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MV-TEST SPACES VERSUS MV-ALGEBRAS

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Abstract. In analogy with effect algebras, we introduce the test spaces and MV-test spaces. A test corresponds to a hypothesis on the propositional system, or, equivalently, to a partition of unity. We show that there is a close correspondence between MV-algebras and MV-test spaces.

Keywords: algebra, effect algebra, MV-algebra, test space, MV-test space, state, weight

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1. Introduction

The event structure of a logical system of a quantum physical model can be identified with a quantum logic [1] or an effect algebra [9]. In classical mechanics it is assumed to be a Boolean algebra. Other important structures which entered mathematics in the fifties by the finding of Chang [2] are MV-algebras. For them we have the famous Mundici's representation theorem [13], [3]. It says that MV-algebras can be viewed as intervals in unital Abelian ℓ-groups. If we relax the lattice structure of unital po-groups, we have intervals which always correspond to effect algebras, but the vice-versa statement does not have to be true.

Let us recall that a partial algebra $E = (E; +, 0, 1)$ is said to be an *effect algebra* if, for all $a, b, c \in E$,

(E1) $a + b$ is defined in E iff $b + a$ is defined, and in this case $a + b = b + a$;

(E2) $a + b, (a + b) + c$ are defined iff $b + c$ and $a + (b + c)$ are defined, and in this case $(a + b) + c = a + (b + c)$;

(E3) for any $a \in E$, there exists a unique element $a' \in E$ such that $a + a' = 1$;

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(E4) if $a + 1$ is defined in E, then $a = 0$.

We define $a \leqslant b$ iff there exists an element $c \in E$ such that $a + c = b$, and then $\leqslant$ is a partial ordering; we write $c := b - a$.

For example, if (G, u) is an Abelian unital po-group with a strong unit u, and if $\Gamma(G, u) := \{g \in G\colon 0 \leqslant g \leqslant u\}$ is endowed with the restriction of the group addition $+$, then $(\Gamma(G, u); +, 0, u)$ is an effect algebra.

Let us recall that an *MV-algebra* is an algebra $M := (M; \oplus, \odot, ^*, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that, for all $a, b, c \in M$, we have $a \odot b = (a^* \oplus b^*)^*$ and

(M1) $a \oplus b = b \oplus a$;

(M2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;

(M3) $a \oplus 0 = a$;

(M4) $a \oplus 1 = 1$;

(M5) $(a^*)^* = a$;

(M6) $a \oplus a^* = 1$;

(M7) $0^* = 1$;

(M8) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.

If we define a partial operation $+$ on M in such a way that $a + b$ is defined in M iff $a \leqslant b^*$ and then set $a + b := a \oplus b$, we see that $(M; +, 0, 1)$ is an effect algebra.

For two elements a and b of an effect algebra E (or an MV-algebra) we write $a \perp b$ iff $a + b$ is defined.

For a finite sequence $F = \{a_i\}_{i=1}^n$ of elements in E we write $\bigoplus F := a_1 + \ldots + a_n$ provided the element $a_1 + \ldots + a_n$ is defined in E. In this case we also say that F is $\bigoplus$-orthogonal.

A finite sequence $F = \{a_i\}_{i=1}^n$ of non-zero elements in E is said to be a *partition of unity* if $a_1 + \ldots + a_n = 1$. Let us recall that our definition of partition of unity is different from that of Mundici [15] (e.g., $\{a, a*\}$ is a partition of unity $(0 \neq a \neq 1)$ but it is not a partition in the sense of Mundici).

It seems that MV-algebras play a role analogous to that of Boolean algebras within quantum logics [6]. For instance, it was observed by Riečanová [16] that every lattice effect algebra E can be covered by MV-subalgebras of E.

Foulis and Randall [10], in 1972, gave a new mathematical foundation of an operational probability theory and statistics based upon a generalization of the conventional notion of sample space in the sense of Kolmogorov [12]. They introduced tests and test spaces which in a "Kolmogorovian fashion" entail a propositional system of the quantum mechanical system. The main notion is a test space with tests. They are defined on the set of outcomes as a natural generalization of hypotheses which can be verified on outcomes after. After introducing an appropriate equivalence, we obtain an algebraic structure describing the quantum mechanical system.

This approach was further generalized in [5] (see also [6] for D-posets and effect algebras).

Inspired by this approach and the intimate relations among effect algebras and MV-algebras, we introduce two kinds of test spaces and MV-test spaces. They are in a close correspondence with MV-algebras. In addition, we show that there is a correspondence between states on MV-algebras and weights on MV-test spaces.

2. Test spaces I

In the present section we introduce test spaces. The definition is motivated by [6], [5]. Another equivalent notion will be introduced in Section 4.

