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ON SET COVARIANCE AND THREE-POINT TEST SETS

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Abstract. The information contained in the measure of all shifts of two or three given points contained in an observed compact subset of $\mathbb{R}^d$ is studied. In particular, the connection of the first order directional derivatives of the described characteristic with the oriented and the unoriented normal measure of a set representable as a finite union of sets with positive reach is established. For smooth convex bodies with positive curvatures, the second and the third order directional derivatives of the characteristic is computed.

Keywords: convex body, set with positive reach, normal measure, set covariance

MSC 2000: 52A22

1. Introduction

Let $X \subseteq \mathbb{R}^d$ be a nonempty compact set and $T \subseteq \mathbb{R}^d$ a nonempty finite set containing the origin. Consider the functional

$$\psi_X(T) = \lambda^d\biggl(\bigcap_{t\in T}(X-t)\biggr),$$

where λ^d denotes the d-dimensional Lebesgue measure. We interpret X as an observed body, T as a test set (probe) and $\psi_X(T)$ is the commonly used morphological characteristic of the erosion of X by T, i.e., the measure of all shifts of T contained in X. Note that $\psi_X(T)$ is the volume of X if T has one point and it coincides with the set covariance (called also covariogram) if T has two points.

If $T = \{0, t_1, \dots, t_n\}$ has $n+1$ points, we shall often write $\psi_X^{(n)}(t_1, \dots, t_n)$ or simply $\psi_X(t_1, \dots, t_n)$ instead of $\psi_X(T)$.

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Usually random instead of deterministic sets are considered in mathematical morphology. If X is a random compact set (in the sense of Matheron [6]) then the mean value $\mathrm{E}\psi_X(T)$ is the basic characteristic of interest. If X is a stationary closed subset of $\mathbb{R}^d$, the mean value must be replaced by the spatial intensity. In this note, the deterministic case will be treated; nevertheless, the results may be applied also in the random setting.

The goal of this note is to present some results concerning the information on X which can be retrieved from the observations $\psi_X(T)$ for two- and three-point test sets T. It was shown by Nagel [7] (and by another method in [9]) that three-point test sets are sufficient to determine a compact set uniquely up to shifts and differences of Lebesgue measure zero. The results presented in this note are restricted to special classes of compact sets.

Let PR denote the family of subsets of $\mathbb{R}^d$ with positive reach (this concept was introduced by Federer [2], see also [12], [13]). For $X \in \mathrm{PR}$, the curvature measures $C_k(X;\cdot)$, $k = 0, 1, \ldots, d-1$, are defined as signed Radon measures on the unit normal bundle nor X. The concept of curvature measures was extended to the family $\mathscr{U}_{\mathrm{PR}}$ of locally finite unions of sets with positive reach such that all their finite intersections have positive reach as well, see [13]. The second coordinate projections of curvature measures, called *spherical area measures*, are denoted by

$$\sigma_k(X;\cdot) = C_k(X;\mathbb{R}^d \times \cdot);$$

in particular, $\sigma_{d-1}(X;\cdot)$ is obtained by integrating the usual unit outer normal to X over the boundary of X (cf. [13, Theorem 4.1]). The symmetrized version of $\sigma_{d-1}(X;\cdot)$,

$$\sigma^*_{d-1}(X;\cdot) = \sigma_{d-1}(X;\cdot) + \sigma_{d-1}(-X;\cdot),$$

is called the (unoriented) normal measure of X. The normal measure is a function of the boundary ∂X only and can be determined from the intensity of the intersections of ∂X with lines of given directions. The determination of its oriented version, $\sigma_{d-1}(X;\cdot)$, is more complicated, see [16], [10], [11], [15].

