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ALMOST-FLAT MODULES

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Abstract. We present general properties for almost-flat modules and we prove that a self-small right module is almost flat as a left module over its endomorphism ring if and only if the class of g-static modules is closed under the kernels.

Keywords: almost-flat module, self-small module, endomorphism ring

MSC 2000: 16E30, 16D40, 20K40

1. Introduction

Albrecht and Goeters introduced the notion of an *almost-flat* module ([3]) in order to obtain a class of torsion free abelian groups which contains the E-flat groups and which is closed under the quasi-isomorphisms. A left R-module A is an almost-flat module if for every right R-module M the group $\operatorname{Tor}_R^1(M,A)$ is bounded, or equivalently, the kernel of $\alpha \otimes 1_A$ is a bounded group for every monomorphism α of right R-modules.

In the second section of this paper we will study the general properties of almost-flat left modules. Proposition 3.2 proves that if $\varphi\colon R \to S$ is a homomorphism of rings such that the groups $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$ are bounded then the extending of scalars functor and, respectively, the restriction of scalars functor, preserve the almost-flat modules. In Section 4 we consider the right modules and the abelian groups which are almost-flat as modules over their endomorphism ring E (namely E-almost-flat modules). A right module is E-flat if and only if the class of the A-static modules is closed under kernels ([1], [10]). In the same way we give a characterization of E-almost-flat modules in terms of q-static modules (Proposition 4.1). Albrecht and Goeters give in [3] an example of a torsion free group A of finite rank which is not E-almost-flat but such that $\mathbb{Q}A$ is $\mathbb{Q}E$-flat. In Proposition 4.5 we prove that if A is a finite ran torsion free abelian group E-finitely generated such that the endomorphism ring E is noetherian then A is E-almost-flat if and only if $\mathbb{Q}A$ is flat (projective) as a $\mathbb{Q}E$-module. In this paper all groups considered will be abelian, all rings will be associative with identity and all ring homomorphisms will be required to preserve the identity elements. All modules will be unitary modules.

2. General results

Let M be a right R-module. We say that a left R-module A is *n-almost-flat relative to M* if for every monomorphism $f\colon K \to M$ of right R-modules, the abelian group $\operatorname{Ker} f \otimes 1_A$ is bounded by n. A is (n-)almost-flat iff it is (n-)almost-flat relative to each right R module. Recall that A is n-almost-flat if and only if $n \operatorname{Tor}^1_R(M, A) = 0$ for every right R-module M ([3]). We observe that a direct sum of modules which are n-almost-flat relative to M is n-almost-flat relative to M, too. We also observe that we can assume that K is finitely generated. As in the study of flat modules we obtain the following two lemmas:

2.1. Lemma. a) *Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be an exact sequence in* Mod-R *and let A be a left R-module n-almost-flat relative to M. Then A is n-almost-flat relative to M' and M''.*

b) *Conversely, if A is m-almost-flat relative to M' and n-almost-flat relative to M'' then A is mn-almost-flat relative to $M' \oplus M''$.*

Proof. a) We assume that A is n-almost-flat relative to M. If $f\colon K \to M'$ is a monomorphism then $\alpha f\colon K \to M$ is monic. It follows that $\operatorname{Ker} f \otimes 1_A$ is bounded by n because it is a subgroup in $\operatorname{Ker} \alpha f \otimes 1_A$ which is bounded by n.

Let $f\colon L \to M''$ be a monomorphism. We obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
M' \otimes_R A \arrow[r, "\delta\otimes_R 1_A"] \arrow[d, equal] & P \otimes_R A \arrow[r, "\gamma\otimes_R 1_A"] \arrow[d, "g\otimes_R 1_A"] & L \otimes_R A \arrow[r] \arrow[d, "f\otimes_R 1_A"] & 0 \\
M' \otimes_R A \arrow[r, "\alpha\otimes_R 1_A"] & M \otimes_R A \arrow[r, "\beta\otimes_R 1_A"] & M'' \otimes_R A \arrow[r] & 0
\end{tikzcd}$$

