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INTEGRAL MULTILINEAR FORMS ON $C(K,X)$ SPACES

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Abstract. We use polymeasures to characterize when a multilinear form defined on a product of $C(K,X)$ spaces is integral.

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1. Introduction and notation

Given a compact Hausdorff space K with Borel σ-algebra Σ, and Banach spaces X and Y, it is well known that an operator $T\colon C(K,X) \longrightarrow Y$ can be represented in terms of a measure $m\colon \Sigma \longrightarrow \mathcal{L}(X;Y^{**})$ verifying certain properties (see for instance [5, § 19]).

In a series of papers (see [7], [8] and the references therein), Dobrakov developed a theory of *polymeasures* (set functions defined on a product of σ-algebras which are separately measures) that can be used to extend the classical Riesz representation theorem to a multilinear setting. With this theory, multilinear operators from a product of $C(K,X)$ spaces into Y can be represented as operator valued polymeasures. This representation theorem can be found in [12, Theorem 1.1]. The theory of polymeasures has been used by different authors, see, f.i., [1], [9], [10], [6] and the references therein.

In [12] we used the above mentioned representation theorem to obtain necessary and sometimes sufficient conditions on the polymeasure Γ representing a multilinear operator T for T to be completely continuous or unconditionally converging. In this note, which can be thought of as a continuation to [12], we use some techniques

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developed in [3] to characterize the integral multilinear forms (see definition below) $T\colon\ C(K_1,X_1)\times\ldots\times C(K_n,X_n)\longrightarrow\mathbb{K}$ in terms of their representing polymeasures Γ.

In this note we follow the notation of [12]. However we recall some basic notation. K, K_i will always be compact Hausdorff spaces and Σ, Σ_i will be their Borel σ-algebras. If X is a Banach space, $C(K,X)$ is the Banach space of the X-valued continuous functions, endowed with the supremum norm. $S(\Sigma,X)$ is the space of the X-valued Σ-simple functions defined on K and $B(\Sigma,X)$ is the completion of $S(\Sigma,X)$ under the supremum norm. It is well known that $C(K,X)^* = \mathrm{bvrca}(\Sigma;X^*)$, the space of regular measures with bounded variation defined on Σ with values in X^*, endowed with the variation norm. We write $\mathrm{bv}(\Sigma;X)$ for the measures from Σ into X with bounded variation and similarly we write $\mathrm{bv}(\Sigma_1,\ldots,\Sigma_n;X)$ for the polymeasures from $\Sigma_1\times\ldots\times\Sigma_n$ into X with bounded variation.

For notation and basic facts concerning polymeasures we refer to [12] and the references therein.

The following two definitions go back to Grothendieck.

Definition 1.1. A multilinear form $T\in\mathcal{L}^k(X_1,\ldots,X_n)$ is integral if $\hat{T}$ (i.e., its linearization) is continuous for the injective (ε) topology on $X_1\otimes\ldots\otimes X_n$. Its norm (as an element of $(X_1\hat{\otimes}_\varepsilon\ldots\hat{\otimes}_\varepsilon X_n)^*$) is the *integral* norm of T, $\|T\|_{\mathrm{int}}:=\|\hat{T}\|_\varepsilon$.

Definition 1.2. An operator $T\in\mathcal{L}(X;Y)$ is integral if the associated bilinear form

$$\begin{aligned} B_T\colon\ X\times Y^* &\longrightarrow\mathbb{K},\\ (x,y)&\mapsto y(T(x)) \end{aligned}$$

is integral. In that case the integral norm of T, $\|T\|_{\mathrm{int}}:=\|B_T\|_{\mathrm{int}}$. $\mathcal{I}(X;Y)$ denotes the Banach space of the integral operators from X into Y, endowed with the integral norm.

We will use the fact that a bilinear form $T\in\mathcal{L}^2(E_1,E_2)$ is integral if and only if any of the two associated linear operators $T_1\in\mathcal{L}(E_1;E_2^*)$ and $T_2\in\mathcal{L}(E_2;E_1^*)$ is integral in the above sense (see, f.i., [4, Chapter VI]).

We will also need the following result from [11].

Proposition 1.3. *Let $T\in\mathcal{L}(C(K,X);Y)$ and let m be its representing measure. Then T is integral if and only if m is $\mathcal{I}(X;Y)$-valued and it has bounded variation when considered with values in this space.*

We will later need the following well known lemma, which can be found, for instance, in [2].