Let X be a nonempty set. The elements of X are called *outcomes*. A function $F\colon I \to X$ is said to be of *finite multiplicity* if, for any $x \in X$, $F^{-1}(x)$ is of finite cardinality. For arbitrary sets I and J and functions $F \in X^I$ and $G \in X^J$ define $F \preccurlyeq G$ iff there is an injection $\sigma\colon I \to J$ such that $F = G \circ \sigma$, i.e. $F(i) = G(\sigma(i))$ for all $i \in I$. If $F \preccurlyeq G$ and $\sigma\colon I \to J$ is a bijection, we say that F and G are *equivalent*, in symbols $F \sim G$. In what follows, we will identify functions which are equivalent.[1] Therefore, we can correctly define a function $F \dot\cup G$ as follows: Let $K = I' \cup J'$, where $I' \cap J' = \emptyset$ and $\varphi\colon I \to I'$, $\psi\colon J \to J'$ are bijections. Then $F \dot\cup G$ is an element of X^K such that

$$F \dot\cup G(k) = \begin{cases} F(i) & \text{if } i \in I,\ k = \varphi(i), \\ G(j) & \text{if } j \in J,\ k = \psi(j). \end{cases}$$

Let us note that the associativity $(F \dot\cup G) \dot\cup H = F \dot\cup (G \dot\cup H)$ holds, and therefore for both sides we can write $F \dot\cup G \dot\cup H$. It can be easily checked that $\preccurlyeq$ is a preorder which becomes a partial order on equivalence classes with respect to $\sim$.

Let $\mathscr{R}(F) := \{F(i)\colon i \in I\}$ denote the range of $F \in X^I$.

Definition 2.1. Let $\mathscr{T} = \{F \in X^I\colon I \in \mathscr{I}\}$, where $X \neq \emptyset$ and $\mathscr{I}$ is a nonvoid family of index sets. We say that the pair $(X, \mathscr{T})$ is a *test space* (of the first type) if the following three conditions are satisfied:

(i) Any $T \in \mathscr{T}$ is of finite multiplicity.

(ii) For every $x \in X$ there is a $T \in \mathscr{T}$ such that $x \in \mathscr{R}(T)$.

(iii) If $S, T \in \mathscr{T}$ and $S \preccurlyeq T$, then $S \sim T$.

Any element of $\mathscr{T}$ is called a *test*.

[1] Observe that if $I = \emptyset$, then $X^I = \{\emptyset\}$.

Lemma 2.2. *If $F \in X^I$ is a test, then $I \neq \emptyset$.*

P r o o f. Let $F = \emptyset \in \mathscr{T}$. Then, for any $T \in \mathscr{T}$, $F \preccurlyeq T$ implies $F \sim T$ and, by (iii) of Definition 2.1, $T = \emptyset$ which is in contradiction with the condition (ii) of Definition 2.1. □

Definition 2.3. Let $(X, \mathscr{T})$ be a test space. Let J be arbitrary and let $G \in X^J$. We say that G is an *event* if there is a test $T \in \mathscr{T}$ such that $G \preccurlyeq T$. Let us denote the set of all events in $\mathscr{T}$ by $\mathscr{E} = \mathscr{E}(X, \mathscr{T})$.

Clearly, $\emptyset \in \mathscr{E}$.

Definition 2.4. Let $(X, \mathscr{T})$ be a test space. We say that two events F and G are

(i) *orthogonal* to each other, in symbols $F \perp G$, if there is a test $T \in \mathscr{T}$ such that $F \dot\cup G \preccurlyeq T$;

(ii) *local complements* of each other, in symbols $F \operatorname{loc} G$, if there is a test $T \in \mathscr{T}$ such that $F \dot\cup G \sim T$;

(iii) *perspective with axis* H, in symbols $F \approx_H G$, if they share a common local complement H.

We write $F \approx_H G$ or $F \approx G$ if the axis is not emphasized.

Lemma 2.5.

(i) *If $\emptyset \approx G$, then $\emptyset \sim G$.*

(ii) *If $T, S \in \mathscr{T}$ are tests, then $T \approx S$ with axis $\emptyset$.*

P r o o f. (i) Every local complement of $\emptyset$ is a test. Therefore, if $\emptyset \approx G$, there is a test T such that $T \dot\cup G$ is a test. But then $T \preccurlyeq T \dot\cup G$ implies $T \sim T \dot\cup G$, which implies $G \sim \emptyset$.

(ii) Observe that the unique local complement of every test is $\emptyset$. □

Definition 2.6. A test space $(X, \mathscr{T})$ is *algebraic* if the following implication holds true: If $F, G, H \in \mathscr{E}$, $F \approx G$ and $F \perp H$, then $G \perp H$.

For simplicity, we usually refer to X rather than to $(X, \mathscr{T})$ when we deal with test spaces. In what follows, let X be a test space.

Lemma 2.7. *A test space X is algebraic if and only if, for $F, G, H \in \mathscr{E}$,*

$$F \approx G, \ F \operatorname{loc} H \Rightarrow G \operatorname{loc} H.$$

P r o o f. Let X be algebraic and let $F, G, H \in \mathscr{E}$, $F \approx G$, $F \operatorname{loc} H$. The algebraicity implies that $G \perp H$. Let Q be a local complement of $G \dot\cup H$. Then

$H \dot{\cup} Q$ loc G implies $H \dot{\cup} Q \perp F$. But since H loc F, $H \dot{\cup} F \preccurlyeq H \dot{\cup} F \dot{\cup} Q$ implies $H \dot{\cup} F \sim H \dot{\cup} F \dot{\cup} Q$, we obtain $Q = \emptyset$.