where P is a pullback. Then the kernels of $g \otimes_R 1_A$ and $\alpha \otimes_R 1_A$ are bounded by n because g and α are monic. If $x \in \operatorname{Ker} f \otimes_R 1_A$ then there is an element $y \in P \otimes_R A$ such that $\gamma \otimes_R 1_A(y) = x$. Because the diagram is commutative, we obtain that $g \otimes_R 1_A(y) \in \operatorname{Ker} \beta \otimes_R 1_A = \operatorname{Im} \alpha \otimes_R 1_A$ and it follows that there exists $z \in M' \otimes_R A$ such that $g \otimes_R 1_A(y) = \alpha \otimes_R 1_A(z) = g \otimes_R 1_A(\delta \otimes_R 1_A(z))$. Then $y - \delta \otimes_R 1_A(z) \in \operatorname{Ker}(g \otimes_R 1_A)$ and it follows that $ny = n\delta \otimes_R 1_A(z)$. We obtain the equalities $nx = \gamma \otimes_R 1_A(ny) = \gamma \otimes_R 1_A(n\delta \otimes_R 1_A(z)) = 0$. Therefore $\mathrm{Ker}(f \otimes_R 1_A)$ is bounded by n.

b) Conversely, if $f\colon\ K \to M' \oplus M''$ is monic then, using a pullback, we obtain a commutative diagram ([9, 12.15]) and applying the Ker-Coker Lemma we obtain an exact sequence $X_1 \to \mathrm{Ker}\, f \otimes_R 1_A \to X_2$ with X_1 bounded by m and X_2 bounded by n. $\square$

2.2. Lemma. *Let A be a left R-module and $n \in \mathbb{N}$. The following conditions are equivalent:*

a) *A is n-almost-flat;*

b) *A is n-almost-flat relative to R;*

c) *if $a_1 \dots a_m \in A$ and $r_1 \dots r_m \in R$ are such that $\sum\limits_{i=1}^{m} r_i a_i = 0$ then there exist $b_1 \dots b_l \in A$ and $s_{i1} \dots s_{il} \in R$, $i = 1 \dots m$, such that $\sum\limits_{j=1}^{l} s_{ij} b_j = n a_i$ and $\sum\limits_{i=1}^{m} r_i s_{ij} = 0$.*

Proof. The equivalence b) $\Leftrightarrow$ c) is proved in a special case in [3, Lemma 3.3], and the proof can be adapted to the general case. The implication a) $\Rightarrow$ b) is obvious. We prove that b), c) $\Rightarrow$ a).

Let X be a set and let $0 \to K \xrightarrow{u} R^{(X)}$ be a monomorphism of right R-modules. We can assume that K is finitely generated and, in this case, there is a positive integer k such that $u(K) \subseteq R^k$.

Let $0 \to K \xrightarrow{u} R^k$ be an exact sequence of right R-modules. We show by induction on k that $\mathrm{Ker}(u \otimes_R 1_A)$ is bounded. For $k = 1$ this is obvious and so it is enough to prove the assertion for $k = 2$. If $\sum\limits_{i=1}^{m} ((r_1^i, r_2^i) \otimes_R a_i) \in \mathrm{Ker}(u \otimes_R 1_A)$ we apply the canonical homomorphism $K \otimes_R A \xrightarrow{u \otimes_R 1_A} R^k \otimes_R A \xrightarrow{h} A^k$ and it follows that $\sum\limits_{i=1}^{m} r_1^i a_i = 0$ and $\sum\limits_{i=1}^{m} r_2^i a_i = 0$. Then there exist $b_j, b'_j \in A$ and $s_{ij}, s'_{ij} \in R$, $j = 1 \dots l$ and $i = 1 \dots m$, such that $\sum\limits_{j=1}^{l} s_{ij} b_j = n a_i$, $\sum\limits_{i=1}^{m} r_1^i s_{ij} = 0$ and $\sum\limits_{j=1}^{l} s'_{ij} b'_j = n a_i$, $\sum\limits_{i=1}^{m} r_2^i s'_{ij} = 0$. We obtain that $n \sum\limits_{i=1}^{m} ((r_1^i, r_2^i) \otimes_R a_i) = \sum\limits_{i=1}^{m} ((r_1^i, r_2^i) \otimes_R \sum\limits_{j=1}^{l} s_{ij} b_j) = \sum\limits_{j=1}^{l} (0, \sum\limits_{i=1}^{m} r_2^i s_{ij}) \otimes_R b_j = \sum\limits_{i=1}^{m} (0, r_2^i) \otimes_R n a_i = \sum\limits_{j=1}^{l} (0, \sum\limits_{i=1}^{m} r_2^i s'_{ij}) \otimes_R b'_j = 0$. Then $\mathrm{Ker}(u \otimes_R 1_A)$ is bounded by n. Then for every set X the left R-module A is n-almost-flat relative to $R^{(X)}$ and, by Lemma 2.1, it follows that A is n-almost-flat. $\square$