It is well known at least for convex sets X that the directional derivative at the origin of the set covariance equals minus the volume of the projection of X in the given direction, and the volumes of projections of a convex set determine the normal measure of X, see e.g. [6]. We shall show in Section 3 that this is true even for $\mathscr{U}_{\mathrm{PR}}$-sets satisfying certain full-dimensionality condition. In analogy to this result, we show in Section 2 that the first order directional derivatives of $\psi^{(2)}_X(\cdot,\cdot)$ for certain $\mathscr{U}_{\mathrm{PR}}$ sets X determine the oriented normal measure of X.

In order to understand better the information obtained in the set covariance of convex bodies, we calculate certain second and third order derivatives of the set covariance for smooth convex bodies with positive Gauss curvature. This yields a partial answer in the planar smooth case to the question of determination of a convex body by its set covariance up to shifts and central reflection, see also Bianchi et al. [1].

2. Three-point test sets

In this section, the function

$$\psi_X(t_1, t_2) = \lambda^d(X \cap (X - t_1) \cap (X - t_2))$$

will be investigated. First, we recall a known result (see also [7, Satz 2.1]).

Theorem ([9, Theorem 4]). *The function $\psi_X(\cdot,\cdot)$ determines the compact set X uniquely up to a translation and a set difference of Lebesgue measure zero.*

Remarks.

1. In [9], the corresponding result was formulated for dilations instead of erosions; nevertheless, the proof given there is based on erosions and the transfer from dilations to erosions is enabled by the inclusion-exclusion formula.
2. It is clear that for any finite T, $\psi_X(T)$ is invariant under translations as well as set differences of Lebesgue measure zero. Therefore, the information obtained from three-point test sets is optimal and no additional information on X is obtained for finite test sets of more than three points.

Now we shall investigate the directional derivatives of $\psi_X(\cdot,\cdot)$ for certain $\mathscr{U}_{\mathrm{PR}}$-sets. We say that $X \in \mathscr{U}_{\mathrm{PR}}$ is *full-dimensional* if there exists no $(x, n) \in \operatorname{nor} X$ such that $(x, -n) \in \operatorname{nor} X$, where $\operatorname{nor} X$ is the unit normal bundle of X (see [13]).

We say that $X \in \mathrm{PR}$ compact has *bounded tangential projections* (cf. [14]) if for any $0 \leqslant k \leqslant d-2$,

$$\sup_{W \in G(d,k+1)} H^k(T_W X) < \infty,$$

where $G(d, k+1)$ is the Grassmannian of $(k+1)$-dimensional linear subspaces of $\mathbb{R}^d$,

$$T_W X = \{p_W x \colon \exists n \in W \cap \mathbb{S}^{d-1}, (x, n) \in \operatorname{nor} X\},$$

p_W denotes the orthogonal projection onto W and H^k denotes the k-dimensional Hausdorff measure. We use the standard notation SO_d for the group of Euclidean rotations in $\mathbb{R}^d$.

Let $\mathscr{U}$ denote the set of all compact $\mathscr{U}_{\mathrm{PR}}$-sets with representation $X = X_1 \cup \ldots \cup X_N$ such that

1. $X_{i_1} \cap \ldots \cap X_{i_k}$ has bounded tangential projections for any $1 \leqslant i_1 < \ldots < i_k \leqslant N$,
2. $H^{d-1}(\partial X_i \cap \partial X_j) = 0$ for any $1 \leqslant i < j \leqslant N$,
3. X is full-dimensional.

Theorem 1. *Let $0 \neq u, v \in \mathbb{R}^d$ be two vectors whose angle is not a rational multiple of π. Then the directional derivatives $d\psi_X^{(2)}(0,0)(\varrho u, \varrho v)$ together with $d\psi_X^{(1)}(0)(\varrho u)$, $\varrho \in \mathrm{SO}_d$, and $\psi_X(0)$ determine $\sigma_{d-1}(X;\cdot)$ for a set $X \in \mathscr{U}$ uniquely.*

Remark. If follows from Theorem 1 that $\sigma_{d-1}(X;\cdot)$ can be determined from two-dimensional sections of X (but, of course, not from one-dimensional sections; these determine the unoriented normal measure only).

