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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF TWO-DIMENSIONAL NEUTRAL DIFFERENTIAL SYSTEMS

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Abstract. We study asymptotic properties of solutions of the system of differential equations of neutral type.

Keywords: neutral equation, oscillatory solution, bounded solution

MSC 2000: 34K15, 34K10

1. Introduction

We consider systems of neutral differential equations of the form

$$\tag{A} \begin{aligned} (x_1(t) - px_1(t-\tau))' &= a_1(t) f_1(x_2(g_2(t))), \\ x_2'(t) &= -a_2(t) f_2(x_1(g_1(t))) \end{aligned}$$

and the following conditions are assumed to hold without further notice:

(a) p, τ are positive numbers, $0 < p \leqslant 1$;

(b) $a_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, $i = 1, 2$, are not identically zero on any subinterval $[T, \infty) \subset (0, \infty)$ and
$$\int^{\infty} a_1(s)\,\mathrm{d}s = \infty;$$

(c) $g_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\lim\limits_{t\to\infty} g_i(t) = \infty, \quad i = 1, 2$;

(d) $f_i \in C(\mathbb{R}, \mathbb{R})$, $f_i(u)u > 0$ for $u \neq 0$ and there exist positive constants K, L such that $|f_1(u)| \geqslant L|u|$, $|f_2(v)| \geqslant K|v|$ for $u, v \in \mathbb{R}$.

The problem of oscillation of neutral functional differential equations has received considerable attention in the last few years (see for example [3] and the references

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cited therein). However, very little has been published on systems of neutral differential equations [1], [6]–[10].

Our aim in the present paper is to establish sufficient conditions under which all proper solutions of (A) are oscillatory. By a proper solution of (A) we mean a continuous vector function $\boldsymbol{x} = (x_1, x_2)$ on $[t_x, \infty)$ such that $x_1(t) - px_1(t-\tau)$, $x_2(t)$ are continuously differentiable, $\boldsymbol{x}$ satisfies system (A) for all sufficiently large $t \geqslant t_x$ and $\sup\{|x_1(t)| + |x_2(t)| \colon t \geqslant T\} > 0$ for any $T \geqslant t_x$. Such a solution is called nonoscillatory if there exists a $T_0 \geqslant t_x$ such that its every component is different from zero for all $t \geqslant T_0$, and it is called oscillatory otherwise.

2. Properties of nonoscillatory solutions

Let $\boldsymbol{x} = (x_1, x_2)$ be a nonoscillatory solution of the system (A). For any $x_1(t)$ we define $u_1(t)$ by

$$u_1(t) = x_1(t) - px_1(t-\tau). \tag{1}$$

It follows from (A) that the function $u_1(t)$ is eventually monotone, so that $u_1(t)$ has to be of constant sign. Therefore, either

$$x_1(t)u_1(t) > 0, \tag{2}$$

or

$$x_1(t)u_1(t) < 0 \tag{3}$$

for all sufficiently large t. Denote by N^+ or N^- respectively the set of all nonoscillatory solutions $\boldsymbol{x} = (x_1, x_2)$ of system (A) such that (2) or (3) is satisfied. Denoting by N the set of all nonoscillatory solutions of (A) we have $N = N^+ \cup N^-$.

If $\boldsymbol{x} \in N^+$ then for every $T \geqslant t_0$ and every integer $n > 0$ there exists $T_n \geqslant T$ such that $t - n\tau \geqslant T$ and

$$|x_1(t)| \geqslant \sum_{j=0}^{n} p^j |u_1(t - j\tau)| \quad \text{for } t \geqslant T_n. \tag{4}$$

Similarly, if $\boldsymbol{x} \in N^-$ then for every $T \geqslant t_0$ and every integer $m > 0$ there exists $T_m \geqslant T$ such that

$$|x_1(t)| \geqslant \sum_{j=1}^{m} \frac{|u_1(t + j\tau)|}{p^j} \quad \text{for } t \geqslant T_m. \tag{5}$$

A simple known lemma given below indicates that an additional restriction upon p may lead to some properties of the nonoscillatory solutions. (See for example [6].)