In [3] the authors prove that the class of almost-flat modules is closed under quasi-isomorphisms and that if R is a unital subring of S, quasi-equal to S, then the restriction of scalars functor preserves the almost-flat modules. If A and B are left R-modules then we say that a R-homomorphism $\alpha\colon A \to B$ is a *q-isomorphism* if $\mathrm{Ker}(\alpha)$ and $\mathrm{Coker}(\alpha)$ are bounded as abelian groups. It is easy to see that two modules A and B are quasi-isomorphic if and only if there exists a q-isomorphism $\alpha\colon A \to B$. The next statement is in fact Lemma 2.2 in [3]. We include a proof here for the sake of completeness.

2.3. Proposition. *If A and B are left R-modules such that there exists a q-isomorphism $\alpha\colon A \to B$ then A is almost-flat if and only if B is almost-flat.*

Proof. Let M be a right R-module and $0 \to U \to P \to M \to 0$ an exact sequence with P a projective right R-module.

We assume in the first case that α is monic and we apply the tensor product functors $U \otimes_R -$ and $P \otimes_R -$ to the exact sequence $0 \to A \to B \to A/B \to 0$. Then we obtain the following commutative diagram:

$$\begin{tikzcd}
 & U \otimes_R A \arrow[r] \arrow[d, "f\otimes 1_A"] & U \otimes_R B \arrow[r] \arrow[d, "f\otimes 1_B"] & U_R \otimes B/A \arrow[r] \arrow[d, "f\otimes 1_{B/A}"] & 0 \\
0 \arrow[r] & P \otimes_R A \arrow[r] & P \otimes_R B \arrow[r] & P \otimes_R B/A \arrow[r] & 0
\end{tikzcd}$$

with exact rows. The Ker-Coker Lemma yields the existence of an exact sequence $\mathrm{Ker}(f \otimes 1_A) \to \mathrm{Ker}(f \otimes 1_B) \to \mathrm{Ker}(f \otimes 1_{B/A})$. Because P is projective, it follows that $\mathrm{Ker}(f \otimes 1_A) \simeq \mathrm{Tor}_R^1(M, A)$ and $\mathrm{Ker}(f \otimes 1_B) \simeq \mathrm{Tor}_R^1(M, B)$. Furthermore, $\mathrm{Ker}(f \otimes 1_A)$ and $\mathrm{Ker}(f \otimes 1_{B/A})$ are bounded because A is almost-flat and B/A is bounded. Consequently, $\mathrm{Tor}_R^1(M, B)$ is bounded.

If α is epic, let C be the kernel of α. Then we obtain a commutative diagram with exact rows:

$$\begin{tikzcd}
 & U \otimes_R C \arrow[r] \arrow[d, "f\otimes 1_C"] & U \otimes_R A \arrow[r] \arrow[d, "f\otimes 1_A"] & U \otimes_R B \arrow[r] \arrow[d, "f\otimes 1_B"] & 0 \\
0 \arrow[r] & P \otimes_R C \arrow[r] & P \otimes_R A \arrow[r] & P \otimes_R B \arrow[r] & 0
\end{tikzcd}$$

There exist isomorphisms $\mathrm{Tor}_R^1(M, A) \simeq \mathrm{Ker}(f \otimes 1_A)$ and $\mathrm{Tor}_R^1(M, B) \simeq \mathrm{Ker}(f \otimes 1_B)$, because P is projective. We apply the Ker-Coker Lemma and obtain an exact sequence $\mathrm{Tor}_R^1(M, A) \to \mathrm{Tor}_R^1(M, B) \to M \otimes C$ with $\mathrm{Tor}_R^1(M, A)$ and $M \otimes C$ bounded as abelian groups. Thus $\mathrm{Tor}_R^1(M, B)$ is bounded.

Consequently, in the general case, if A is almost-flat then $\alpha(A)$ is almost flat and, because $B/\alpha(A)$ is bounded, it follows that B is almost-flat.