Proof. Let K be the triangle with vertices 0, u, v. K satisfies the assumption of [11, Theorem 3.1] and it is not difficult to see that the pair X, T satisfies the assumptions of [11, Corollary 4.2]. In fact, using the boundedness of tangential projections of X, we may apply [14, Theorem 2] (eventually its proof) to verify the assumptions of [11, Corollary 4.2] for the pair X, L, where L is either the two-dimensional subspace spanned by K, or any of the one-dimensional subspaces spanned by the edges of K. It follows that X, K satisfy the assumptions of [11, Corollary 4.2] as well. Hence, applying [11, Theorem 3.1, Corollary 4.2], we see that the derivatives of the functions

$$\varepsilon \mapsto \lambda^d(X \oplus \varepsilon\varrho K), \quad \varrho \in \mathrm{SO}_d,$$

at zero determine uniquely $\sigma_{d-1}(X,\cdot)$. By using the elementary inclusion and exclusion formula, one can express the volume of the dilation of X by a three-point set by means of the volumes of erosion by subsets of this three-point set. Therefore, the information on X assumed in the Theorem yields the derivatives of the functions

$$\varepsilon \mapsto \lambda^d(X \oplus \varepsilon\varrho\{0, u, v\}), \quad \varrho \in \mathrm{SO}_d.$$

The proof is completed by applying Lemma 2 below. □

For the notion of rectifiability, we refer to Federer's book [3].

Lemma 1. *Let $Y \subseteq \mathbb{R}^d$ be countably $(H^{d-1}, d-1)$-rectifiable and H^{d-1}-measurable and let S be a segment in $\mathbb{R}^d$. Then*

$$\lambda^d(\{z \in \mathbb{R}^d :\ H^0(Y \cap (z + \varepsilon S)) \geqslant 2\}) = o(\varepsilon), \quad \varepsilon \to 0.$$

Proof. Suppose without loss of generality that S is a unit segment centred at the origin. Let L be the line containing S and $L^\perp$ the $(d-1)$-subspace orthogonal to L. For $y \in L^\perp$, set

$$k(y) = H^0(Y \cap (y + L)).$$

It is well known that the mapping k is measurable, $k(y) < \infty$ for λ^{d-1}-a.a. $y \in L^\perp$ and

$$\int_{L^\perp} k(y)\lambda^{d-1}(\mathrm{d}y) \leqslant H^{d-1}(Y) < \infty$$

(the last integral is the total projection content of Y onto $L^{\perp}$). Further, to a given $y \in L^{\perp}$ we attach

$$\delta(y) = \inf\{|u - v|\colon\ u, v \in Y \cap (y + L),\ \ u \neq v\}.$$

The mapping δ is measurable as well and since $k(y) < \infty$ implies $\delta(y) > 0$, it fulfills $\delta(y) > 0$ for a.a. $y \in L^{\perp}$. We have

$$\begin{aligned}
&\lambda^d\{z\colon\ H^0(Y \cap \varepsilon(z + \varepsilon S)) \geqslant 2\} \\
&\quad = \int_{L^{\perp}} \lambda^1\{z \in y + L\colon\ H^0(Y \cap (z + \varepsilon S)) \geqslant 2\}\lambda^{d-1}(\mathrm{d}y) \\
&\quad \leqslant \int_{\{\delta(y) \leqslant \varepsilon\}} \varepsilon k(y)\lambda^{d-1}(\mathrm{d}y).
\end{aligned}$$