Lemma 1.4. *Let Σ be a σ-algebra, X a Banach space and $Y \subset X^*$ a subspace norming X. If $m\colon \Sigma \longrightarrow X$ is a strongly additive and $\sigma(X,Y)$-regular measure, then m is regular.*

If $\Gamma\colon \Sigma_1 \times \ldots \times \Sigma_n \longrightarrow X$ is a polymeasure, we define its *variation*

$$v(\Gamma)\colon \Sigma_1 \times \ldots \times \Sigma_n \longrightarrow [0,+\infty]$$

by

$$v(\Gamma)(A_1,\ldots,A_n) = \sup\left\{ \sum_{j_1=1}^{m_1} \ldots \sum_{j_n=1}^{m_n} \|\Gamma(A_{1,j_1},\ldots,A_{n,j_n})\| \right\}$$

where $(A_{i,j_i})_{j_i=1}^{m_i}$ is a Σ_i-partition of A_i $(1 \leqslant i \leqslant n)$.

The following lemma can be found in [3].

Lemma 1.5. *Let X be a Banach space, $\Omega_1,\ldots,\Omega_n$ sets and $\Sigma_1,\ldots,\Sigma_n$ σ-algebras defined on them. Let now $\gamma\colon \Sigma_1 \times \ldots \times \Sigma_n \longrightarrow X$ be a polymeasure and let $\varphi_1\colon \Sigma_1 \longrightarrow \mathrm{pm}(\Sigma_2,\ldots,\Sigma_n;X)$ be the measure given by $\varphi_1(A_1)(A_2,\ldots,A_n) = \gamma(A_1)(A_2,\ldots,A_n)$. Then $v(\gamma) < \infty$ if and only if φ_1 takes values in $\mathrm{bvpm}(\Sigma_2,\ldots\Sigma_n;X)$ and $v(\varphi_1) < \infty$ when we consider the variation norm in the image space. In that case, $v(\varphi_1)(A_1) = v(\gamma)(A_1,\Omega_2,\ldots,\Omega_n)$ and $v(\varphi_1(A_1))(A_2,\ldots A_n) \leqslant v(\gamma)(A_1, A_2,\ldots,A_n)$. Of course the role played by the first variable could be played by any of the other ones.*

2. The result

We can present now our main result. In the following we write $B_0(K_1 \times \ldots \times K_n)$ for the σ-algebra of the Borel sets of $C(K_1 \times \ldots \times K_n)$.

Proposition 2.1. *Let $T \in \mathcal{L}^k(C(K_1,X_1),\ldots,C(K_n,X_n))$ and let Γ be its representing polymeasure. Then the following are equivalent:*

a) *The polymeasure $\Gamma\colon \Sigma_1 \times \ldots \times \Sigma_n \longrightarrow (X_1\hat{\otimes}_\pi \ldots \hat{\otimes}_\pi X_n)^*$ can be extended to a measure $m \in \mathrm{bvrca}(B_0(K_1 \times \ldots \times K_n);(X_1\hat{\otimes}_\varepsilon \ldots \hat{\otimes}_\varepsilon X_n)^*)$ (which implies in particular that Γ is $(X_1\hat{\otimes}_\varepsilon \ldots \hat{\otimes}_\varepsilon X_n)^*$-valued).*
b) *Γ is $(X_1\hat{\otimes}_\varepsilon \ldots \hat{\otimes}_\varepsilon X_n)^*$-valued and $v(\Gamma) < \infty$, when we consider the integral norm in the image space.*
c) *T is integral.*

Moreover, in that case $v(\Gamma) = v(\mu) = \|T\|_{\rm int}$, so there is an isometric isomorphism between $(C(K_1, X_1)\hat{\otimes}_\varepsilon \ldots \hat{\otimes}_\varepsilon C(K_n, X_n))^*$ and the space of separately regular polymeasures with bounded variation defined on $\Sigma_1 \times \ldots \times \Sigma_n$ with values in $(X_1\hat{\otimes}_\varepsilon \ldots \hat{\otimes}_\varepsilon X_n)^*$, endowed with the variation norm.

P r o o f. (c) $\Rightarrow$ (a): If

$$T\colon C(K_1, X_1) \times \ldots \times C(K_n, X_n) \longrightarrow \mathbb{K}$$

is integral, then we can consider the continuous linear operator

$$T'\colon C(K_1)\hat{\otimes}_\varepsilon X_1 \hat{\otimes}_\varepsilon \ldots \hat{\otimes}_\varepsilon C(K_n)\hat{\otimes}_\varepsilon X_n \longrightarrow \mathbb{K}$$

and, using the associativity of the injective tensor product and the fact that $C(K_1)\hat{\otimes}_\varepsilon \ldots \hat{\otimes}_\varepsilon C(K_n) \approx C(K_1 \times \ldots \times K_n)$, we can define the integral operator

$$T_1\colon C(K_1 \times \ldots \times K_n) \longrightarrow (X_1\hat{\otimes}_\varepsilon \ldots \hat{\otimes}_\varepsilon X_n)^*.$$

Let $\mu \in \operatorname{bvrca}(B_0(K_1 \times \ldots \times K_n); (X_1\hat{\otimes}_\varepsilon \ldots \hat{\otimes}_\varepsilon X_n)^*)$ be the representing measure of T_1.