Conversely, let $0 \to K \xrightarrow{f} M$ be a monomorphism in Mod-R. Then the following diagram is commutative:

$$\begin{tikzcd}
K \otimes_R A \arrow[r, "f\otimes 1_A"] \arrow[d, "1_K\otimes\alpha"] & M \otimes_R A \arrow[d, "1_M\otimes\alpha"] \\
K \otimes_R B \arrow[r, "f\otimes 1_B"] & M \otimes_R B
\end{tikzcd}$$

Then the kernel of $f \otimes 1_A$ is bounded because $\operatorname{Ker}(1_K \otimes \alpha)$ and $\operatorname{Ker} f \otimes 1_B$ are bounded, hence A is almost-flat. □

3. Change of rings

Let $\varphi\colon R \to S$ be a ring homomorphism. As in [7, IV.9], we consider the pair of adjoint functors (φ^*, φ_*), where $\varphi^*\colon R\text{-Mod} \to S\text{-Mod}$, $\varphi^*(C) = S \otimes_R C$, is the extension of scalars functor and $\varphi_*\colon S\text{-Mod} \to R\text{-Mod}$, $\varphi_*(A) = A$, is the restriction of scalars functor. Then there exist natural transformations $\xi\colon \varphi^*\varphi_* \to 1_{S\text{-Mod}}$ and $\zeta\colon 1_{R\text{-Mod}} \to \varphi_*\varphi^*$ which are defined in the following way: if $A \in S$-Mod then $\xi_A\colon \varphi^*\varphi_*(A) \to A$, $s\otimes a \mapsto sa$, and if $C \in R$-Mod then $\zeta_C\colon C \to \varphi_*\varphi^*(C)$, $c \mapsto 1\otimes c$.

3.1. Lemma. *Let $\varphi\colon R \to S$ be a ring homomorphism such that the abelian groups* $\operatorname{Ker}\varphi$ *and* $S/\operatorname{Im}\varphi$ *are bounded. Then:*

a) *If $A \in S$-Mod then the natural homomorphism ξ_A is a q-isomorphism.*

b) *If $C \in R$-Mod then the natural homomorphism ζ_C is a q-isomorphism.*

P r o o f. a) If $A \in S$-Mod then ξ_A is obtained from the composition of the canonical homomorphisms $S \otimes_R A \to S \otimes_S A \to A$. The kernel of the canonical homomorphism $S \otimes_R A \to S \otimes_S A$ is bounded because it is equal to the composition $S \otimes_R A \to S \otimes_{\operatorname{Im}(\varphi)} A \to S \otimes_S A$ and we observe that the first arrow is an isomorphism by [5] while the second one has a bounded kernel by [3]. Therefore ξ_A is a q-isomorphism.

b) Analogously, for every $C \in R$-Mod we have in R-Mod a commutative diagram with canonical homomorphisms:

$$\begin{tikzcd}
R \otimes_R C \arrow[dr] & \\
C \arrow[u] \arrow[r] & S \otimes_R C
\end{tikzcd}$$

We observe that the kernel of the epimorphism $R \otimes_R C \to \mathrm{Im}(\varphi) \otimes_R C$ is bounded and that $\mathrm{Im}(\varphi) \otimes_R C \to S \otimes_R C$ has a bounded cokernel. The proof is complete. $\square$

3.2. Proposition. *Let $\varphi\colon R \to S$ be a ring homomorphism such that $\operatorname{Ker}\varphi$ and $S/\mathrm{Im}(\alpha)$ are bounded as abelian groups. Then:*

a) *A left R-module C is almost-flat if and only if the left S-module $\varphi^*(C) = S \otimes_R C$ is almost-flat.*

b) *A left S-module A is almost-flat if and only if the left R-module $\varphi_*(A) = A$ is almost-flat.*

P r o o f. a) Suppose that C is an almost flat left R-module. Let $0 \to M \xrightarrow{f} N$ be an exact sequence in Mod-S. It follows that f induces a monomorphism in Mod-R. We obtain the commutative diagram

$$\begin{tikzcd}
M \otimes_R C \arrow[r, "f \otimes_R 1_C"] \arrow[d] & N \otimes_R C \arrow[d] \\
M \otimes_S (S \otimes_R C) \arrow[r, "f \otimes_S 1_{S \otimes_R C}"] & N \otimes_S (S \otimes_R C)
\end{tikzcd}$$

Then $\operatorname{Ker}(f \otimes_S 1_{S \otimes_R C})$ is bounded, because the vertical arrows are isomorphisms and $\operatorname{Ker}(f \otimes_R 1_C)$ is bounded. Hence $S \otimes_R C$ is almost flat.