Lemma 1. *Let $0 < p < 1$ hold and $\boldsymbol{x} \in N^-$. Then $\lim_{t\to\infty} x_1(t) = 0$, $\lim_{t\to\infty} u_1(t) = 0$.*

3. Main results

Theorem 1. *Let $0 < p < 1$ and let the following assumptions hold:*

(i) $g_1'(t) > 0$, $t \geqslant t_0$;

(ii) *there exist an integer number $n \geqslant 0$ and $T \geqslant t_0$ such that $g_2(g_1(t) - i\tau) \leqslant t$ for $t \geqslant T$, $i = 0, \ldots, n$.*

If

$$\int_T^\infty \left(a_2(s) g_1(s) - \frac{g_1'(s)}{4KLg_1(s)\sum_{i=0}^n p^i a_1(g_1(s) - i\tau)} \right) \mathrm{d}s = \infty \tag{6}$$

then for every nonoscillatory solution (x_1, x_2) of (A) its both components tend to zero for $t \to \infty$.

Proof. Let $\boldsymbol{x} = (x_1, x_2)$ be a nonoscillatory solution of (A) and let $x_1(t) > 0$ for $t \geqslant t_0$. It follows from the system (A) that $x_2(t)$ is decreasing and hence there exists such a $t_1 \geqslant t_0$ that there are two possibilities for $x_2(t)$:

1. $x_2(t) < 0$ for $t \geqslant t_1$,
2. $x_2(t) > 0$ for $t \geqslant t_1$.

Assume that 1 holds. Then there exist a constant $c < 0$ and $t_2 \geqslant t_1$ such that $x_2(t) \leqslant c$, $x_2(g_2(t)) \leqslant c$ for $t \geqslant t_2$. Using (d) and the first equation of the system (A) we get

$$u_1(t) - u_1(t_2) \leqslant Lc \int_{t_2}^t a_1(s)\,\mathrm{d}s.$$

Letting $t \to \infty$, in view of (b) we have $\lim_{t\to\infty} u_1(t) = -\infty$. This means that $\boldsymbol{x} \in N^-$, which contradicts Lemma 1.

We assume now that 2 holds and consider the following cases:

a) Let $\boldsymbol{x} \in N^-$. The function $x_2(t)$ is positive, decreasing and so there exists $\lim_{t\to\infty} x_2(t) = d \geqslant 0$. We shall show that $d = 0$. Suppose the contrary. Then there exists $t_2 \geqslant t_1$ such that $x_2(t) \geqslant d$, $x_2(g_2(t)) \geqslant d$ for $t \geqslant t_2$. Using (d) we obtain from the first equation of (A)

$$u_1(t) - u_1(t_2) \geqslant Ld \int_{t_2}^t a_1(s)\,\mathrm{d}s.$$

With regard to (b), letting $t \to \infty$ we have $\lim_{t\to\infty} u_1(t) = \infty$, which contradicts the negativity of $u_1(t)$. Therefore $\lim_{t\to\infty} x_2(t) = 0$. Because of $\boldsymbol{x} \in N^-$, using Lemma 1 we have $\lim_{t\to\infty} u_1(t) = 0$, $\lim_{t\to\infty} x_1(t) = 0$.

b) Let $\boldsymbol{x} \in N^+$. We define the function

$$F(t) = \frac{x_2(t)g_1(t)}{\sum\limits_{i=0}^{n} p^i u_1(g_1(t) - i\tau)}, \quad t \geqslant t_1.$$

Then $F(t) \geqslant 0$ and using the first equation of (A) we have

$$\text{(7)}\quad F'(t) = \frac{-a_2(t)f_2(x_1(g_1(t)))g_1(t)}{\sum_{i=0}^{n} p^i u_1(g_1(t) - i\tau)} + \frac{g_1'(t)}{g_1(t)}\left(F(t) - F^2(t)\frac{\sum_{i=0}^{n} p^i u_1'(g_1(t) - i\tau)}{x_2(t)}\right).$$

In view of (4), (d) there exist an integer number $n \geqslant 0$ and $t_2 \geqslant t_1$ such that

$$\text{(8)}\qquad f_2(x_1(g_1(t))) \geqslant K\sum_{i=0}^{n} p^i u_1(g_1(t) - i\tau), \quad t \geqslant t_2.$$