Since $\{\delta(y) \leqslant \varepsilon\} \searrow \{\delta(y) = 0\}$ and $\lambda^{d-1}(\{\delta(y) = 0\}) = 0$, we conclude that

$$\lim_{\varepsilon \searrow 0} \int_{\{\delta(y) \leqslant \varepsilon\}} k(y)\lambda^{d-1}(\mathrm{d}y) = 0,$$

which completes the proof. $\square$

Lemma 2. *Let K be a triangle in $\mathbb{R}^d$ with vertices u, v, w and let $X \in \mathscr{U}$. Then*

$$\lambda^d((X \oplus \varepsilon K) \setminus (X \oplus \varepsilon\{u, v, w\})) = o(\varepsilon), \quad \varepsilon \to 0_+.$$

Proof. Denote by L the linear hull of K and let $z \in \mathbb{R}^d$ be such that $z + \varepsilon K$ hits X but $z + \varepsilon\{u, v, w\}$ does not. Then, by the full-dimensionality of X, at least one of the following three cases occurs:

1. ∂X hits one of the sides of $z + \varepsilon K$ twice,
2. X osculates with the linear hull of one of the sides of $z + \varepsilon K$ (i.e., there exists $(x, m) \in \operatorname{nor} X$ with $x \in z + \varepsilon K$ and m perpendicular to a side of $z + \varepsilon K$),
3. $X \cap (z + \varepsilon K)$ is a relatively (in $z + L$) isolated component of $X \cap (L + z)$.

Let N_1^{ε}, N_2^{ε}, N_3^{ε} denote, in turn, the sets of shifts z with properties 1, 2, 3. We have $\lambda^d(N_1^{\varepsilon}) = o(\varepsilon)$ by Lemma 1 since ∂X is compact and $(H^{d-1}, d-1)$-rectifiable. Using the boundedness of tangential projections and [14, Theorem 2], we get $\lambda^d(N_2^{\varepsilon}) = 0$. Finally, note that, given $y \in L^{\perp}$,

$$\lambda^2(N_3^{\varepsilon} \cap (y + L)) \leqslant \varepsilon^2\lambda^2(K)C(y),$$

where $C(y)$ is the number of isolated components of $X \cap (L + y)$. Clearly $C(y)$ is bounded from above by the zeroth absolute curvature measure of $X \cap (y + L)$,

$V_0^{\mathrm{abs}}(X \cap (y+L))$, and, hence,

$$\lambda^d(N_3^\varepsilon) = \int_{L^\perp} \lambda^2(N_3^\varepsilon \cap (y+L))\, \lambda^{d-2}(\mathrm{d}y) \\ \leqslant \varepsilon^2 \lambda^2(T) \int_{L^\perp} V_0^{\mathrm{abs}}(X \cap (y+L)) \lambda^{d-2}(\mathrm{d}y),$$

the last integral being bounded again by the boundedness of tangential projections and [14, proof of Theorem 2]. □

3. Set covariance

In this section we shall investigate the information contained in the function

$$\psi_X(u) = \lambda^d((X \cap (X-u)), \quad u \in \mathbb{R}^d.$$

First note that, besides translations and set differences of Lebesgue measure zero, ψ is also invariant with respect to central reflection. But, even when ignoring the translations, differences of Lebesgue measure zero and reflections, a compact set is not determined by the set covariance, as has been shown independently by H. Rost in [8] and Lešanovský and Rataj [4], see also [5]. E.g., the sets $[0,5] \cup [6,9] \cup [11,12]$ and $[0,2] \cup [4,10] \cup [11,12]$ in $\mathbb{R}$ have the same set covariance but they are different even after translations and reflection.

For $u \in \mathbb{S}^{d-1}$, the *total projection* of X in the direction u is defined by

$$TP_X(u) = \frac{1}{2} \int_{\operatorname{nor} X} |u \cdot n| C_{d-1}(X; d(x,n)).$$

It is well known that the directional derivative of the covariance equals the total projection, at least for convex sets X. We present a proof of this fact in a more general setting, for full-dimensional $\mathscr{U}_{\mathrm{PR}}$-sets.