Conversely, if M is a right R-module and $0 \to K \to P \to M \to 0$ is an exact sequence in Mod-R with P projective, we obtain the commutative diagram

$$\begin{tikzcd}
K \otimes_R \varphi_*\varphi^*(C) \arrow[r] \arrow[d] & P \otimes_R \varphi_*\varphi^*(C) \arrow[r] \arrow[d] & M \otimes_R \varphi_*\varphi^*(C) \arrow[r] \arrow[d] & 0 \\
\varphi^*(K) \otimes_S \varphi^*(C) \arrow[r] & \varphi^*(P) \otimes_S \varphi^*(C) \arrow[r] & \varphi^*(M) \otimes_S \varphi^*(C) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. We observe that the vertical arrows are isomorphisms and that the kernel of $K \otimes_R \varphi_*\varphi^*(C) \to P \otimes_R \varphi_*\varphi^*(C)$ is $\mathrm{Tor}_R^1(M, \varphi_*\varphi^*(C))$. The kernel of $\varphi^*(K) \otimes_S \varphi^*(C) \to \varphi^*(P) \otimes_S \varphi^*(C)$ is a homomorphic image of the bounded group $\mathrm{Tor}_S^1(\varphi^*(M), \varphi^*(C))$, so it is a bounded group. It follows that $\varphi_*\varphi^*(C)$ is almost-flat as an R-module. Moreover, $\varphi_*\varphi^*(C)$ is q-isomorphic with C and, by Proposition 2.3, we conclude that C is almost-flat as an R-module.

b) Let A be a left S-module. It is almost-flat if and only if the left S-module $\varphi^*\varphi_*(A)$ is almost flat, because they are q-isomorphic (by Proposition 2.3). From a), it follows that A is almost-flat as an S-module if and only if $\varphi_*(A)$ is almost-flat as an R-module. $\square$

The E-flat R-modules are characterized by the property that the class of the A-static modules is closed under the kernels ([1], [10]). We obtain a similar result for E-almost flat R-modules.

Let A be a right R-module. We denote by H_A : Mod-$R \to$ Mod-E the functor $\mathrm{Hom}_R(A,-)$ and by $\mathrm{T}_A\colon$ Mod-$E \to$ Mod-R the functor $-\otimes_E A$. Then a canonical natural R-homomorphism $\theta_B\colon \mathrm{T}_A\,\mathrm{H}_A(B) \to B$, $\theta_B(f\otimes a) = f(a)$, exists for every right R-module B. We say that $B \in$ Mod-R is *q-A-static* if θ_B is a q-isomorphism. We denote by $q\mathcal{C}_A$ the class of q-A-static modules. Observe that the class $q\mathcal{C}_A$ is closed under the q-isomorphisms. In [4] Arnold and Murley introduced the notion of a self-small module in order to obtain a module A such that the class of A-static modules is closed under the direct sums of copies of A.

4.1. Proposition. *Let R be a ring and let A be a self-small right R-module. Then the following conditions are equivalent:*

a) *A is almost flat as an E-module;*

b) *$q\mathcal{C}_A$ is closed under the kernels;*

c) *if Λ and Γ are countable sets and $\alpha\colon A^{(\Lambda)} \to A^{(\Gamma)}$ is an R-homomorphism then $\mathrm{Ker}(\alpha) \in q\mathcal{C}_A$;*

d) *there exists a nonzero integer n such that if $m \in \mathbb{N}$ and $\alpha\colon A^m \to A$ is an R-morphism then the cokernel of $\theta_{\mathrm{Ker}\,\alpha}$ is bounded by n.*

Proof. In order to prove that a) $\Rightarrow$ b) we consider two q-static modules C and G and an exact sequence $0 \to B \xrightarrow{\alpha} C \xrightarrow{\beta} G$. Then we obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{Tor}^1(\mathrm{Im}\,\mathrm{H}_A(\beta), A) \arrow[r] & \mathrm{T}_A\,\mathrm{H}_A(B) \arrow[r, "\mathrm{T}_A\,\mathrm{H}_A(\alpha)"] \arrow[d, "\theta_B"] & \mathrm{T}_A\,\mathrm{H}_A(C) \arrow[r, "\mathrm{T}_A\,\mathrm{H}_A(\beta)"] \arrow[d, "\theta_C"] & \mathrm{T}_A(\mathrm{Im}\,\mathrm{H}_A(\beta)) \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & B \arrow[r] & C \arrow[r] & G &
\end{tikzcd}$$

with $\theta = \theta_G\,\mathrm{T}_A(i)$ and $i\colon \mathrm{Im}\,\mathrm{H}_A(\beta) \to \mathrm{H}_A(G)$ the inclusion map. It follows that $\mathrm{Ker}\,\theta$ is bounded because $\ker\theta_G$ and $\ker \mathrm{T}_A(i)$ are bounded. We apply the Ker-Coker Lemma and get that $\mathrm{Coker}\,\theta_B$ is bounded. We observe that in the first commutative square the groups $\mathrm{Ker}\,\mathrm{T}_A\,\mathrm{H}_A(\alpha)$ and $\mathrm{Ker}\,\theta_C$ are bounded, thus $\mathrm{Ker}\,\theta_B$ is bounded.