Taking into account the monotonicity of x_2, (ii), (d) we obtain from the first equation of the system (A)

$$\text{(9)}\qquad \begin{aligned} u_1'(g_1(t) - i\tau) &= a_1(g_1(t) - i\tau)f_1(x_2(g_2(g_1(t) - i\tau))) \\ &\geqslant La_1(g_1(t) - i\tau)x_2(t), \quad t \geqslant t_2. \end{aligned}$$

Combining (7), (8), (9) we get

$$\begin{aligned} F'(t) \leqslant & -Ka_2(t)g_1(t) \\ & + L\sum_{i=0}^{n} p^i a_1(g_1(t) - i\tau)\frac{g_1'(t)}{g_1(t)}\left(\frac{F(t)}{L\sum_{i=0}^{n} p^i a_1(g_1(t) - i\tau)} - F^2(t)\right) \\ \leqslant & -Ka_2(t)g_1(t) + \frac{g_1'(t)}{4Lg_1(t)\sum_{i=0}^{n} p^i a_1(g_1(t) - i\tau)}, \quad t \geqslant t_2. \end{aligned}$$

Integration of the last inequality from t_2 to t yields

$$\text{(10)}\quad F(t) \leqslant F(t_3) - K\int_{t_2}^{t}\left(a_2(s)g_1(s) - \frac{g_1'(s)}{4LKg_1(s)\sum_{i=0}^{n} p^i a_1(g_1(s) - i\tau)}\right)\mathrm{d}s.$$

Letting $t \to \infty$ then by virtue of (6) we have $F(t) \to -\infty$, which contradicts the positivity of $F(t)$. $\square$

The conclusion of Theorem 1 can by strengthened as follows.

Theorem 2. *In addition to the conditions of Theorem* 1, *assume that*

(i) $g_2(t) \leqslant t$,

(ii) *there exists an integer number* $m \geqslant 1$ *such that* $g_1(t) + m\tau < t$.

If

$$\limsup_{t\to\infty} \int_{g_1(t)+m\tau}^{t} a_1(s) \int_{g_2(s)}^{t} a_2(u)\,\mathrm{d}u\,\mathrm{d}s > \frac{1}{KL}\frac{p^m(1-p)}{1-p^m} \tag{11}$$

then every proper solution of (A) *is oscillatory.*

Proof. Taking the proof of Theorem 1 into account, it is sufficient to show the impossibility of the case 2a).

Suppose the contrary. Let the system (A) have a solution with the properties $x_2(t) > 0$, $x_1(t) > 0$, $u_1(t) < 0$ for $t \geqslant T$, T sufficiently large. With regard to (A) $x_2(t)$ is a decreasing function and $u_1(t)$ is an increasing function and from (5) we get

$$x_1(g_1(t)) \geqslant -Au_1(g_1(t)+m\tau), \quad t \geqslant T, \tag{12}$$

where $A = \sum\limits_{j=1}^{m} 1/p^j$. Integrating the second equation of (A) from $s \geqslant T$ to $t > s$ and using the monotonicity of $u_1(t)$, $g_1(t)$, (d) and (12) we have

$$-x_2(s) \leqslant x_2(t) - x_2(s) \leqslant KAu_1(g_1(t)+m\tau)\int_s^t a_2(u)\,\mathrm{d}u.$$

Putting this inequality into the first equation of (A) we get

$$u_1'(s) \geqslant -KLAu_1(g_1(t)+m\tau)a_1(s)\int_{g_2(s)}^{t} a_2(u)\,\mathrm{d}u$$

and integration from $g_1(t)+m\tau$ to t yields

$$\begin{aligned} -u_1(g_1(t)+m\tau) &\geqslant u_1(t) - u_1(g_1(t)+m\tau) \\ &\geqslant -KLAu_1(g_1(t)+m\tau)\int_{g_1(t)+m\tau}^{t} a_1(s)\int_{g_2(s)}^{t} a_2(u)\,\mathrm{d}u\,\mathrm{d}s \end{aligned}$$

and so

$$1 \geqslant KLA\int_{g_1(t)+m\tau_1}^{t} a_1(s)\int_{g_2(s)}^{t} a_2(u)\,\mathrm{d}u\,\mathrm{d}s,$$

which contradicts (11). $\square$

Remark 1. If the system (A) is equivalent to the differential equation

$$(x(t)-px(t-\tau))'' + a(t)x(g(t)) = 0$$

then Theorems 1, 2 generalize the results for this equation given in the paper [2].