From regularity it follows that, for every $(A_1, \ldots, A_n) \in \Sigma_1 \times \ldots \times \Sigma_n$,

$$\mu(A_1 \times \ldots \times A_n) = \Gamma(A_1, \ldots, A_n),$$

i.e., that μ extends Γ.

(a) $\Rightarrow$ (b) is clear (observe that $v(\Gamma) \leqslant v(m)$).

(b) $\Rightarrow$ (c): We reason by induction on n. If $n = 1$ there is nothing to prove. Let us consider $n = 2$ and let T and Γ be as in the hypothesis.

Let $\varphi_1\colon \Sigma_1 \longrightarrow \operatorname{bv}(\Sigma_2; (X_1\hat{\otimes}_\varepsilon X_2)^*)$ be the measure associated to Γ given by $\varphi_1(A_1)(A_2) = \Gamma(A_1, A_2)$. Since $v(\Gamma) < \infty$, Lemma 1.5 assures that φ_1 is indeed $\operatorname{bv}(\Sigma_2; (X_1\hat{\otimes}_\varepsilon X_2)^*)$-valued and has bounded variation with values in this space.

Claim 1. *For every $A_1 \in \Sigma_1$, $\varphi_1(A_1)$ is a regular measure.*

Every measure of bounded variation is strongly additive ([4, Proposition I.1.15]). So, by Lemma 1.4, to prove the claim we just need to check that, for every $g \in X_1 \otimes X_2$, $g \circ \varphi_1(A_1)$ is regular. So, let us first suppose that $g = x_1 \otimes x_2$. Then

$$(x_1 \otimes x_2) \circ \varphi_1(A_1)(A_2) = \Gamma(A_1, A_2)(x_1, x_2).$$

Since Γ is weak*-separately regular (see [12, Theorem 1.1]) we get that $(x_1 \otimes x_2) \circ \varphi_1(A_1)$ is regular. From here the result follows easily for a general $g \in X_1 \otimes X_2$ and the claim is established. As a consequence of it we obtain that φ_1 is $\operatorname{bvrca}(\Sigma_2, (X_1\hat{\otimes}_\varepsilon X_2)^*)$-valued.

Claim 2. *The measure*

$$\varphi_1\colon\ \Sigma_1 \longrightarrow \mathrm{bvrca}(\Sigma_2,(X_1\hat{\otimes}_\varepsilon X_2)^*)$$

is regular.

We have that $C(K_2,X_1\hat{\otimes}_\varepsilon X_2)$ is isometrically isomorphic to a subspace of $B(\Sigma_2,X_1\hat{\otimes}_\varepsilon X_2)$, which in turn is isometrically isomorphic to a subspace of $C(K_2,X_1\hat{\otimes}_\varepsilon X_2)^{**}\approx(\mathrm{bvrca}(\Sigma_2,(X_1\hat{\otimes}_\varepsilon X_2)^*))^*$. Moreover, $S(\Sigma_2,X_1\hat{\otimes}_\varepsilon X_2)$ is a dense subspace of $B(\Sigma_2,X_1\hat{\otimes}_\varepsilon X_2)$ and $S(\Sigma_2,X_1\otimes X_2)$ is a dense subspace of $S(\Sigma_2,X_1\hat{\otimes}_\varepsilon X_2)$. So, $S(\Sigma_2,X_1\otimes X_2)\subset C(K_2,X_1\hat{\otimes}_\varepsilon X_2)^{**}$ is a subspace norming $\mathrm{bvrca}(\Sigma_2,(X_1\hat{\otimes}_\varepsilon X_2)^*)$. Therefore, by reasonings analogous to the proof of Claim 1, we just need to prove that, for every $s\in S(\Sigma_2,X_1\otimes X_2)$, $s\circ\varphi_1$ is regular. This follows again from the separate weak*-continuity of Γ, considering first $s=\chi_A(x_1\otimes x_2)$, then $s=\chi_A g$ for any $g\in X_1\otimes X_2$ and finally $s=\sum\limits_{m=1}^{n}\chi_{A_m}g_m$ for any $A_m\in\Sigma_2$ and $g_m\in X_1\otimes X_2$.