Theorem 2. *For $X \in \mathscr{U}_{\mathrm{PR}}$ bounded full-dimensional and $u \in \mathbb{S}^{d-1}$ we have*

$$\mathrm{d}\psi_X(0)(u) = -TP_X(u).$$

P r o o f. Let $\partial_R X$ denote the set of all "regular" boundary points of X, i.e., the set of points $x \in \partial X$ at which the tangent cone $\operatorname{Tan}(\partial X, x)$ is a $(d-1)$-dimensional subspace of $\mathbb{R}^d$. Due to the full-dimensionality assumptions, there exists a unique unit outer normal $n = n(x)$ with $(x,n) \in \operatorname{nor} X$ at any regular boundary point $x \in \partial_R X$. We have $H^{d-1}(\partial X \setminus \partial_R X) = 0$; this follows from [13, Theorem 4.1] after applying the area formula to the projection $(x,n) \mapsto x$ whose Jacobian vanishes if $x \notin \partial_R X$. Further, denote

$$\partial_{u-} X = \{x \in \partial_R X : \; u \cdot n(x) > 0\}$$

and notice that

$$TP_X(u) = \int_{\partial_{u-} X} u \cdot n(x) H^{d-1}(\mathrm{d}x).$$

For $x \in \partial X$, denote by

$$q_u(x) = \inf\{t > 0 : \; x - tu \notin X \;\text{ or }\; x + tu \in X\}$$

(the maximum t such that the segment $[x - tu, x + tu]$ meets X exactly in the half-segment $[x - tu, x]$). Note that the half-segment $[x - \min\{t, q_u(x)\}u, x]$ is "lost" when X is intersected with $X - tu$. Thus we get using the Fubini theorem

$$\lambda^d(X) - \lambda^d(X \cap (X - tu)) \geqslant \int_{\partial_{u-} X} \min\{t, q_{u-}(x)\} \, u \cdot n(x) H^{d-1}(\mathrm{d}x).$$

On the other hand, we have clearly

$$\lambda^d(X) - \lambda^d(X \cap (X - tu)) \leqslant \int_{\partial_{u-} X} tu \cdot n(x) H^{d-1}(\mathrm{d}x).$$

Note that since $u \cdot n(x) > 0$ and X is fulldimensional, $q_{u-}(x) > 0$ whenever $x \in \partial_{u-} X$. Thus $t^{-1} \min\{t, q_{u-}(x)\}$ converges monotonely upwards to 1 as t tends to 0 from the right for any $x \in \partial_{u-} X$, and we get the assertion by the monotone convergence theorem. $\square$

It is well known that the total projections determine the normal measure uniquely (see, e.g., 6, Theorem 4.5.1]). Thus we get the following result.

Corollary 1. *The first order derivative at the origin of the set covariance determines uniquely the normal measure $\sigma^*_{d-1}(X; \cdot)$ of a full-dimensional $\mathscr{U}_{\mathrm{PR}}$-set $X \subseteq \mathbb{R}^d$.*

In the remainder of this note we shall investigate higher order derivatives of the set covariance at the origin. We shall limit ourselves to smooth convex bodies. Assume thus that X is a full-dimensional convex body in $\mathbb{R}^d$, ∂X is C^2-smooth and that the Gauss curvature $K(x)$ is positive at any $x \in \partial X$. Thus, there exists a unique outer unit normal $n(x)$ at any $x \in \partial X$ and the Gauss map $x \mapsto n(x)$ is invertible.

