 has degree 1 and thus $d_t(G) = 1$ and $d_t(G) = d_t(\overline{G})$.

If none of the vertices of Q has degree p, then there exists a vertex $x_1 \in P$ nonadjacent to y_1 and a vertex $x_2 \in P$ non-adjacent to y_2 . We have $x_1 \neq x_2$, otherwise this vertex would be isolated. Both x_1 , x_2 have degree 1 and thus $d_t(G) = 1$. If we put $D_1 = \{x_1, y_1\}$, $D_2 = (A - \{x_1\}) \cup \{y_2\}$, then $\{D_1, D_2\}$ is a total domatic partition of \overline{G} and thus $d_t(\overline{G}) \ge 2$ and $d_t(\overline{G}) \ne d_t(G)$. If both vertices of Q have degree p, then choose $x \in P$ and put $D'_1 = \{x, y_1\}$, $D'_2 = (A - \{x\}) \cup \{y_1\}$. The partition $\{D'_1, D'_2\}$ is domatic in G and thus $d_t(G) = 2$ (the degrees of vertices of Pare equal to 2). In \overline{G} both vertices of Q have degree 1 and thus $d_t(\overline{G}) = 1$ and $d_t(G) \ne d_t(\overline{G})$.

Now we prove a lemma.

Lemma 3. Let G be a bipartite graph without isolated vertices and with bipartition classes P, Q, let p = |P|, q = |Q|, $p \ge q \ge 2$. Then $d(\overline{G}) \ge \lfloor \frac{1}{2}q \rfloor$.

Proof. The sets P, Q induce complete subgraphs in G. Denote $r = \lfloor \frac{1}{2}q \rfloor$. Choose an arbitrary partition $\{Q_1, \ldots, Q_r\}$ of Q such that at most one class has three elements and all others have two elements each; such a partition has r classes. As $p \ge q$, also p can be partitioned into r classes, each of which has at least two elements. Let this partition be $\{P_1, \ldots, P_r\}$. Then $\{P_1 \cup Q_1, \ldots, P_r \cup Q_r\}$ is a domatic partition of G, which implies the assertion.

Now we prove the existence theorem.

Theorem 4. Let p, q, s be positive integers, $p \ge q \ge 3$. There exists a bipartite graph G with the bipartition classes P, Q such that |P| = p, |Q| = q and $d_t(G) = d_t(\overline{G}) = s$ if and only if $\frac{1}{2}q \le s \le \frac{3}{4}q$.

Proof. Let $\frac{1}{2}q \leq s \leq \frac{3}{4}q$. First we shall investigate the case $s = \frac{1}{2}q$; then obviously q is even. Denote $r = \frac{1}{2}q$. Take two disjoint sets $P = \{x_1, \ldots, x_p\}$, $Q = \{y_1, \ldots, y_p\}$; the vertex set of G will be $V(G) = P \cup Q$. Join each vertex of P with each vertex of Q by an edge, except the pairs $\{x_1, y_i\}$ for $i = 1, \ldots, r$. Thus G is constructed. The vertex x_1 has degree $\frac{1}{2}q$ and thus $d_t(G) \leq \frac{1}{2}q$. Put $D_i = \{x_{r+i}, y_{r+i}\}$ for $i = 1, \ldots, r-1$ and $d_r = V(G) - \bigcup_{i=1}^{r-1} D_i$. The partition $\{D_1, \ldots, D_r\}$ is total domatic in G and thus $d_t(G) = r = \frac{1}{2}q$. In \overline{G} no subset of P is total dominating and thus each total dominating set in \overline{G} has a non-empty intersection with Q. If this intersection consists of one element, then this element must be some of the vertices y_1, \ldots, y_r and moreover this total dominating set must contain a vertex of P adjacent to this vertex; such a vertex is only x_1 . Therefore a total domatic partition of \overline{G} can contain at most one class having only one vertex in common with Q, all others must have at least two. The number of classes is at most r and $d_t(\overline{G}) \leq r$. There exists the same total domatic partition of G as in the proof of Lemma 3 and thus $d_t(G) = r = \frac{1}{2}q$ and $d_t(G) = d_t(\overline{G})$.

Now let $\lfloor \frac{1}{2}q \rfloor + 1 \leq \frac{3}{4}q$; we will denote $r = \lfloor \frac{1}{2}q \rfloor$. Take again $V(G) = P \cup Q$, where $P = \{x_1, ..., x_p\}, Q = \{y_1, ..., y_q\}$. Let m = 2s - q; we have $2 \le m \le r$. We construct first the complement \overline{G} . It contains the edges $x_i y_i$ for $i = 1, \ldots, m$ and in addition the edges $x_i y_{2m+j}$, where $1 \leq j \leq p-2m$, $j \equiv i \pmod{m}$, again for $i = 1, \ldots, m$ and for all j satisfying the condition (such j need not exist). Further, \overline{G} obviously contains all edges joining two vertices of P and all edges joining two vertices of Q. In \overline{G} no subset of P is total dominating and thus each total dominating set in \overline{G} must have a non-empty intersection with Q. This intersection may consist of one vertex, only if this vertex is adjacent in \overline{G} to a vertex of P; moreover, the mentioned total dominating set must contain also a vertex of P adjacent to this vertex. Only the vertices x_1, \ldots, x_m are adjacent in \overline{G} to vertices of Q and thus in each total domatic partition of \overline{G} at most m classes have one vertex in common with Q; the others have at least two and the number of classes is at most m + q $\frac{1}{2}(q-m) = s$. Therefore $d_t(\overline{G}) \leq s$. Let $L_i = \{y_{m+2i-1}, y_{m+2i}\}$ for $i = 1, \ldots, n$ s-m. Let $\{M_1,\ldots,M_{s-m}\}$ be an arbitrary partition of $P-\{x_1,\ldots,x_m\}$ into s-m classes. Put $\overline{D}_i = \{x_i, y_i\}$ for $i = 1, \ldots, m, \overline{D}_i = L_{i-m} \cup M_{i-m}$ for $i = 1, \ldots, m, \overline{D}_i = L_{i-m} \cup M_{i-m}$ $m+1,\ldots,m+s$. The partition $\{\overline{D}_1,\ldots,\overline{D}_s\}$ is a total domatic partition of \overline{G} and $d_t(\overline{G}) = s.$