The converse implication follows from the fact that if two events are orthogonal, then one of them can be enlarged to a local complement of the other. □

Lemma 2.8. *If X is algebraic, then $\approx$ is transitive on $\mathscr{E}$, hence $\approx$ is an equivalence relation.*

Proof. Suppose $F, G, H \in \mathscr{E}$ with $F \approx G$ and $G \approx H$. Let K be the axis for $G \approx H$. By Lemma 2.7, K is a local complement of F, hence $F \approx H$ with axis K. □

3. MV-test spaces

In the present section we show how MV-algebras can be identified with test spaces. Here by tests and test spaces we understand the corresponding notions defined in Section 2. In an analogous way we can also use the notions which will be defined in Section 4.

We say that a test space $(X, \mathscr{T})$ possesses the *strong Riesz property* if, given four events E_1, E_2, F_1, F_2 with $E_1 \perp E_2$ and $F_1 \perp F_2$ such that $E_1 \dot{\cup} E_2 \approx F_1 \dot{\cup} F_2$, there exist four events C_{11}, C_{12}, C_{21}, C_{22} such that $E_1 \approx C_{11} \dot{\cup} C_{12}$, $E_2 \approx C_{21} \dot{\cup} C_{22}$, $F_1 \approx C_{11} \dot{\cup} C_{21}$, $F_2 \approx C_{21} \dot{\cup} C_{22}$, and if there exists an event C such that $C \preccurlyeq C_{12}$ and $C \preccurlyeq C_{21}$, then $C \approx \emptyset$.

In what follows we show that any algebraic test space possessing the strong Riesz property will give rise to an MV-algebra, and conversely, each MV-algebra gives us an algebraic test space possessing the strong Riesz property. Hence, any algebraic test space possessing the strong Riesz property is said to be an *MV-test space*.

Definition 3.1. Let X be an algebraic test space. If $F \in \mathscr{E}$, we define $\pi(F) := \{G \in \mathscr{E} \colon G \approx F\}$ and refer to $\pi(F)$ as the *proposition affiliated with* F. The set

$$\Pi = \Pi(X) := \{\pi(F) \colon F \in \mathscr{E}\} \tag{3.1}$$

is called the *logic* of the test space X.

We define $0, 1 \in \Pi$ by

$$0 = \pi(\emptyset), \quad 1 = \pi(T), \tag{3.2}$$

where T is any test.

The following two theorems are the key results of the section.

Theorem 3.2. *Let $(X, \mathcal{T})$ be an MV-test space. Then $\Pi(X)$ with $0 := \pi(\emptyset)$, $1 = \pi(T)$, where T is a test, can be organized into an MV-algebra.*

Proof. We define a partial binary operation $+$ on $\Pi(X)$ as follows. For two elements $a = \pi(A)$ and $b = \pi(B)$, $a + b$ is defined in $\Pi(X)$ and equals to $c = \pi(C)$ iff there exists an event $C' \in \pi(C)$ such that there are $A' \in \pi(A)$ and $B' \in \pi(B)$ with $A'\dot{\cup}B' \approx C'$. The algebraicity of the test space yields that our partial binary operation $+$ is well-defined. In addition, it is commutative associative, with the neutral element 0. Also, $M := \Pi(X)$ is an effect algebra. For any $a = \pi(A)$ we define $a^* = \pi(A')$, where A' is a local complement of A.

The partial addition $+$ implies a partial binary relation $\leqslant$ on $\Pi(X)$ defined as follows: $a \leqslant b$ iff there exists an event $c \in M$ such that $a + c = b$. In that case, we define $c = b - a$.

Suppose now that $a_1 + a_2 = b_1 + b_2$ in M. Our hypotheses yield that there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in M$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$, $b_2 = c_{12} + c_{22}$ and $c_{12} \wedge c_{21} = 0$. Using this property, we can prove (see [7, Prop. 3.3]) that our effect algebra is a lattice. Indeed, since $a + a^* = b + b^* = 1$ for all $a, b \in M$, there exist four elements $c_{11}, c_{12}, c_{21}, c_{22} \in M$ such that $a = c_{11} + c_{12}$, $a^* = c_{21} + c_{22}$, $b = c_{11} + c_{21}$, $b^* = c_{12} + c_{22}$ and $c_{12} \wedge c_{21} = 0$. Then $c_{12} \vee c_{21} = c_{12} + c_{21}$ and $c_{11} + (c_{12} \vee c_{21}) = (c_{11} + c_{12}) \vee (c_{11} + c_{21}) = a \vee b$.