Let $u \in \mathbb{S}^{d-1}$ be fixed. The mapping $f\colon \; t \mapsto \psi_X(tu)$ is differentiable at $t > 0$ small enough with

$$f'(t) = -TP_{X \cap (X - tu)}(u) = -\frac{1}{2} \int_{\mathbb{S}^{d-1}} |u \cdot v| \sigma_{d-1}(X \cap (X - tu); \mathrm{d}v)$$

(see Theorem 2). The last integral is equal to

$$\int_{\partial X} |u \cdot n(x)| g(\mathbf{1}_X(x-tu) + \mathbf{1}_X(x+tu)) H^{d-1}(\mathrm{d}x),$$

hence

$$f'(t) - f'(0) = \frac{1}{2} \int_{\partial X} |u \cdot n(x)| \mathbf{1}_{X^C}(x-tu) \mathbf{1}_{X^C}(x+tu) H^{d-1}(\mathrm{d}x).$$

Denote $P = \{x \in \partial X \colon\ n(x) \cdot u = 0\}$; P is a $(d-2)$-dimensional C^2-submanifold of ∂X (note that u is not a tangent vector to P at any $z \in P$ since, otherwise, the Gauss curvature at x would have to be zero). Let C be the cylinder over P in the direction u. Let p denote the mapping which assigns to $y \in C$ the point $x \in \partial X$ such that y is the metric projection of x onto C. p is a bijection defined on some neighbourhood of P in C. Let, further, q denote the Gauss map on C. Then we have by the area and co-area formulas

$$\begin{aligned}
&f'(t) - f'(0) \\
&\quad = \frac{1}{2} \int_C |u \cdot n(p(y))| J_{d-1} p(y) \mathbf{1}_{X^C}(p(y) - tu) \mathbf{1}_{X^C}(p(y) + tu) H^{d-1}(\mathrm{d}y) \\
&\quad = \frac{1}{2} \int_{\mathbb{S}_{d-2}(u^\perp)} \int_{q^{-1}\{v\}} |u \cdot n(p(y))| \frac{J_{d-1} p(y)}{J_{d-2} q(y)} \mathbf{1}_{X^C}(p(y) - tu) \mathbf{1}_{X^C}(p(y) + tu) \\
&\qquad\qquad \times H^1(\mathrm{d}y) H^{d-2}(\mathrm{d}v).
\end{aligned}$$

For given $v \in \mathbb{S}_{d-2}(u^\perp)$, let $z(v)$ denote the point of P with $n(z(v)) = v$ and let h_v be the function

$$h_v(s) = (z(v) - p(z(v) + su)) \cdot v$$

defined for small s (the graph of h_v parametrizes the section of ∂X by the plane through $z(v)$ spanned by the vectors u, v). h_v is a convex C^2-function with $h_v'(0) = 0$ and $h_v''(0) = k_{\langle u \rangle}(v)$, where $k_{\langle u \rangle}(v)$ denotes the normal curvature of X at $z(v)$ in the direction u. We have further

$$\mathbf{1}_{X^C}(p(y) - tu) \mathbf{1}_{X^C}(p(y) + tu) = \begin{cases} \mathbf{1}_{\{f(s-t) < f(s)\}}, & s > 0, \\ \mathbf{1}_{\{f(s+t) > f(s)\}}, & s < 0, \end{cases}$$

for $y = z(v) + su$. Thus we have

$$f'(t) - f'(0) = \frac{1}{2} \int_{\mathbb{S}_{d-2}(u^\perp)} \int_{\xi_v - t}^{\xi_v} |u \cdot n(p(z(v) + su))| \frac{J_{d-1} p(z(v) + su)}{J_{d-2} q(z(v) + su)} \, \mathrm{d}s H^{d-2}(\mathrm{d}v),$$

where $\xi_v > 0$ is such that $h_v(\xi_v - t) = h_v(\xi_v)$. We find easily that $\xi_v = \frac{1}{2}t + o(t)$ and $h_v(\xi_v - t) = \frac{1}{8} k_{\langle u \rangle}(v) t^2 + o(t^2)$, $h_v(\xi_v^+) = \frac{1}{8} k_{\langle u \rangle}(v) t^2 + o(t^2)$, $t \to 0$. We shall need later that the remainder terms above are of order $o(t^2)$ uniformly in v; this can be guaranteed by the Taylor formula if ∂X is C^3-smooth.