Also each total dominating set in G has a non-empty intersection with Q. It has one vertex in common with Q, only if this vertex has degree p in Q; otherwise it has at least two. There are m vertices of degree p in Q, namely y_{m+1}, \ldots, y_{2m} . Analogously as in the case of \overline{G} we have $d_t(G) \leq m + \frac{1}{2}(q-m) = s$. Put $D_i = \{x_{m+i}, y_{m+i}\}$ for $i = 1, \ldots, m$. Further, for q even (and thus also m even) put $D_i = \{x_{2(i-m)-1}, x_{2(i-m)}; y_{2(i-m)-1}, y_{2(i-m)}\}$ for $i = m+1, \ldots, \frac{3}{2}m$, $D_i = \{x_{2(i-m)-1}, x_{2(i-m)}, y_{2(i-m)}\}$ for $i = \frac{3}{2}m+1, \ldots, s$. For q odd we have $D_i = \{x_{2(i-m)-1}, x_{2(i-m)-1}, y_{2(i-m)}\}$ for $i = m+1, \ldots, \frac{1}{2}(3m-1), D_i = \{x_m, x_{2m+1}, y_m, y_{2m+1}\}$ for $i = \frac{1}{2}(3m+1), D_i = \{x_{2i-m-1}, x_{2i-m}, y_{2i-m-1}, y_{2i-m}\}$ for $i = \frac{1}{2}(3m+1) + 1, \ldots, s$. Then $\{D_1, \ldots, D_s\}$ is a total domatic partition of G and we have $d_t(G) = d_t(\overline{G}) = s$.

Now consider the cases when a does not satisfy the above mentioned inequality. By Lemma 3 for $s < \lfloor \frac{1}{2}q \rfloor$ the required graph does not exist. For q odd consider the case $s = \lfloor \frac{1}{2}q \rfloor = \frac{1}{2}(q-1) < \frac{1}{2}q$. We have $d_t(\overline{G}) = s$ in the case when G is a complete bipartite graph $K_{p,q}$, but then $d_t(G) = q \neq s$. Suppose that G is a bipartite graph on P, Q with |P| = p, |Q| = q which is not $K_{p,q}$. Then there exists $x \in P$ and $y \in Q$ such that x, y are non-adjacent in G and thus adjacent in \overline{G} . Let $\{L_1, \ldots, L_s\}$ be a partition of $Q - \{y\}$ into two-element sets, let $\{M_1, \ldots, M_s\}$ be a partition of $P - \{x\}$ into sets with at least two vertices. Put $D_i = L_i \cup M_i$ for $i = 1, \ldots, s, D_{s+1} = \{x, y\}$. The partition $\{D_1, \ldots, D_{s+1}\}$ is total domatic in \overline{G} and $d_t(\overline{G}) \geq s+1$. This excludes the case $s = \frac{1}{2}(q-1)$. Suppose $s > \frac{3}{4}q$. With the notation introduced above, we have $m = 2s - q > \frac{1}{2}q$. As we have seen in the first part of the proof, for $d_t(G) = s$ we must have at least m vertices of degree p in Q; they are non-adjacent to any vertex in G. For $d_t(\overline{G}) = s$ we must have at least m vertice of Q which are adjacent to some vertex of P in \overline{G} . As $m > \frac{1}{2}q$, these two conditions cannot be satisfied simultaneously and thus for $s > \frac{3}{4}q$ the required graph does not exist.

At the end we prove a theorem which concerns graphs in general, not only bipartite graphs.

Theorem 5. No disconnected graph G with $d_t(G) = d_t(\overline{G})$ exists.

Proof. Let G be a disconnected graph. If G contains isolated vertices, then $d_t(G)$ is not defined; therefore suppose that G has no isolated vertex. Let H_1 be a connected component of G with the minimum number of vertices; let $H_2 = G - H_1$. Let h be the number of vertices of H_1 . In \overline{G} each vertex of H_1 is adjacent to each vertex of H_2 . Let the vertices of H_1 be v_1, \ldots, v_h and choose h pairwise distinct vertices w_1, \ldots, w_h in H_2 . Put $\overline{D}_i = \{v_i, w_i\}$ for $i = 1, \ldots, h - 1$ and $\overline{D}_h = V(G) - \bigcup_{i=1}^{n-1} \overline{D}_i$. Then $\{\overline{D}_1, \ldots, \overline{D}_h\}$ is a total domatic partition of \overline{G} and $d_t(\overline{G}) \ge h$. The total domatic number of G is the minimum of total domatic numbers of the connected components of G and thus $d_t(G) \le d_t(H_1)$. Any total dominating set in a graph has at least two vertices and thus $d_t(G) \le d_t(H_1) \le \frac{1}{2}h < h \le d(\overline{G})$.

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