Theorem 3. *Let* $p=1$ *and let the assumptions* (i), (ii) *and* (11) *of Theorem* 1 *hold. Then first component of every nonoscillatory solution* $\boldsymbol{x}=(x_1,x_2)$ *of* (A) *is bounded.*

Proof. Let $\boldsymbol{x}=(x_1,x_2)$ be a nonoscillatory solution of (A) and let $x_1(t)>0$ for $t\geqslant t_0$. Then $x_2(t)$ is a decreasing function and there exists such a $t_1\geqslant t_0$ that there are two possibilities for $x_2(t)$:

1. $x_2(t)<0$ for $t\geqslant t_1$,
2. $x_2(t)>0$ for $t\geqslant t_1$.

We consider the case 1. Then $x_2(t)<x_2(t_1)=c<0$ and the first equation of (A) and (d) imply

$$u_1(t)-u_1(t_2)\leqslant Lc\int_{t_2}^{t} a_1(s)\,\mathrm{d}s, \quad t\geqslant t_2\geqslant t_1.$$

In view of (b) we see that $u_1(t)<0$, $\lim\limits_{t\to\infty} u_1(t)=-\infty$. Therefore $x_1(t)<x_1(t-\tau)$ for all large t. This implies that $x_1(t)$ is bounded, which contradicts $\lim\limits_{t\to\infty} u_1(t)=-\infty$.

Now we assume that 2 holds and we consider the following cases:

a) Let $\boldsymbol{x}\in N^+$. Then $u_1(t)$ is a decreasing function and by (4), (A), (d) the inequalities

$$x_1(g_1(t))\geqslant \sum_{i=0}^{n} u_1(g_1(t)-i\tau), \tag{13}$$

$$x_2'(t)\leqslant -Ka_1(t)\sum_{i=0}^{n} u_1(g_1(t)-i\tau),$$

$$u_1'(g_1(t)-i\tau)\geqslant La_1(g_1(t)-i\tau)x_2(t)$$

hold for $t\geqslant T$. Analogously as in the proof of Theorem 1 we define the function

$$F(t)=\frac{x_2(t)g_1(t)}{\sum_{i=0}^{n} u_1(g_1(t)-i\tau)}\geqslant 0.$$

In a similar manner as in the proof of Theorem 1 in the case 2b), using the inequalities (13) we get (10), which leads to contradicion.

b) Let $\boldsymbol{x}\in N^-$. Then $x_1(t)<x_1(t-\tau)$, which implies that $x_1(t)$ is bounded. $\square$

Theorem 4. *In addition to the conditions of Theorem 3 assume that*

(i) $g_2(t) \leqslant t$,

(ii) *there exists an integer number* $m \geqslant 1$ *such that* $g_1(t) + m\tau < t$.

If

$$\limsup_{t\to\infty} \int_{g_1(t)+m\tau}^{t} a_1(s) \int_{g_2(s)}^{t} a_2(u)\,\mathrm{d}u\,\mathrm{d}s > \frac{1}{mKL} \tag{14}$$

then every proper solution of (A) *is oscillatory.*

P r o o f. Let $\boldsymbol{x} = (x_1, x_2)$ be a nonoscillatory solution of (A) and let $x_1(t) > 0$ for $t \geqslant t_0$. Taking the proof of Theorem 3 into account, it is sufficient to show that the case 2b) is impossible. On the contrary we suppose that $x_1(t) > 0$, $u_1(t) < 0$, $x_2(t) > 0$ for $t \geqslant T \geqslant t_0$. By virtue of (5) and the monotonicity of u_1 we get inequality (12) from the proof of Theorem 2 in the form

$$x_1(g_1(t)) \geqslant -mu_1(g_1(t) + m\tau).$$

Repeating the corresponding part of the proof of Theorem 2 we obtain

$$mKL \int_{g_1(t)+m\tau}^{t} a_1(s) \int_{g_2(s)}^{t} a_2(u)\,\mathrm{d}u\,\mathrm{d}s \leqslant 1,$$

which contradicts (14). □

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