Further, we note that the influence of the curvature of C on the Jacobian $J_{d-1}p$ can be neglected for our purposes; in fact, we have

$$J_{d-1}p(z(v)+su) = |v \cdot n(p(z(v)+su))|^{-1} + O(s), \quad s \to 0,$$

where the remainder $O(s)$ is uniform in v. The Jacobian $J_{d-2}q(z(v)+su)$ equals the product of all $d-2$ nonzero principal curvatures of C at $z(v)$; we shall denote this number by $K_C(v)$. If dimension $d=2$ we set $K_C(v)=1$ since clearly $J_0q(v)=1$. Using the fact that

$$\left|\frac{u \cdot n(p(z(v)+su))}{v \cdot n(p(z(v)+su))}\right| = h_v'(s),$$

we can write

$$\begin{aligned}
f'(t) - f'(0) &= \frac{1}{2}\int_{\mathbb{S}_{d-2}(u^\perp)} \int_{\xi_v - t}^{\xi_v} (|h_v'(s)| + O(s))\,\mathrm{d}s \frac{1}{K_C(v)} H^{d-2}(\mathrm{d}v) \\
&= \frac{1}{2}\int_{\mathbb{S}_{d-2}(u^\perp)} (h_v(\xi_v - t) + h_v(\xi_v) + o(t^2)) \frac{1}{K_C(v)} H^{d-2}(\mathrm{d}v) \\
&= \frac{t^2}{8}\int_{\mathbb{S}_{d-2}(u^\perp)} \frac{k_{\langle u\rangle}(v)}{K_C(v)} H^{d-2}(\mathrm{d}v) + o(t^2).
\end{aligned}$$

We conclude the following

Theorem 3. *Let X be a C^3-smooth convex body in $\mathbb{R}^d$ with positive Gauss curvature and let $u \in \mathbb{S}^{d-1}$. Then $d^2\psi_X(0)(u,u) = 0$ and*

$$\mathrm{d}^3\psi_X(0)(u,u,u) = \frac{1}{4}\int_{\mathbb{S}_{d-2}(u^\perp)} \frac{k_{\langle u\rangle}(v)}{K_C(v)} H^{d-2}(\mathrm{d}v).$$

Remarks.

1. In $\mathbb{R}^3$, $K_C(v)$ is equal to the normal curvature of the cylinder C at $z(v)$ in the direction perpendicular to u. If, for example, X is the ball in $\mathbb{R}^3$ of radius R, then clearly $K_C(v) = k_{\langle u\rangle}(v) = R^{-1}$ for any unit vectors $u \perp v$, thus $\mathrm{d}^3\psi_X(0)(u,u,u) = \frac{1}{2}\pi$ by Theorem 3, which agrees with the directly obtained formula
$$\psi_X(tu) = \pi\Big(\frac{4}{3}R^3 - R^2 t + \frac{1}{12}t^3\Big).$$
2. In $\mathbb{R}^2$, the assumption of C^2-smoothness of X is sufficient since $\mathbb{S}^0(u^\perp)$ is finite and no uniform property is needed. In this case, we get the expression
$$\mathrm{d}^3\psi_X(0)(u,u,u) = \frac{1}{4}(k(u_+^\perp) + k(u_-^\perp)),$$

where $k(u_+^\perp)$, $k(u_-^\perp)$ are the curvatures of ∂X at the two boundary points with normal perpendicular to u. The first directional derivatives of ψ_X at 0 determine the normal measure of X which has density $k(u_+^\perp)^{-1} + k(u_-^\perp)^{-1}$, $u \in \mathbb{S}^1$. It follows that both the first and the third order directional derivatives of a C^2-smooth planar convex body with positive curvature determine the (unordered) pairs of curvatures $k(u_+^\perp), k(u_-^\perp)$ at any $u \in \mathbb{S}^{d-1}$ uniquely, which was shown independently in [1].

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