The implication b) $\Rightarrow$ c) is obvious.

c) $\Rightarrow$ d). Suppose that there exists a family of R-homomorphisms $\{\alpha_i\colon A^{m_i} \to A\}_{i\in\mathbb{N}}$, $m_i \in \mathbb{N}$, and that we can find an ascending chain of integers $0 < n_0 < \ldots < n_i < \ldots$, $i \in \mathbb{N}$, such that $n_i\,\mathrm{Coker}\,\theta_{\mathrm{Ker}\,\alpha_i} = 0$. We also suppose that if $i < j$ then

$n_i \operatorname{Coker} \theta_{\operatorname{Ker} \alpha_j} \neq 0$. Let us denote $\alpha = \bigoplus_{i\in\mathbb{N}} \alpha_i\colon \bigoplus_{i\in\mathbb{N}} A^{m_i} \to \bigoplus_{i\in\mathbb{N}} A$. We claim that the group $\operatorname{Coker} \theta_{\operatorname{Ker} \alpha}$ is not bounded.

We observe that $\operatorname{Ker} \alpha = \bigoplus_{i\in\mathbb{N}} \operatorname{Ker} \alpha_i$. Since A is self-small and for every index i the kernel of α_i is a submodule of A^{m_i}, we obtain $\mathrm{H}_A(\operatorname{Ker} \alpha) \simeq \bigoplus_{i\in\mathbb{N}} \mathrm{H}_A(\operatorname{Ker} \alpha_i)$ and from the definitions of the homomorphisms $\theta_{\operatorname{Ker} \alpha_i}$ it follows that $\theta_{\operatorname{Ker} \alpha} = \bigoplus_{i\in\mathbb{N}} \theta_{\operatorname{Ker} \alpha_i}$. Consequently, $\operatorname{Coker} \theta_{\operatorname{Ker} \alpha}$ is not bounded and this contradicts c).

d) $\Rightarrow$ a). We consider $f_1, \dots, f_m \in E(A)$ and $a_1, \dots, a_m \in A$ such that $\sum_{i=1}^{m} f_i(a_i) = 0$. Let $\alpha\colon A^m \to A$ be the homomorphism which is induced by $f_1, \dots, f_m$. It follows that $(a_1, \dots, a_m) \in \operatorname{Ker}(\alpha)$.

The diagram

$$\begin{tikzcd}
 & \mathrm{T}_A \mathrm{H}_A(\operatorname{Ker}(\alpha)) \arrow[r] \arrow[d, "\theta_{\operatorname{Ker} \alpha}"] & \mathrm{T}_A \mathrm{H}_A(A^m) \arrow[r] \arrow[d] & \mathrm{T}_A \mathrm{H}_A(A) \arrow[d] \\
0 \arrow[r] & \operatorname{Ker}(\alpha) \arrow[r] & A^m \arrow[r] & A
\end{tikzcd}$$

is commutative with exact rows and $n(a_1, \dots, a_m) \in \operatorname{Im} \theta_{\operatorname{Ker} \alpha}$, because the cokernel of $\theta_{\operatorname{Ker} \alpha}$ is bounded by n. Then we can find an element $\sum_{j=1}^{k} g_j \otimes b_j \in \mathrm{T}_A \mathrm{H}_A(\operatorname{Ker}(\alpha))$ such that $\sum_{j=1}^{k} g_j(b_j) = n(a_1, \dots, a_m)$. In this case, we can write $g_j = (g_{j1} \dots g_{jm})$ with $g_{ji} \in E(A)$, for all i, j, because $\mathrm{H}_A(\operatorname{Ker}(\alpha)) \subseteq \mathrm{H}_A(A^m)$. Therefore we can find endomorphisms $g_{ji} \in E(A)$ and elements $b_j \in A$, $i = 1 \dots m$, $j = 1 \dots k$ such that $\sum_{j=1}^{k} g_{ij}(b_j) = na_i$ and $\sum_{i=1}^{m} f_i g_{ij} = 0$ ($g_j \in \mathrm{H}_A(\operatorname{Ker}(\alpha))$ implies $\alpha g_j = 0$). We conclude that A is E-almost-flat. $\square$