Moreover, it is possible to show (see [8, Prop. 8.7]) that, for all $a, b \in M$,

$$a - (a \wedge b) = (a \vee b) - b.$$

Hence, by [8, Thm 8.8], M can be organized into an MV-algebra $(M; \oplus, \odot, ^*, 0, 1)$ as follows:

$$a \oplus b := (a^* - (a^* \wedge b))^*,$$
$$a \odot b := a - (a \wedge b^*).$$

□

Theorem 3.3. *Let M be an MV-algebra. Then there exists an MV-test space $(X, \mathcal{T})$ such that M is an MV-algebra isomorphic to $\Pi(X)$.*

Proof. Let $X = M \setminus \{0\}$ and let $\mathcal{T}$ be the set of all finite partitions of 1 in M consisting of nonzero elements. We claim that $(X, \mathcal{T})$ is a test space. Indeed, we have $\{1\} \in \mathcal{T}$, and if $a \neq 0$, $a \neq 1$, then $\{a, a^*\} \in \mathcal{T}$ and $a \in \mathcal{R}(\{a, a^*\})$ which proves that $(X, \mathcal{T})$ is a test space. Moreover, $(X, \mathcal{T})$ is algebraic. Indeed, let F, G, H be events such that $F \approx G$ and $F \operatorname{loc} H$. Then $F = \{a_1, \dots, a_n\}$, $G = \{b_1, \dots, b_m\}$, $H = \{c_1, \dots, c_k\}$. Let $K = \{d_1, \dots, d_l\}$ be the axis for $F \approx G$. Now $\bigoplus F + \bigoplus K = 1$, $\bigoplus G + \bigoplus K = 1$, $\bigoplus F + \bigoplus H = 1$, so that $\bigoplus H = \bigoplus K$ and $\bigoplus G + \bigoplus H = 1$, hence $G \operatorname{loc} H$. Because every MV-algebra possesses the strong Riesz property, so does the test space. Thus the test space is an MV-test space.

Let $\Pi(X)$ be the logic of X. By Theorem 3.2, $\Pi(X)$ is an MV-algebra. Define a mapping $\varphi\colon M \to \Pi(X)$ by

$$\varphi(a) = \begin{cases} \pi(\{a\}) & \text{if } a \neq 0, \\ \pi(\emptyset) & \text{if } a = 0. \end{cases} \tag{3.3}$$

The mapping φ preserves the partial addition $+$ from M onto $\Pi(X)$. Indeed, let $a + b \in M$. If one of a, b is 0, then $\varphi(a) \perp \varphi(b)$ is evident. If $a \neq 0 \neq b$, then either $a = b^*$ or $a \neq b^*$. Thus $\{a, b\}$ or $\{a, b, (a+b)^*\}$, respectively, are partitions of 1, which implies that $\varphi(a) \perp \varphi(b)$ and $\varphi(a+b) = \varphi(a) + \varphi(b)$.

Let now $\pi(\{a\}) \perp \pi(\{b\})$. There is a partition of 1 containing a and b, hence $a \perp b$ in M. This proves that φ is injective. To prove that φ is onto, let $A = \{a_1, \ldots, a_n\}$ be an event. Then A is $\bigoplus$-orthogonal, and we put $a = \bigoplus\limits_{i=1}^{n} a_i$. Then

$$\pi(A) = \pi(\{a_1, \ldots, a_n\}) = \bigoplus_{i=1}^{n} \pi(\{a_i\}) = \pi(\{a\}) = \varphi(a),$$

hence φ is onto.

Moreover, we show that φ preserves $\vee$. It is clear that $\varphi(a \vee b) \geqslant \varphi(a), \varphi(b)$. If $\varphi(x) \geqslant \varphi(a), \varphi(b)$, then $x \geqslant a, b$, so that $x \geqslant a \vee b$ and $\varphi(x) \geqslant \varphi(a \vee b)$. According to the rest of the proof of Theorem 3.2, φ is the MV-isomorphism in question. $\square$

Let $\varphi\colon X \to Y$ be a mapping. For $E \in X^I$ we denote by $\varphi(E)$ the function

$$\varphi(E) := \varphi \circ E \in Y^I,$$

that is,

$$\varphi(E)(i) = \varphi(E(i)), \quad i \in I. \tag{3.4}$$

A mapping $\varphi\colon X \to Y$ is called a *homomorphism* of test spaces $(X, \mathscr{T})$ and $(Y, \mathscr{S})$ if $\varphi(T) \in \mathscr{S}$ for every $T \in \mathscr{T}$. Evidently, if $E, F \in \mathscr{E}(X)$ and $E \perp F$, then $\varphi(E)$, $\varphi(F) \in \mathscr{E}(Y)$, $\varphi(E) \perp \varphi(F)$ and $\varphi(E \mathbin{\dot\cup} F) = \varphi(E) \mathbin{\dot\cup} \varphi(F)$ (we identify events related by $\sim$). Also, $E \approx F$ implies $\varphi(E) \approx \varphi(F)$.