Therefore $\varphi_1\colon\ \Sigma_1\longrightarrow\mathrm{bvrca}(\Sigma_2,(X_1\hat{\otimes}_\varepsilon X_2)^*)$ is regular and $v(\varphi_1)=v(\Gamma)<\infty$. Observe that $\mathrm{bvrca}(\Sigma_2,(X_1\hat{\otimes}_\varepsilon X_2)^*)=C(K_2,X_1\hat{\otimes}_\varepsilon X_2)^*=(X_1\hat{\otimes}_\varepsilon C(K_2,X_2))^*=I(X_1;C(K_2,X_2)^*)$.

So, we can consider the operator $T_{\varphi_1}\colon\ C(K_1,X_1)\longrightarrow C(K_2,X_2)^*$ defined by $T_{\varphi_1}(f)=\int f\,\mathrm{d}\varphi_1$ and, according to Proposition 1.3, T_{φ_1} is integral (and $\|T_{\varphi_1}\|=v(\varphi_1)=v(\gamma)$). Hence, the bilinear form

$$\widetilde{T_{\varphi_1}}\colon\ C(K_1,X_1)\times C(K_2,X_2)\longrightarrow\mathbb{K}$$

is integral.

Let

$$\overline{T_{\varphi_1}}\colon\ B(\Sigma_1,X_1)\times B(\Sigma_2,X_2)\longrightarrow\mathbb{K}$$

be the extension of $\widetilde{T_{\varphi_1}}$ given by [12, Theorem 1.1]. Then, for every $x_i\in X_i$, $A_i\in\Sigma_i$ $(1\leqslant i\leqslant 2)$, we have

$$\begin{aligned}\overline{T}_{\varphi_1}(x_1\chi_{A_1},x_2\chi_{A_2})&=\varphi_1(A_1)(A_2)(x_1\otimes x_2)\\&=\Gamma(A_1,A_2)(x_1\otimes x_2)=\overline{T}(x_1\chi_{A_1},x_2\chi_{A_2}),\end{aligned}$$

where $\overline{T}$ is the extension of T given by [12, Theorem 1.1].

Therefore $\overline{T}_{\varphi_1}=\overline{T}$, so $\widetilde{T_{\varphi_1}}=T$ and $\|T\|_{\mathrm{int}}=\|\widetilde{T_{\varphi_1}}\|_{\mathrm{int}}=v(\varphi_1)=v(\gamma)$, which finishes the proof in the case $n=2$.

Let us now suppose the result to be true for $n=1$, consider

$$T\colon\ C(K_1,X_1)\times\ldots\times C(K_n,X_n)\longrightarrow\mathbb{K}$$

and let its associated polymeasure Γ be as in the hypothesis. Let

$$\varphi_1 \colon \Sigma_1 \longrightarrow \mathrm{bvpm}(\Sigma_2, \ldots, \Sigma_n; (X_1 \hat{\otimes}_\varepsilon \ldots \hat{\otimes}_\varepsilon X_n)^*)$$

be the measure associated to Γ given by $\varphi_1(A_1)(A_2, \ldots, A_n) = \Gamma(A_1, \ldots, A_n)$. By Lemma 1.5 we get that φ_1 is well defined and with bounded variation. Similarly to the proof of Claim 1 above it can be proved that, for every $A_1 \in \Sigma_1$, the polymeasure $\varphi_1(A_1)$ is separately regular. Call Z the space of separately regular polymeasures with bounded variation defined on $\Sigma_2 \times \ldots \times \Sigma_n$ and with values in $(X_1 \hat{\otimes}_\varepsilon \ldots \hat{\otimes}_\varepsilon X_n)^*$. Note that the induction hypothesis tells us that

$$\begin{aligned} Z &= C(K_2 \times \ldots \times K_n, X_1 \hat{\otimes}_\varepsilon X_2 \hat{\otimes}_\varepsilon \ldots \hat{\otimes}_\varepsilon X_n)^* \\ &= (X_1 \hat{\otimes}_\varepsilon C(K_2, X_2) \hat{\otimes}_\varepsilon \ldots \hat{\otimes}_\varepsilon C(K_n, X_n))^* \\ &= I(X_1; (C(K_2, X_2) \hat{\otimes}_\varepsilon \ldots \hat{\otimes}_\varepsilon C(K_n, X_n))^*). \end{aligned}$$

Now we can continue similarly to the proof of the case $n = 2$ to prove that φ_1 is regular, and the proof finishes similarly to the case $n = 2$. □

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