In [2], Urlich Albrecht proved that a torsion free group A is E-flat if and only if for every right E-module M the A-socle of $\operatorname{Tor}_E^1(M, A)$, $\mathrm{S}_A(\operatorname{Tor}_E^1(M, A)) = \operatorname{Im} \theta_{\operatorname{Tor}_E^1(M,A)}$, is zero. We obtain a similar characterization for E-almost-flat torsion free groups:

4.2. Proposition. *Let A be a torsion free abelian group. Then A is E-almost-flat if and only if $\mathrm{S}_A(\operatorname{Tor}_E^1(M, A))$ is bounded for every right E-module M.*

Proof. Observe that $\mathrm{S}_A(\operatorname{Tor}_E^1(M, A))$ is bounded if and only if the group $\mathrm{H}_A(\operatorname{Tor}_E^1(M, N))$ is bounded. Because for every E-module M the torsion part $t(M)$ of M^+ is a submodule we have the exact sequence $\operatorname{Tor}_E^1(t(M), A) \to \operatorname{Tor}_E^1(M, A) \to \operatorname{Tor}_E^1(M/t(M), A)$. Thus, in order to prove that for a right E-module M the group

$\mathrm{Tor}_E^1(M,A)$ is bounded, we can assume that the group M^+ is a torsion group or a torsion free group.

We suppose that $\mathrm{Tor}_E^1(M,A)$ is not bounded. If its torsion part is not bounded we will have two cases:

In the first case we assume that there exists an infinite set S of primes p such that $\mathrm{Tor}_E^1(M,A)[p] \neq 0$. If $p \in S$ and $A = pA$ then the multiplication by p becomes an automorphism of $\mathrm{Tor}_E^1(M,A)$ and this is impossible. We obtain that for every $p \in S$ the group A is not p-divisible. Then for every $p \in S$ we obtain a canonical nonzero homomorphism $A \to A/pA \to \mathrm{Tor}_E^1(M,A)[p]$ which has the order p in $\mathrm{H}_A(\mathrm{Tor}_E^1(M,A))$. Then $\mathrm{S}_A(\mathrm{Tor}_E^1(M,A))$ is not bounded and this is not possible.

In the second case we suppose that there exists a prime p such that for all positive integers k, the group $\mathrm{Tor}_E^1(M,A)[p^k]$ is not zero. It follows that there is an integer k such that $p^kA = p^{k+1}A$ because if for every k the exponent of A/p^kA is p^k then like in the first case we can find in $\mathrm{H}_A(\mathrm{Tor}_E^1(M,A))$ elements of order p^k for all k. Then the group A would be p-divisible. As in the first case we obtain a contradiction.

Hence, for every right E-module M, the torsion part of $\mathrm{Tor}_E^1(M,A)$ is bounded and it follows that $\mathrm{Tor}_E^1(M,A) = B \oplus C$ where B is a bounded group and C is a torsion free group.

Assume that M^+ is torsion. Then $\mathrm{Tor}_E^1(M,A)$ is bounded. Let $0 \to U \to P \to M \to 0$ be an exact sequence in Mod-E with P a projective right E-module. Then the groups $\mathrm{Tor}_E^2(M,A)$ and $\mathrm{Tor}_E^1(U,A)$ are isomorphic. The first group is a torsion group and the other is a direct sum of a bounded group and a torsion free group. Thus $\mathrm{Tor}_E^2(M,A)$ is bounded.

If M^+ is a torsion free group, we obtain an exact sequence $\mathrm{Tor}_E^2(T,A) \to \mathrm{Tor}_E^1(M,A) \xrightarrow{\gamma} \mathrm{Tor}_E^1(\mathbb{Q}\otimes M,A) \to \mathrm{Tor}_E^1(T,A)$ with T an E-module such that the group T^+ is a torsion group. We observe that $\mathrm{Tor}_E^1(\mathbb{Q}\otimes M,A)$ is a torsion free divisible group, because the multiplication with any positive integer is an automorphism, and that the group $\mathrm{Tor}_E^2(T,A)$ is bounded. It follows that $\mathrm{Tor}_E^1(M,A) = B \oplus D$ where B is bounded and D is torsion free and divisible. Therefore, if D is not zero we obtain that $\mathrm{H}_A(\mathrm{Tor}_E^1(M,A)) \simeq \mathrm{H}_A(B) \oplus \mathrm{H}_A(\bigoplus \mathbb{Q})$ and it is not bounded. Then $\mathrm{Tor}_E^1(M,A)$ is bounded and this completes the proof. $\square$

The following proposition gives a characterization for the E-almost-flat almost complete decomposable groups.