Proposition 3.4. *Let φ be a homomorphism of two test spaces $(X, \mathscr{T})$ and $(Y, \mathscr{S})$. If $(X, \mathscr{T})$ is an MV-test space, then so is $(\varphi(X), \varphi(\mathscr{T}))$, where $\varphi(\mathscr{T})$ stands for $\{\varphi(T)\colon T \in \mathscr{T}\}$.*

Proof. For any $y \in \varphi(X)$ there is an $x \in X$ with $y = \varphi(x)$. If $T \in \mathscr{T}$ is such that $x \in \mathscr{R}(T)$, then $\varphi(x) \in \mathscr{R}(\varphi(T))$. This proves that $(\varphi(X), \varphi(\mathscr{T}))$ is a test space. If now $E \in X^I$, $F \in X^J$ are such that $\varphi(E) \perp \varphi(F)$, then $I \cap J = \emptyset$, so that $E \perp F$. Moreover, $\varphi(E \mathbin{\dot\cup} F) = \varphi(E) \mathbin{\dot\cup} \varphi(F)$. If $\varphi(E) \operatorname{loc} \varphi(F)$, then $E \perp F$ implies that there is a $G \in \mathscr{E}(X)$ such that $E \mathbin{\dot\cup} G \mathbin{\dot\cup} F \in \mathscr{T}$. But then $\varphi(E) \mathbin{\dot\cup} \varphi(G) \mathbin{\dot\cup} \varphi(F) \in \varphi(\mathscr{T})$ and also $\varphi(E) \mathbin{\dot\cup} \varphi(F) \in \varphi(\mathscr{T})$, which yields $\varphi(G) = \emptyset$. Hence $G = \emptyset$ and $E \operatorname{loc} F$. This proves that $E \operatorname{loc} F$ iff $\varphi(E) \operatorname{loc} \varphi(F)$, from which we easily derive that $(\varphi(X), \varphi(\mathscr{T}))$ is algebraic iff $(X, \mathscr{T})$ is algebraic. Moreover, we easily see that if $(X, \mathscr{T})$ is an MV-test space, then so is $(\varphi(X), \varphi(\mathscr{T}))$. $\square$

Proposition 3.5. *Let $(X, \mathscr{T})$ and $(Y, \mathscr{S})$ be MV-test spaces and let $\varphi\colon X \to Y$ be a homomorphism. Then the mapping $\psi\colon \Pi(X) \to \Pi(Y)$ defined by*

$$\psi(\pi(E)) := \pi(\varphi(E)) \quad (E \in \mathscr{E}(X)), \tag{3.5}$$

is a homomorphism of effect algebras.

Proof. Clearly, $\psi(1) = 1$. Further, $\pi(F) \perp \pi(G)$ implies $F \perp G$ and $\varphi(F) \perp \varphi(G)$, so that $\psi(\pi(F)) \perp \psi(\pi(G))$ and $\varphi(F \mathbin{\dot\cup} G) = \varphi(F) \mathbin{\dot\cup} \varphi(G)$, which gives $\psi(\pi(F) + \pi(G)) = \psi(\pi(F)) + \psi(\pi(G))$. $\square$

Let us recall that if $h\colon M_1 \to M_2$ is a homomorphism of MV-algebras, then the restriction of h onto $M_1 \setminus \{0\}$ is a homomorphism of MV-test spaces $(M_1 \setminus \{0\}, \mathscr{T})$ and $(M_2 \setminus \{0\}, \mathscr{S})$, where $\mathscr{T}$ and $\mathscr{S}$ are the sets of all partitions in M_1 and M_2, iff h is injective. Hence we can define the following two categories.

Let $\mathscr{MV}_0$ be the category of MV-algebras whose objects are MV-algebras and morphisms are injective homomorphisms of MV-algebras. By $\mathscr{MVT}$ we denote the category of MV-test spaces whose objects are MV-test spaces and morphisms are injective homomorphisms of MV-test spaces. It would be interesting to see whether these categories are equivalent. This question seems to be open.

4. Test spaces II

In the present section we describe an alternative and equivalent way how to define test spaces. This approach is motivated by [11].

Let X be a nonempty set describing a system, elements of X being called *outcomes*, and let $\mathscr{T} \subseteq \{0, 1, 2, \ldots\}^X$. We call $(X, \mathscr{T})$ a *test space* (of the second type) if

(1) for any $x \in X$ there exists a $T \in \mathscr{T}$ such that $T(x) \neq 0$;

(2) if $S, T \in \mathscr{T}$ with $S \leqslant T$ (i.e., $S(x) \leqslant T(x)$ for all $x \in X$), then $S = T$.

The elements of $\mathscr{T}$ are called *tests*.

We call a function $F \in \{0, 1, 2, \ldots\}^X$ an *event* if $F \leqslant T$ for some $T \in \mathscr{T}$, and let $\mathscr{E} = \mathscr{E}(X, \mathscr{T})$ be the set of all events. We say that $F, G \in \mathscr{E}$ are

(i) *mutually exclusive* ($F \perp G$) if $F + G \in \mathscr{E}$;

(ii) *local complements* of each other ($F \operatorname{loc} G$) if $F + G \in \mathscr{T}$,

(iii) *perspective* ($F \approx G$) if they share a local complement. We say that a test space $\mathscr{T}$ is *algebraic* if the following holds true: if $F, G, H \in \mathscr{E}$, $F \approx G$ and $H \perp F$, then $H \perp G$.

For two events $F, G \in \mathscr{E}$ we write $F \preccurlyeq G$ iff there is a mutually exclusive event $H \perp F$ such that $F + H = G$.

Let $O\colon X \to \{0, 1, 2, \ldots\}$ be defined by $O(x) = 0$ for all $x \in X$. Then O is an event, while $O \leqslant T$ for any test T, i.e., $O \in \mathscr{E}$. In addition, we always have $T \in \mathscr{E}$. We recall that if T is a test, then $T \neq O$. If not, since $O \leqslant S$ for any $S \in \mathscr{T}$, we would have $S = O$ by (2) which is a contradiction with (1) of the definition of the test space.

It is worth recalling that if $\mathscr{T} \subseteq \{0, 1\}^X$, then the tests are crisp subsets of X because they are characteristic functions of subsets of X. They then correspond to two-valued reasoning, i.e. to the situation when the system is defined precisely. If $\mathscr{T} \subseteq \{0, 1, \ldots, n\}^X$, then the tests correspond to the situation when the system is described by multivalued logic, e.g. by an $(n+1)$-valued logic using fuzzy sets from

$$L_{n+1} := \{0, 1/n, 2/n, \ldots, n/n\}$$

as is shown in the following examples.

Example 4.1. Let B be a Boolean algebra of subsets of a nonvoid set Ω. Define $X = B \setminus \{\emptyset\}$. Let $A_1, \ldots, A_n$ be a decomposition of Ω, i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$, and $A_1 \cup \ldots \cup A_n = \Omega$. Define a function $F_{A_1, \ldots, A_n}\colon X \to \{0, 1\}$ by

$$F_{A_1, \ldots, A_n}(A) = \begin{cases} 1 & \text{if } A = A_i \text{ for some } i = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Then the system $\mathscr{T}$ of all such functions $F_{A_1, \ldots, A_n}$ is an algebraic test space.

The following two examples are connected with the Ulam game with one lie (n lies), [3], i.e. a game with two players where one player is asking the smallest possible number of questions to find an unknown number from the known set of numbers, and the second player can only answer "yes" or "no", being allowed at most one lie (n lies).

Example 4.2. Let us consider a three-valued logic, i.e., the Łukasiewicz logic connected with the truth values system $L_3 = \{0, 1/2, 1\}$. Assume that our system is described by a two-valued set $X = \{x_1, x_2\}$. Define two functions F_1 and F_2 as mappings from X into $\{0, 1, 2\}$ by

$$F_1(x) = \begin{cases} 2 & \text{if } x = x_1, \\ 0 & \text{if } x = x_2, \end{cases}$$

and

$$F_2(x) = \begin{cases} 0 & \text{if } x = x_1, \\ 1 & \text{if } x = x_2. \end{cases}$$

Then $\{F_1, F_2\}$ is an algebraic test space connected with a three-valued reasoning.

Example 4.3. Let us consider an $(n+1)$-valued logic, i.e., the Łukasiewicz logic connected with the truth values system $L_{n+1} = \{0, 1/n, 2/n, \ldots, 1\}$. Assume that our system is described by an n-element set $X = \{x_1, \ldots, x_n\}$. Define for any n-tuple $(k_1, \ldots, k_n)$ of integers from the set $\{0, 1, \ldots, n\}$ with $1k_1 + 2k_2 + \ldots + nk_n = n$ a function $F_{k_1,\ldots,k_n}\colon X \to \{0, 1, \ldots, n\}$ by

$$F_{k_1,\ldots,k_n}(x) = k_i \quad \text{if } x = x_i \text{ for some } i = 1, \ldots, n.$$

Then the system $\mathscr{T}$ of all such functions $F_{k_1,\ldots,k_n}$ is an algebraic test space connected with $(n+1)$-valued reasoning.

It is interesting to recall that the test spaces are in fact all partitions of the truth-value which are possible in the given system using two-valued or multivalued reasoning for the description of our logical system.

Let us recall that the following results have similar proofs as those in Section 2. We include them here only to make the exposition self-contained.

Proposition 4.4. *A test space $(X, \mathscr{T})$ is algebraic if and only if, for all $F, G, H \in \mathscr{E}$,*

$$F \approx G,\ F \operatorname{loc} H \Rightarrow G \operatorname{loc} H.$$

Proof. Let $(X, \mathscr{T})$ be an algebraic test space, and let $F, G, H \in \mathscr{E}$, $F \approx G$, $F \operatorname{loc} H$. The algebraicity implies that $G \perp H$. Let Q be a local complement of

$G+H$. Then $(H+Q)\operatorname{loc} G$ implies $(H+Q)\perp F$. But since $(H+F)\preccurlyeq H+F+Q$, we get $Q=O$, i.e. $G\operatorname{loc} H$.