4.3. Proposition. *Let A be a torsion free almost complete decomposable group. Then A is E-almost-flat if and only if the critical type-set T of A satisfies the following condition: if τ_1 and τ_2 are critical types such that they have an upper bound in T then they have a lower bound in T.*

P r o o f. Let C be a torsion free complete decomposable group quasi-equal to A. If A is E-almost flat then C is E-almost-flat ([3]). According to [3] it follows that $\mathbb{Q}\otimes C$ is flat over $\mathbb{Q}\otimes E$ and this assertion is equivalent to the fact that C is E-flat ([8]). The conclusion follows by [6].

The converse is obvious. $\square$

From the proof we obtain the next corollary.

4.4. Corollary.

a) *A complete decomposable group is E-almost-flat if and only if it is E-flat.*

b) *An almost complete decomposable group is E-almost-flat if and only if $\mathbb{Q}A$ is a flat $\mathbb{Q}E$-module.*

4.5. Proposition. *Let A be a finite rank torsion free group E-finitely-generated such that the endomorphism ring E is noetherian. Then A is E-almost-flat if and only if $\mathbb{Q}A$ is flat as a $\mathbb{Q}E$-module.*

P r o o f. If A is E-almost-flat then $\mathbb{Q}A$ is flat as a $\mathbb{Q}E$-module (see [3]). Conversely, $\mathbb{Q}A$ is a finite generated $\mathbb{Q}E$-module and it follows that it is finitely presented flat $\mathbb{Q}E$-module, hence it is projective.

Because E is noetherian we have a natural isomorphism $\mathrm{Hom}_{\mathbb{Q}E}(\mathbb{Q}A,\mathbb{Q}E)\simeq\mathbb{Q}\otimes\mathrm{Hom}_E(A,E)$ and we can consider the elements $\overline{g_j}=\frac{m_j}{n_j}\otimes g_j\in\mathbb{Q}\otimes\mathrm{Hom}_E(A,E)$ and $\overline{b_j}=\frac{p_j}{q_j}\otimes b_j\in\mathbb{Q}A$ with $j=1\dots n$ such that the pairs $=\big((\frac{m_j}{n_j}\otimes g_j),(\frac{p_j}{q_j}\otimes b_j)\big)$ form a dual basis for $\mathbb{Q}A$. Let k and l be respectively the products of the numbers n_j, q_j, with $j=1\dots n$. We can assume that the elements of the dual basis have the form $\overline{g_j}=\frac{m_j}{k}\otimes g_j$ and $\overline{b_j}=\frac{p_j}{l}\otimes b_j$ respectively.

We consider now $f_i\in E$ and $a_i\in A$, $i=1\dots m$, such that $\sum\limits_{i=1}^{m}f_i(a_i)=0$. For every index i we have that $\overline{a_i}=1\otimes a_i=\sum\limits_{j=1}^{n}\overline{g_j}(\overline{a_i})(\overline{b_j})=\sum\limits_{j=1}^{n}\frac{m_jp_j}{kl}\otimes g_j(a_i)(b_j)$. Then $1\otimes kla_i=1\otimes\sum\limits_{j=1}^{n}m_jp_jg_j(a_i)(b_j)$ and it follows that, for every index i, $kla_i=\sum\limits_{j=1}^{n}m_jp_jg_j(a_i)(b_j)$ because A is torsion free.

Moreover, $\sum\limits_{i=1}^{n}f_i\circ(k_jg_j(a_i))=\sum\limits_{i=1}^{n}k_jf_i\circ g_j(a_i)=k_jg_j\Big(\sum\limits_{i=1}^{n}f_i(a_i)\Big)=0$ where $k_j=m_jp_j$ for all j.

We conclude that there exist endomorphisms $k_jg_j(a_i)$ and elements $b_j\in A$ such that $kla_i=\sum\limits_{j=1}^{n}k_jg_j(a_i)(b_j)$ and $\sum\limits_{i=1}^{n}f_i\circ(k_jg_j(a_i))=0$, hence A is E-almost-flat. $\square$

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