Suppose the converse and let $F\approx G$ and $H\perp F$. Then $H+F\in\mathscr{E}$, and there is a local complement F' to $H+F$, i.e. $H+F+F'\in\mathscr{T}$. Hence $(H+F')\operatorname{loc} F$ and by the assumptions, $G\operatorname{loc}(H+F')$ so that $H\perp G$. $\square$

Proposition 4.5. *Let $(X,\mathscr{E})$ be an algebraic test space. Then the relation $\approx$ is an equivalence on the space of all events $\mathscr{E}=\mathscr{E}(X,\mathscr{T})$.*

Proof. Let F be an event. There is a test T such that $F\leqslant T$. Then $F'=T-F$ is a local complement of F, hence $F\approx F$.

It is evident that if $F\approx G$ then $G\approx F$.

Assume $F\approx G$ and $G\approx H$. By Proposition 4.4, if K is a local complement to H, so is it for G and consequently for F. $\square$

Let us say that a test space $(X,\mathscr{T})$ possesses the *strong Riesz property* if, given four events E_1,E_2,F_1,F_2 with $E_1\perp E_2$ and $F_1\perp F_2$ such that $E_1+E_2\approx F_1+F_2$, there exist four events C_{11}, C_{12}, C_{21}, C_{22} such that $E_1\approx C_{11}+C_{12}$, $E_2\approx C_{21}+C_{22}$, $F_1\approx C_{11}+C_{21}$, $F_2\approx C_{21}+C_{22}$, and if there exists an event C such that $C\preccurlyeq C_{12}$ and $C\preccurlyeq C_{21}$, then $C\approx\emptyset$.

Like in Section 3 we define, for an algebraic test space $(X,\mathscr{T})$,

$$\pi(F):=\{G\in\mathscr{E}:\ G\approx F\}$$

and refer to $\pi(F)$ as the *proposition* (or an *event*) affiliated with F. The set

$$\Pi=\Pi(X,\mathscr{T}):=\{\pi(F):\ F\in\mathscr{E}\}$$

is called the *event structure* of the test space $(X,\mathscr{T})$.

We define $0,1\in\Pi$ by

$$0=\pi(O),\quad 1=\pi(T),$$

where T is any test.

We say that a test space $(X,\mathscr{T})$ (of the second type) possesses the *strong Riesz property* if, given four events E_1,E_2,F_1,F_2 with $E_1\perp E_2$ and $F_1\perp F_2$ such that $E_1+E_2\approx F_1+F_2$, there exist four events C_{11}, C_{12}, C_{21}, C_{22} such that $E_1\approx C_{11}+C_{12}$, $E_2\approx C_{21}+C_{22}$, $F_1\approx C_{11}+C_{21}$, $F_2\approx C_{21}+C_{22}$, and if there exists an event C such that $C\preccurlyeq C_{12}$ and $C\preccurlyeq C_{21}$, then $C\approx\emptyset$.

An algebraic test space (of the second type) satisfying the strong Riesz property is called an *MV-test space* (of the second type). Comparing both types of MV-test spaces we will see that they are essentially identical. It follows from the next result because results analogous to Theorem 3.2 and Theorem 3.3 are also true for this type of test spaces.

Theorem 4.6. *Let M be an MV-algebra. Then there exists an MV-test space $(X, \mathscr{T})$ such that $\Pi(X, \mathscr{T})$ is an MV-algebra isomorphic to M. Conversely, if $(X, \mathscr{T})$ is an MV-test space, then $\Pi(X)$ can be organized into an MV-algebra.*

Proof. The proof follows similar lines to those in Theorems 3.2 and 3.3. We outline only the main ideas.

Let M be an MV-algebra and let $(M; +, 0, 1)$ be the effect algebra derived from the MV-algebra M, where $+$ is the partial addition taken from $\oplus$ via $a + b := a \oplus b$ whenever $a \leqslant b^*$. Set $X = E \setminus \{0\}$. We define a test space $\mathscr{T}$ as the system of all functions $F\colon X \to \{0, 1, 2, \ldots\}$ such that F is zero for all but a finite number of elements of X and

$$\sum_{a \in X} F(a)a = 1. \tag{4.1}$$

In other words, any non-zero value of $F(a)a$ is multiplicity of the element a in the partition of 1.

We claim that $(X, \mathscr{T})$ is an algebraic test space. Indeed, let $a \in X$. Then either there is an integer $n \geqslant 1$ such that $na = 1$, or there is no such an integer. In the former case we define F_a by $F_a(a) = n$ and $F_a(x) = 0$ for $x \neq a$. In the latter case, we put $F_a(a) = 1$, $F_a(a') = 1$, and $F_a(x) = 0$ if $x \notin \{a, a'\}$. Then $F_a \in \mathscr{T}$ and $F_a(a) \neq 0$.

If now $F \leqslant G$ for $F, G \in \mathscr{T}$, then

$$1 = \sum_{a \in X} F(a)a \leqslant \sum_{a \in X} G(a)a = 1,$$

which proves $F = G$.

It is evident that a function $F\colon X \to \{0, 1, 2, \ldots\}$ is an event iff F takes nonzero values only for finitely many elements of X, and

$$\sum_{a \in X} F(a)a \in E.$$

Hence $F' \in \mathscr{E}(X, \mathscr{T})$ is a local complement of F iff

$$\sum_{a \in X} F'(a)a + \sum_{a \in X} F(a)a = 1.$$

Hence if $F \approx G$ and $H \operatorname{loc} F$, then $H \operatorname{loc} G$.

Let $\Pi(X, \mathscr{T})$ be the event structure affiliated with the algebraic test space $(X, \mathscr{T})$ which, analogously to Theorem 3.3, is an effect algebra.

The function $F_a\colon\ X \to \{0, 1, 2, \ldots\}$ defined by

$$F_a(x) = \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{otherwise} \end{cases}$$

is an element of $\mathscr{E}$. Define a mapping $\varphi\colon\ E \to \Pi(X, \mathscr{T})$ by

$$\varphi(a) = \begin{cases} \pi(F_a) & \text{if } a \neq 0, \\ \pi(O) & \text{if } a = 0. \end{cases}$$

The mapping φ preserves $+$ in E. Indeed, let $a, b \in E$ and $a + b \in E$. If one of them is 0, then $\varphi(a) + \varphi(b)$ is defined in $\Pi(X, \mathscr{T})$. If $a \neq 0 \neq b$, then either $a = b'$ or $a \neq b'$ and $a + b = 1$ or $a + b + (a + b)' = 1$, respectively. Hence $\varphi(a) + \varphi(b)$ is defined, and $\varphi(a) + \varphi(b) = \varphi(a + b)$.

Let now $\varphi(F_a) \perp \varphi(F_b)$. There exists a partition of 1 containing a and b, hence $a \perp b$ in E. This proves that φ is injective and preserves $+$ both in E and in $\Phi(X, \mathscr{T})$.

To show that φ is surjective, let F be an event in $\mathscr{E}$. Then F is nonzero for finitely many elements of E and $\sum\limits_{a \in X} F(a)a \in E$. Hence

$$\varphi(a) = \pi(F_a) = \pi(F),$$

proving that φ is onto.

Using now arguments similar to those in Theorem 3.3, we can prove that $\Pi(X)$ is an MV-algebra isomorphic to M.

The converse statement is now clear; its proof follows the main steps of the proofs of Theorems 3.2 and 3.3. □

5. States and Weights

A state on an MV-algebra M is said to be a mapping $s\colon\ M \to [0, 1]$ such that $s(1) = 1$ and $s(a \oplus b) = s(a) + s(b)$ whenever $a \leqslant b^*$.

States on MV-algebras were introduced by F. Chovanec [4] and by D. Mundici [14] with the intent of capturing the notion of "average degree of truth" of a proposition, see also [17].

It is important to recall that every MV-algebra possesses at least one state, see [14].

Let $(X, \mathscr{T})$ be an (algebraic) test space (of the second type). A *weight* on X is a function $\omega\colon\ X \to [0, 1]$ such that, for any test $T \in \mathscr{T}$,

$$\omega(T) := \sum_{x \in X} T(x)\omega(x) = 1. \tag{5.1}$$

In what follows we show that there is a correspondence between states on MV-algebras and weights in MV-test spaces.

Theorem 5.1. *Let ω be a weight on an MV-test space $(X,\mathscr{T})$. The mapping $s\colon \Pi(X,\mathscr{T})\to[0,1]$ defined via*

$$s(\pi(F)) := \sum_{x\in X} F(x)\omega(x), \quad \pi(F)\in\Pi(X,\mathscr{T}), \tag{5.2}$$

is a state on the MV-algebra $\Pi(X,\mathscr{T})$. Conversely, if s is a state on an MV-algebra M, then the function $\omega\colon M\setminus\{0\}\to[0,1]$ defined by $\omega(x):=s(x)$, $x\in M\setminus\{0\}$, is a weight on the MV-test space $(M\setminus\{0\},\mathscr{T})$, where $\mathscr{T}$ is the system of all tests defined by (4.1).

P r o o f. Let $a=\varphi(F)=\varphi(F_1)$ and let F' be a local complement of F, so that it is a local complement of F_1 as well. Hence the function s defined by (5.2) is well-defined in view of

$$\sum_{x\in X} F(x)\omega(x)+\sum_{x\in X} F'(x)\omega(x)=1=\sum_{x\in X} F_1(x)\,\omega(x)+\sum_{x\in X} F'(x)\omega(x).$$

The additivity of s is now clear.

Conversely, let s be a state on an effect algebra E. Using the ideas of the proof of Theorem 4.6 and the algebraic test spaces (3.3), wee see that setting $\omega(x):=s(x)$, $x\neq 0$, gives us a weight on $X=E\setminus\{0\}$ and $\mathscr{T}$. $\square$

If we take an MV-test space of the first kind, then a weight is defined as a function $\omega\colon X\to[0,1]$ for an MV-test space $(X,\mathscr{T})$ such that, for every $E\in\mathscr{T}$, $E\in X^I$, we have

$$\omega(E) := \sum_{i\in I}\omega(E(i)) = 1.$$

It is evident that we can establish an analogous assertion among weights on MV-test spaces (of the first kind) and states on MV-algebras as in Theorem 5.1.

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